Valued Difference Fields

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Valued Fields

A valued field is given by a field K, an ordered abelian group Γ, a surjective group homomorphism $v : K^\times \to \Gamma$, such that $v(x-y) \geqslant \min\{v(x), v(y)\}$ (ultrametric triangle inequality). We extend v on K by setting $v(0) = \infty$, and we extend Γ to $\Gamma \cup \{\infty\}$.

In this talk (K, v) will denote a valued field.
important properties:

- $v(1) = v(-1) = 0$
- $v(x) \neq v(y) \Rightarrow v(x-y) = \min\{v(x), v(y)\}$.
  $\Rightarrow$ for a polynomial $P = \sum_i X^i a_i$ and $x \in K$, $v(P(x)) = \min_i\{v(x^i a_i)\} = \min_i\{iv(x) + v(a_i)\}$ if for all $i \neq j$ we have $v(a_j x^i) \neq v(a_i x^j)$.
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This information can be given by $k := K_{\geqslant 0}/K_{>0}$ which is a field , the **residue field of** K. In fact $K_{\geqslant 0}$ is a local ring denoted by $\mathcal{O}_v$, the **valuation ring** of (K, v), and $K_{>0}$ is its maximal ideal.

Characteristic of $(K, v) := (char(K), char(k))$.
In this talk we are interested in equal characteristic (p, p) where $p \in \mathbb{P} \cup \{0\}$.

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Examples

Let k be a field, $k(t)$ is valued by setting $v(t) = 1$, $v_{|k^\times} = 0$.
Hahn Fields: for a field k and ordered abelian group Γ, we set:
$k((\Gamma)) := \{\sum_\gamma a\gamma t^\gamma \mid a_\gamma \in k, \{\gamma \mid a_\gamma \neq 0\}$ is well ordered$\}$
$v(\sum_\gamma a\gamma t^\gamma) :=$ the first γ such that $a_\gamma \neq 0$:
For example: *Laurent Series* $k((\mathbb{Z})) = \{\sum_{i=i_0}^{\infty} a_i t^i\}$, *Puiseux series* $\bigcup_{n>0} k((\frac{1}{n}\mathbb{Z}))$

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Compatible couples of functions

Definition

Let (K, v) be a valued field. A couple of functions (f, f_v) where $f : K \to K$ and $f_v : v(K) \to v(K)$ is said to be **compatible** if $v \circ f = f_v \circ v$.

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Valued difference fields

- If $\sigma \in Aut(K)$ with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$ then σ induces automorphisms:
  σ_v on $v(K)$ and $\overline{\sigma}$ on k; (σ, σ_v) is compatible and σ_v strickly increasing
- $(k, \bar{\sigma})$ is a difference field

In this case we say that (K, v, σ) is a **valued difference field.**
Several people studied valued difference fields,
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σ-polynomials and $\mathbb{Z}[\sigma]$-module $v(K)$

σ-polynomials: A finite sum of σ-monomials which are of the form

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where a is said to be the **coefficient** of M and the $n+1$-tuple $(i_0, i_1, \dots, i_n)$ be the **index** of M, denoted by $ind(M)$. We consider $n+1$ tuples of integers under the partial ordering induced by $\mathbb{N}$.

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Polynomial couples $(P, \cdot P)$

For $P = \sum_j M_j$ a σ-polynomial and for $\gamma \in \Gamma$ we set
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An element $a \in K$ is said to be **regular** for a (σ-) polynomial P, if $v(P(a)) = v(a) \cdot P$, otherwise we say that it is irregular.

Remark

A "regular non-zero root" does not make sense and 0 is always a regular root of any polynomial without constant term.

We will consider polynomials without constant term and equations of type $P(x) = b$ ($b \neq 0$) and say that "a is a regular solution" if $P(a) = b$ with a regular for P.

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σ-linear polynomials and Kaplansky fields

A linear σ-polynomial is one of the form:
$a_n\sigma^n(x) + \cdots + a_1 x$.

If (K, v) is of characteristic (p, p) $(p > 0)$, and perfect, then (K, v) is already a difference valued field with $Frob : x \mapsto x^p$
An **additive polynomial** is an linear Frob-polynomial, i.e. is of the form:
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Definition

A valued field (K, v) is said to be **Kaplansky** if $v(K)$ is p-divisible and if every equation of the form $P(x) = b$ where $P \in k[X]$, is additive, has solutions in k; it is said to be algebraically maximal if it has no proper algebraic extension with same residue field and same value group (that is it has no *immediate* algebraic extension).

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Two very similar caracterization of algebraically maximal Kaplasky fields

Theorem (O.)

A Kaplansky field is algebraically maximal if and only if every equation of the form $P(x) = b$ $(b \neq 0)$, where $P \in K[X]$ is additive, has a regular solution.

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Finding regular elements

Problem : Jump values

$Jump(P) := \{v(x) \mid \text{ x irregular for P}\}$

example: Take $P(X) := X^p - X$, $K := \mathbb{F}_p(t)$. $Jump(P) = \{0\}$ and every $x \in K$ with $v(x) = 0$ is irregular.

$Jump(P)$ is finite $\Rightarrow$ P is "continious" :

for every pseudo-Cauchy (p.c.) sequence $(a_\rho)_\rho$ in K, with a limit a, $(P(a_\rho))_\rho$ has limit $P(a)$.

If $P \in K[X]$ or if P is any σ-polynomial with σ contractive ($:\sigma_v(\gamma) > n\gamma$ for all $\gamma > 0$ and $n \in \mathbb{N}$) then $Jump(P)$ is finite.

if σ is not contractive this can be drastically false: beacause $a_{\rho+1} - a_\rho$ can be always irregular for P.

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Solution

We suppose $\overline{\sigma}^n \neq Id$ on k, for all $n \in \mathbb{N} \setminus \{0\}$.

Lemma

Given a p.c. sequence $(a_\rho)_\rho$ in K, $a \in K$, such that $(a_\rho)_\rho$ converges to a and a σ-polynomial P, we can find a p.c. sequence $(b_\lambda)_\lambda$ such that $(a_\rho)_\rho$ and $(b_\lambda)_\lambda$ have same limits , $(P(b_\rho))_\rho$ converges to $P(a)$.

Proof.

(Main trick) Using above assumption we can find $(b_\lambda)_\lambda$ such that $b_{\lambda+1} - b_\lambda$ is eventually regular for P. □

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We suppose $\overline{\sigma}^n \neq Id$ on k, for all $n \in \mathbb{N} \setminus \{0\}$.

Lemma

Given a p.c. sequence $(a_\rho)_\rho$ in K, $a \in K$, such that $(a_\rho)_\rho$ converges to a and a σ-polynomial P, we can find a p.c. sequence $(b_\lambda)_\lambda$ such that $(a_\rho)_\rho$ and $(b_\lambda)_\lambda$ have same limits , $(P(b_\rho))_\rho$ converges to $P(a)$.

Proof.

(Main trick) Using above assumption we can find $(b_\lambda)_\lambda$ such that $b_{\lambda+1} - b_\lambda$ is eventually regular for P. □

We need more...

From now on we consider the case of equal characteristic $(0, 0)$.

Definition

Given a σ-polynomial P we denote $Lin(P)$ the σ-linear part of P. Let $a \in K$, we say that (P, a) is in σ-hensel configuration if there exists $\gamma \in \Gamma$ such that

1. $v(P(a)) = \gamma \cdot Lin(P)$
2. $\gamma \cdot M < \gamma \cdot M'$ whenever M, M' are monomials of P such that $(0, \ldots, 0) \neq ind(M) < ind(M')$.

Definition

We say that an valued difference field extension of (K, v, σ) is σ-algebraic if all its elements are given by roots of σ-polynomials. (K, v, σ) is said to be σ-algebraically maximal if it has no proper valued difference σ-algebraic extension with same residue field and same value group.

Finding regular solutions: σ-hensenlianty

Lemma

Suppose that (K, v, σ) is σ-algebraically maximal and $(k, \bar{\sigma})$ is **linearly difference closed**, *that is: for every $\bar{\sigma}$-linear Q, and $c \in k$ the equation $Q(x) = c$ has solution in k.*
Conclusion: *For every σ-polynomial P and $b \in K^{\times}$ if for some $a \in K$ such that $v(P(a)) = b$, (P, a) is in σ-hensel configuration then there is a regular solution of the equation $P(x) = b$.*

Definition

(K, v, σ) is said to be σ-henselian if the conclusion of the above lemma holds.

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Draft Results

- All σ-algebraically maximal extensions of a valued difference field with a linearly difference closed residue field are isomorphic.
- (A-K,E) principle for holds for the class of σ-henselian valued difference fields of characteristic $(0, 0)$ with linearly difference closed residue field.

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