Pseudoconvex Domains: Where Holomorphic Functions Live

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Why Complex Analysis

- Beautiful theory
- Applications to Pure Math (PDE's, Geometry, Number Theory, ...)
- Applications to Applied Math (Fourier Analysis, Residue Theorem, Numerical Analysis, ...)
- Applications to other fields (Physics, Engineering, ...)

Real Differentiable Functions

$f:(a,b)\to\mathbb{R}$ is (real) differentiable at $p\in(a,b)$ if the following limit exists

$$f'(p)=\lim_{\mathbb{R}\ni h\to 0}\frac{f(p+h)-f(p)}{h}.$$

f is differentiable on (a,b) $\overset{\not\Leftarrow}{\Rightarrow}$ f is continuous on (a,b).

$f(x)=\begin{cases} x^2\sin(1/x), & x\neq 0\\ 0, & x=0\end{cases}$ is differentiable on $\mathbb{R}$ but $f\notin C^2(\mathbb{R})$.

In fact, $C(\mathbb{R})\supsetneq C^1(\mathbb{R})\supsetneq C^2(\mathbb{R})\supsetneq\cdots\supsetneq C^\infty(\mathbb{R})$.

Complex Numbers

Complex Numbers: $\mathbb{C}=\{x+iy : x,y\in\mathbb{R}\}$ where $i^2=-1$.

Fundamental Theorem of Algebra: Every polynomial (with complex coefficients) of degree n has n roots, counting multiplicity.

For example, z^2+1 has two roots $\pm i$ yet x^2+1 has no real roots.

Euler's Formula: $e^{i\theta}=\cos\theta+i\sin\theta$. Then

$$\begin{aligned}\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2) =& e^{i(\theta_1+\theta_2)}\\ =& e^{i\theta_1}e^{i\theta_2}\\ =& (\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\\ =& (\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)\\ & +i(\cos\theta_1\sin\theta_2+\cos\theta_2\sin\theta_1).\end{aligned}$$

Complex Differentiable Functions

Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ be a function, and $p \in U$. Then f is complex differentiable at $p \in U$ if the following limit exists

$$f'(p) = \lim_{\mathbb{C} \ni h \to 0} \frac{f(p+h) - f(p)}{h}.$$

f is complex differentiable (holomorphic) on U if it is complex differentiable at p for every $p \in U$.

Fact: f is holomorphic if and only if $f_{\bar{z}} = 0$ (CR-equations).

$\mathbb{C}$-Differentiable Versus $\mathbb{R}$-Differentiable

Let $F_{\mathbb{C}}$ be complex differentiable and $f_{\mathbb{R}}$ be real differentiable. Then

- $F_{\mathbb{C}}$ is C^∞-smooth but not necessarily $f_{\mathbb{R}}$.
- $F_{\mathbb{C}}$ is analytic but not necessarily $f_{\mathbb{R}}$.
- $F_{\mathbb{C}}$ has max modulus principle but not necessarily $f_{\mathbb{R}}$.
- $F_{\mathbb{C}}$ has integral representation formula (Cauchy integral formula) but not such thing exists for $f_{\mathbb{R}}$.
- $F_{\mathbb{C}}$ satisfies Cauchy-Riemann equations (a PDE):

$$(F_{\mathbb{C}})_{\bar{z}} = 0 \Leftrightarrow u_x = v_y \text{ and } u_y = -v_x \text{ for } F_{\mathbb{C}} = u + iv.$$

$\mathbb{R}$-analytic versus $\mathbb{C}$-analytic

$\dfrac{1}{1+x^2}$ is real analytic on $\mathbb{R}$.

$\dfrac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$ converges for $|x| < 1$ only.

$\dfrac{1}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k z^{2k}$ converges for $|z| < 1$ only.

$\dfrac{1}{1+z^2}$ is not defined at $\pm i$.

The obstruction for analyticity is "detectable" in the complex plane but not necessarily in $\mathbb{R}$.

Why Analysis in $\mathbb{C}^n$

Range: "Could anyone seriously argue that it might be sufficient to train a mathematics major in calculus of functions of one real variable without expecting him or her to learn at least something about partial derivatives, multiple integrals, and some higher dimensional version of the Fundamental Theorem of Calculus? Of course not, the real world is not one-dimensional! But neither is the complex world ..."

"Aside from questions of applicability, shouldn't the pure mathematician's mind wonder about the restriction to functions of only one complex variable? It should not surprise anyone that there is a natural extension of complex analysis to the multivariable setting. What is surprising is the many new and intriguing phenomena that appear when one considers more than one variable. Indeed, these phenomena presented major challenges to any straightforward generalization of familiar theorems... "

Differentiation in $\mathbb{C}^n$

$f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $p \in U$ if there exists a linear function $T : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbb{R}^n \ni h \to 0} \frac{|f(p+h) - f(p) - Th|}{\|h\|} = 0.$$

Fact: If f and all of its partial derivatives are continuous then f is differentiable.

Definition: $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic ($\mathbb{C}$-analytic) if $f \in C(U)$ and

$$f_{\bar{z}_j} = 0 \text{ for } j = 1, 2, \ldots, n.$$

$\mathbb{C}$ versus $\mathbb{C}^n$

- Holomorphic functions
- Cauchy integral formula and its consequences
- Identity principle
- Riemann mapping theorem
- Domain on holomorphy

Riemann Mapping Theorem

In $\mathbb{C}$: $\left\{ \begin{array}{l} \text{There are two non-conformal simply connected} \\ \text{domains: } \mathbb{D} \text{ and } \mathbb{C}. \end{array} \right.$

In $\mathbb{C}^n, n \geq 2$: $\left\{ \begin{array}{l} \text{There are infinitely many non-conformal} \\ \text{simply connected domains.} \end{array} \right.$

[Poincaré] $\left\{ \begin{array}{l} \text{The unit ball and the unit bidisc in } \mathbb{C}^2 \\ \text{are non-conformal.} \end{array} \right.$

Domain of Holomorphy

A domain $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy if for all $p \in \partial\Omega$ there exists $F \in H(\Omega)$ such that F has no holomorphic extension through p.

Example: $\mathbb{C} \setminus \{0\}$ is a domain of holomorphy.

Example: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is a domain of holomorphy. In fact, the series,

$$\sum_{n=1}^{\infty} \frac{z^{2^n}}{2^n}$$

has no extension through any boundary point of $\mathbb{D}$.

Fact: In $\mathbb{C}$ every open set is a domain of holomorphy.

Hartogs Phenomena

[Hartogs] $\mathbb{C}^2 \setminus \overline{B(0,1)}$ is not a domain of holomorphy.

Sketch of Proof: Let f be holomorphic on $\Omega = \mathbb{C}^2 \setminus \overline{B(0,1)}$. Define

$$F(z,w) = \frac{1}{2\pi i}\int_{|\xi|=10} \frac{f(z,\xi)}{w-\xi}d\xi$$

Then F is holomorphic on $\{|z| < \infty, |w| < 10\}$.

Cauchy Integral Formula $\Rightarrow f = F$ for $\{|w| < 5, 2 < |z| < 3\} \subset \Omega$.

Identity Principle $\Rightarrow f = F$ where defined.

Therefore, F extends f as holomorphic onto $\overline{B(0,1)}$.

Examples in $\mathbb{C}^2$

Example 1: $\mathbb{D}^2$ is a domain of holomorphy.

$$f_p(z) = \frac{1}{z_1 - p_1} \text{ if } |p_1| = 1 \text{ and } f_p(z) = \frac{1}{z_2 - p_2} \text{ if } |p_2| = 1.$$

Example 2: $B(0,1)$ is a domain of holomorphy.

$$f_p(z) = \frac{1}{z_1\overline{p}_1 + z_2\overline{p}_2 - 1}.$$

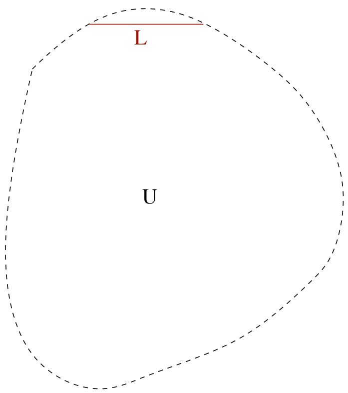
The Levi Problem

Levi Problem: $\left\{\begin{array}{l}\text{Is there a geometric characterization}\\ \text{of domain of holomorphy?}\end{array}\right.$

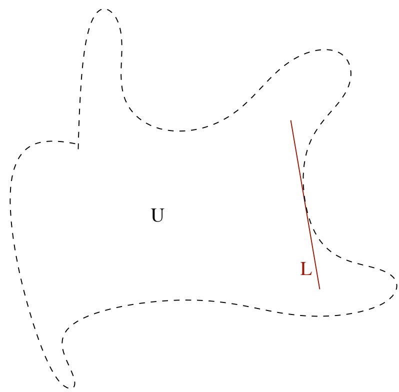
[Oka, Norguet, Bremermann] Yes. It is pseudoconvexity.

A smooth domain $\Omega \subset \mathbb{C}^n$ is said to be pseudoconvex if its Levi form is nonnegative on complex tangential directions on boundary points, $b\Omega$, of Ω.

Convexity



A convex domain



A non-convex domain

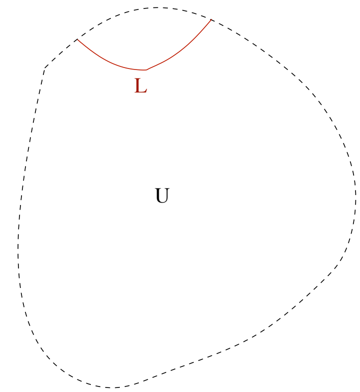
L is a linear image of the unit interval, [0,1].

Pseudoconvexity

A pseudoconvex domain is "convex with respect to holomorphic image of complex discs". (L is holomorphic image of the unit disc)

Convexity $\Rightarrow$ Pseudoconvexity

[Kohn-Nirenberg] Pseudoconvex domains may not be convexifiable.



pseudoconvex domain

Convex $\Rightarrow$ Pseudoconvex

Example 3: Convex Domains in $\mathbb{C}^2$.
Let Ω be a convex domain and $p \in \partial\Omega$. Then

S1: Ω is pseudoconvex $\Leftrightarrow \Omega_p = \Omega - p$ is pseudoconvex.

S2: Ω_p is pseudoconvex $\Leftrightarrow \left\{ \begin{array}{l} \Omega_p^\theta = \{(z_1 e^{i\theta_1}, z_2 e^{i\theta_2}) : z \in \Omega_p\} \\ \text{is pseudoconvex.} \end{array} \right.$

S3: Choose θ so that the (real) normal for Ω_p^θ at 0 is along y_2-axis.

S4: Choose $f(z) = 1/z_2$. Then
Ω_p^θ is pseudoconvex at 0 $\Rightarrow \Omega$ is pseudoconvex at p.

Properties of Pseudoconvexity

Intersection

Ω_1, Ω_2 are pseudoconvex $\Rightarrow \Omega_1 \cap \Omega_2$ is pseudoconvex.

Increasing Union [Behnke-Stein]

$\Omega_j \subset \Omega_{j+1}$ are pseudoconvex for all $j \Rightarrow \bigcup_{j=1}^{\infty} \Omega_j$ is pseudoconvex.

Product

Ω_1, Ω_2 are pseudoconvex $\Rightarrow \Omega_1 \times \Omega_2$ is pseudoconvex.

Locality

$$\Omega \text{ is pseudoconvex} \Leftrightarrow \begin{cases} \text{for every } p \in \partial\Omega \text{ there exists} \\ r > 0 \text{ such that } \Omega \cap B(p,r) \\ \text{is pseudoconvex.} \end{cases}$$

$$\Omega \text{ is pseudoconvex} \Leftrightarrow \begin{cases} \text{there exists } F \in H(\Omega) \text{ with no} \\ \text{holomorphic extension through} \\ \text{any boundary point.} \end{cases}$$

Equivalent Conditions

Let $\Omega \subset \mathbb{C}^n$ be a domain. TFAE

1. Ω is a domain on holomorphy,
2. Ω is pseudoconvex,
3. Ω has a continuous plurisubharmonic exhaustion function: $\{\phi < c\} \Subset \Omega$ where ϕ is continuous plurisubharmonic,
4. Ω has an exhaustion of smooth pseudoconvex domains.

Hull Condition

- $PSH(\Omega)$: continuous plurisubharmonic functions on Ω
- K : compact set in Ω

$$\widehat{K} = \Big\{ z \in \Omega : \phi(z) \leq \sup\{\phi(w) : w \in K\} \text{ for any } \phi \in PSH(\Omega) \Big\}.$$

Example: If $K = S^1$ then $\widehat{K} = \mathbb{D}$. Ω satisfies the hull condition: $\widehat{K} \Subset \Omega$ whenever $K \Subset \Omega$.

$$\Omega \text{ is pseudoconvex } \Leftrightarrow \Omega \text{ satisfies the hull condition.}$$

The $\overline{\partial}$-problem

Let $1 \leq q \leq n$.

We say $\overline{\partial}$ is solvable on $(0,q)$-forms if
Given $f \in C^\infty_{(0,q)}(\Omega)$ with $\overline{\partial} f = 0$ there exists $u \in C^\infty_{(0,q-1)}(\Omega)$ with

$$\overline{\partial} u = f.$$

$$\Omega \text{ is pseudoconvex } \Leftrightarrow \begin{cases} \overline{\partial} \text{ is solvable on } (0,q)\text{-forms} \\ \text{for all } 1 \leq q \leq n. \end{cases}$$

Conclusions

1. Several Complex Variables is very different from complex analysis in one variable.
2. Several Complex Variables has strong connections to PDE's, potential theory, geometry, and analysis.
3. Pseudoconvexity (the "home of holomorphic functions") is a fundamental notion in Several Complex Variables and has many interesting properties.

Further Reading

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