Model Theory and Nilpotence in Groups with Bounded Chains of Centralizers

Paul Baginski

Smith College

Antalya Algebra Days XIV
Cesme, Turkey
May 19, 2012

Joint work with Tuna Altınel of Université Claude Bernard Lyon 1.

Let G be a group, $A \subseteq G$. The **centralizer** of A in G is $C_G(A) = \{g \in G \mid [g, a] = 1 \ \forall a \in A\}$.

A group has **bounded chains of centralizers**, denoted $\mathfrak{M}_C$, if every chain of centralizers

$$1 < C_G(A_1) < C_G(A_2) < \ldots < C_G(A_n)$$

is finite. If there is a uniform bound n on such chains, then the least such $n \geq 1$ is the **centralizer dimension** of G, denoted $\dim(G)$. In this case G has **finite centralizer dimension** $d = \dim(G)$.

- $\dim(G) = 1$ iff G is abelian.
- Ascending vs. descending chains does not matter.

Examples of $\mathfrak{M}_C$ groups.

- Abelian groups
- Torsion-free hyperbolic groups
- Linear groups over fields
- Free groups
- Other familiar infinite groups from group theory
- Stable groups
- rosy groups with NIP
- certain pseudofinite groups

$\mathfrak{M}_C$ is not an elementary class, but being fcd (of a fixed dimension) is.

Subgroups of $\mathfrak{M}_C$ (fcd) groups are $\mathfrak{M}_C$ (fcd of equal or lesser dimension

Quotients of $\mathfrak{M}_C$ or fcd groups are NOT guaranteed to be $\mathfrak{M}_C$ or fcd. Even $G/Z(G)$ could fail to be $\mathfrak{M}_C$.

So induction on the length of chains is much harder.

Can we get a hold of nilpotent subgroups of $\mathfrak{M}_C$ groups?

Motivation: (Poizat) If G is a stable group, and H is a nilpotent subgroup of G, then H is contained in a definable nilpotent subgroup E of G of the same nilpotence class as H.

How close can we get to this in $\mathfrak{M}_C$?

Suppose H is nilpotent and normal. Then H is contained in $F(G)$, the Fitting subgroup, generated by all nilpotent normal subgroups of G.

Theorem (Derakhshan, Wagner 1997). In an $\mathfrak{M}_C$ group G, $F(G)$ is nilpotent.

Theorem (Wagner 1999). In an $\mathfrak{M}_C$ group G, $F(G)$ is definable (with no parameters) and equals the set of bounded left Engel elements.

Uses an important result of Bludov on locally nilpotent $\mathfrak{M}_C$ groups to get the missing step.

Theorem (Ould-Houcine & our referee, 2011) In any group G, if $F(G)$ is nilpotent then it is $\emptyset$-definable.

All of these results rely heavily on relating normal nilpotent subgroups to Engel conditions. Do not have this benefit for

Theorem (Altınel, B. 2011). In an $\mathfrak{M}_C$ group G, if H is a nilpotent subgroup of class n, then there exists a definable subgroup E of G that contains H and is also nilpotent of class n. Furthermore E is normalized by all elements that normalize H.

Groups with fcd: for each d and n, there is a formula $\phi_{d,n}(x, \overline{y})$ with $\ell(y) = dn$ such that if G has dimension d and $H \leq G$ nilpotent of class n, then we can take $E = \phi_{d,n}(G, \overline{a})$ for some $\overline{a} \in G$.

Proof?

- No elementary extensions
- No quotients
- No Engel conditions
- YES, lots of Three Subgroups Lemma to make the most of our commutator identities.

Iterated centralizers

In any group G, we have the upper central series:

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$$

where $Z_{k+1}(G) = \{g \in G \mid [g, G] \leq Z_k(G)\}$. The group $Z_n(G)$ is the n**th center** of G.

We can relativize this chain construction to any subgroup H of G.

$$1 = C_G^0(H) \leq C_G^1(H) \leq C_G^2(H) \leq \dots$$

where

$$C_G^{k+1}(H) = \{g \in G \bigcap_{m \leq k} N_G(C_G^k(H)) \mid [g, H] \leq C_G^k(H)\}.$$

The group $C_G^n(H)$ is the n**th iterated centralizer** of H in G.

The iterated centralizers of H in G.

$$1 = C_G^0(H) \leq C_G^1(H) \leq C_G^2(H) \leq \dots$$

where

$$C_G^{k+1}(H) = \{g \in \bigcap_{m \leq k} N_G(C_G^k(H)) \,|\, [g, H] \leq C_G^k(H)\}.$$

So $C_G^1(H) = C_G(H)$,
$C_G^2(H) = \{g \in N_G(C_G(H)) \,|\, [g, H] \subseteq C_G(H)\}$, etc.

For each n, $C_G^n(H) \cap H = Z_n(H)$.

So if H is nilpotent of class n, $C_G^n(H) \geq Z_n(H) = H$.

Theorem (Altınel, B. 2011). If G is an $\mathfrak{M}_C$ group and $H \leq G$, then the nth iterated center $C_G^n(H)$ is definable with parameters from H.

For each n, there is a uniform definition for $C_G^n(H)$ across groups of dimension d involving dn parameters.

Given $H \leq G$ of nilpotence class n, $C_G^n(H) \geq H$ and it is definable. Yet it need not be nilpotent. Do not even necessarily have $Z_n(C_G^n(H)) \geq H$.

Need a different construction. Observe: $H \leq C_G(C_G(H))$. So rather than iterate *centralizers*, iterate *centralizers of centralizers*.

$$\begin{array}{ccccccccc} 1 & \leq & C_G(H) & \leq & C_G^2(H) & \leq & C_G^3(H) & \leq & \dots \\ \shortparallel & & \cup & & \cup & & \cup & & \\ Z_0(H) & \leq & Z_1(H) & \leq & Z_2(H) & \leq & Z_3(H) & \leq & \dots \end{array}$$

Versus

$$E_0 = G \quad \geq \quad E_1 = C_G(C_G(H)) \quad \geq E_2 \geq E_3 \geq \dots \geq H$$

Since each E_k contains H, we can compute iterated centralizers *inside* E_k.

Our construction guarantees: for all $n \leq j \leq k$

$$C^n_{E_k}(H) = Z_n(E_k) = Z_n(E_j) = C^n_{E_j}(H)$$

$$
\begin{array}{llccccccc}
E_0\colon & 1 & & \leq C^1_{E_0}(H) & & \leq C^2_{E_0}(H) & & \leq C^3_{E_0}(H) & \leq \\
\cup & \parallel & & \cup & & & & & \\
E_1\colon & 1 & \leq & C^1_{E_1}(H)Z_1(E_1) & & \leq C^2_{E_1}(H) & & \leq C^3_{E_1}(H) & \leq \\
\cup & \parallel & & \cup\parallel & & \cup & & & \\
E_2\colon & 1 & \leq & C^1_{E_2}(H)Z_1(E_2) & \leq & C^2_{E_2}(H)Z_2(E_2) & & \leq C^3_{E_2}(H) & \leq \\
\cup & \parallel & & \cup\parallel & & \parallel & & \cup & \\
E_3\colon & 1 & \leq & C^1_{E_3}(H)Z_1(E_3) & \leq & C^2_{E_3}(H)Z_2(E_3) & \leq & C^3_{E_3}(H)Z_3(E_3) & \leq \\
\cup & \parallel & & \cup\parallel & & \parallel & & \parallel & \\
E_4\colon & 1 & \leq & C^1_{E_4}(H)Z_1(E_4) & \leq & C^2_{E_4}(H)Z_2(E_4) & \leq & C^3_{E_4}(H)Z_3(E_4) & \leq C \\
\cup & \parallel & & \cup\parallel & & \parallel & & \parallel & \\
E_5\colon & 1 & \leq & C^1_{E_5}(H)Z_1(E_5) & \leq & C^2_{E_5}(H)Z_2(E_5) & \leq & C^3_{E_5}(H)Z_3(E_5) & \leq C
\end{array}
$$

Our construction guarantees: for all $n \leq j \leq k$

$$C^n_{E_k}(H) = Z_n(E_k) = Z_n(E_j) = C^n_{E_j}(H)$$

Problem with iterated centralizers as definable envelopes was:

• $H \leq C^n_G(H)$, but $C^n_G(H)$ not necessarily nilpotent and may not have $H \leq Z_n(C^n_G(H))$.

With our E_k, we have for H of nilpotence class n

$$H \leq C^n_{E_n}(H) = Z_n(E_n),$$

so $Z_n(E_n)$ is our nilpotent envelope.

Definition of E_k

How to generalize $C_G(C_G(H))$?

$$E_0 := G$$

$$E_{k+1} := \{g \in E_k \mid [g, C_{E_k}^{k+1}(H)] \leq C_{E_k}^{k}(H)\}$$

Example:

$$E_1 = \{g \in E_0 = G \mid [g, C_G^1(H)] \leq C_G^0(H) = 1\} = C_G(C_G(H))$$

Why E_k definable? This definition guarantees that for all $n \leq k$

$$C_{E_k}^{n}(H) = Z_n(E_k)$$

So

$$E_{k+1} := \{g \in E_k \mid [g, C_{E_k}^{k+1}(H)] \leq Z_k(E_k)\}$$

Also, for any $A \subseteq C_{E_k}^{k+1}(H)$ with $C_G(A) = C_G(C_{E_k}^{k+1}(H))$, we have

Continuations?

Theorem (Poizat) If G is a stable group, and H is a solvable subgroup of G, then H is contained in a definable solvable subgroup E of G of the same derived length as H.

True for $\mathfrak{M}_C$?