Galois Cohomology, Spectral Sequences, and Class Field Theory

Matteo Paganin

Sabancı University

May 19, 2012

Let G be a group, we denote by $\mathcal{C}_G$ the category of G-modules. That is, the category of abelian groups A endowed with a G-action.
This is the same as considering the category $\mathbb{Z}[G]$-mod, hence $\mathcal{C}_G$ is a category like R-mod, for a particular kind of R.
It is somehow natural to consider the functor

$$\begin{aligned} (\ \ )^G \quad : \mathcal{C}_G \quad &\longrightarrow \quad \mathcal{A}b \\ A \quad &\longmapsto \quad A^G = \{a \in A \mid ga = a,\ \forall g \in G\} \end{aligned}$$

The functor $(\ \ )^G$ can also be viewed as $\mathrm{Hom}_G(\mathbb{Z},\ \ )$.

Example (Main - cheating)

Let $(K; +, *)$ be a field. We denote by G_K the absolute Galois group of K, that is the Galois group of the extension K^s/K.
By definition,

- $(K^s, +)$ is a G_K-module,
- $((K^s)^\times, *)$ is a G_K-module.

By construction, we have

- $(K^s)^{G_K} = K$,
- $((K^s)^\times)^{G_K} = K^\times$.

Let G be a group, we denote by $\mathcal{C}_G$ the category of G-modules. That is, the category of abelian groups A endowed with a G-action.
This is the same as considering the category $\mathbb{Z}[G]$-mod.
It is somehow natural to consider the functor

$$\begin{array}{rcl} (\quad)^G \ : \mathcal{C}_G & \longrightarrow & \mathcal{A}b \\ A & \longmapsto & A^G = \{a \in A \mid ga = a, \ \forall g \in G\} \end{array}$$

- the category $\mathcal{C}_G$ has enough injectives;
- the functor $(\quad)^G$ is left-exact: for every *exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0, \quad f \text{ inj., } g \text{ surj., and } \mathrm{Im}(f) = \ker(g),$$

  the sequence

$$0 \to A^G \xrightarrow{f} B^G \xrightarrow{g} C^G, \quad f \text{ inj. and } \mathrm{Im}(f) = \ker(g)$$

  is also exact.

Hence, we can define the *right derived functors* of $(\quad)^G$, that are usually denoted by

$$\mathrm{H}^n(G, \quad).$$

There is an explicit description for $\mathrm{H}^n(G, A)$, given a G-module A, when n is small:

- for $n = 0$, we have $\mathrm{H}^0(G, A) = A^G$
- for $n = 1$, we have the following:

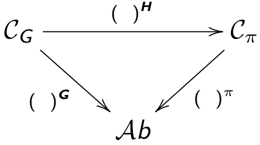
$$\mathrm{H}^1(G, A) = \frac{Z^1(G, A)}{B^1(G, A)} = \frac{\{f : G \to A \mid f(gh) = hf(g) + f(h), \forall g, h \in G\}}{\{f : G \to A \mid \exists a \in A \mid f(g) = ga - a, \forall g \in G\}}$$

Example

Assume A is a trivial G-module.Then

- $\mathrm{H}^0(G, A) = A^G = A$,
- $\mathrm{H}^1(G, A) = \dfrac{Z^1(G, A)}{B^1(G, A)} = \dfrac{\mathrm{Hom}_{\mathbb{Z}}(G, A)}{\langle 0 \rangle} = \mathrm{Hom}_{\mathbb{Z}}(G, A)$.

Let G be again any group. Fix a subgroup H of G. A G-module is also a H-module in natural way. Hence $\mathrm{H}^n(H, A)$ are computable.
If, moreover, H is normal, let us denote by π the quotient G/H. Then, A^H is also π-module.
The functors $(\ )^G$, $(\ )^H$, and $(\ )^\pi$ are related by the following diagram:



Likewise, $\mathrm{H}^n(H, A)$ has a natural structure of π-modules for every n.
This decomposition of the functor $(\ )^G$ is useful also when computing its cohomology.

One of the main tool to deal with cohomology are spectral sequences. The one we are interested in is the so called Lyndon-Hochschild-Serre spectral sequence. We summarize the main results in the following:

Theorem

Let G be a group and H a normal subgroup. For any G-module A, there exists a spectral sequence E_r^{pq} such that the second level is

$$E_2^{pq} = \mathrm{H}^p(\pi, \mathrm{H}^q(H, A)).$$

Moreover, E_r^{pq} converges to $\mathrm{H}^{p+q}(G, A)$. The standard notation is:

$$E_2^{pq} = \mathrm{H}^p(\pi, \mathrm{H}^q(H, A)) \Rightarrow \mathrm{H}^{p+q}(G, A).$$

Tate said: ''Number Theory is the study of $G_{\mathbb{Q}}$''.

From now on, we assume that G is a *profinite* group. If we regard a G-module A as a discrete topological space, we can restrict our attention to the G-modules with a *continuous* action.We obtain the followings:

- the category $\mathcal{C}_G$ of *discrete G-modules* is still an abelian category,
- the category $\mathcal{C}_G$ has still enough injectives,
- the functor $(\ \ )^G$ is still left exact,
- the groups $\mathrm{H}^n(G, A)$ are *torsion groups* for any discrete G-module A and for every $n > 0$,
- the Lyndon-Hochschild-Serre spectral sequence remains defined and it keeps all the properties stated (adding the requirement for the subgroup H to be closed).

Example (Main)

Let $(K;+,*)$ be a field, $G_K = \mathrm{Gal}(K^s/K)$ the absolute Galois group of K.

- $((K^s)^\times, *)$ is a G_K-module.
- $\mathrm{H}^0(G_K, (K^s)^\times) = ((K^s)^\times)^{G_K} = K^\times$,
- $\mathrm{H}^1(G_K, (K^s)^\times) = 0$, by the so-called Hilbert 90 theorem,
- $\mathrm{H}^2(G_K, (K^s)^\times) = \mathrm{Br}(K)$, the Brauer group of K. It can be proved that $\mathrm{Br}(K)$ classifies the central simple algebras over K.

Let G be group and H a closed normal subgroup. We also assume that H is open, that is of finite index, hence, for the finite group $\pi = G/H$, the Tate cohomology $\hat{\mathrm{H}}^n(\pi, \ )$, $n \in \mathbb{Z}$ is defined (but explicitly not in this talk).

$$\text{General term of level 2 for LHS:} \qquad E_2^{pq} = \mathrm{H}^p(\pi, \mathrm{H}^q(H, A)).$$

Why p can't be negative?

Definition

Let G be a profinite group. We say that G has *cohomological dimension smaller than n* if $\mathrm{H}^q(G, A) = 0$ for every torsion G-module A and for every $q > n$.
We write $cd(G) \leq n$.

Definition

Let G be a profinite group. We say that G has *strict cohomological dimension smaller than n* if $\mathrm{H}^q(G, A) = 0$ for every G-module A and for every $q > n$.
We write $scd(G) \leq n$.

We have that $\mathrm{scd}(G) - \mathrm{cd}(G)$ is either 0 or 1.
Moreover, if H is an open subgroup of G, then $\mathrm{cd}(H) = \mathrm{cd}(G)$ and $\mathrm{scd}(H) = \mathrm{scd}(G)$.

Theorem

Let G be a profinite group and H an open normal subgroup of G. Denote by π the quotient G/H. Then, for any discrete G-module A, there exists a spectral sequence such that the second level has the form:

$$E_2^{pq} = \hat{\mathrm{H}}^p(\pi, \mathrm{H}^q(H, A)).$$

We denote this spectral sequence by $\hat{E}_r^{pq}$ and we call it the Tate cohomology spectral sequence. Moreover, if G has finite cohomological dimension, we have that

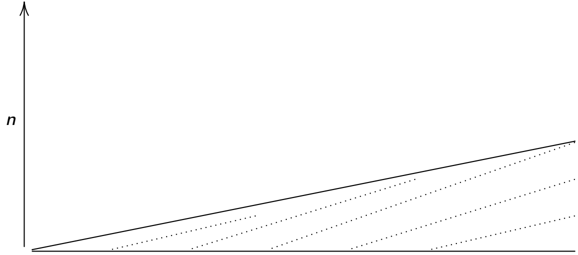
$$\hat{E}_r^{pq} \Rightarrow 0.$$

Definition

A *spectral sequence* in an abelian category $\mathcal{C}$ is a family $\{E_r^{pq}, d_r^{pq}\}_{p,q,r\in\mathbb{Z},r>a}$, where the E_r^{pq} are objects of $\mathcal{C}$ and $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ are morphisms with the following relations:

$$d_r^{pq} \circ d_r^{p-r,q+r-1} = 0 \quad \text{and} \quad E_{r+1}^{pq} = \ker(d_r^{pq})/\mathrm{Im}(d_r^{p-r,q+r-1}).$$

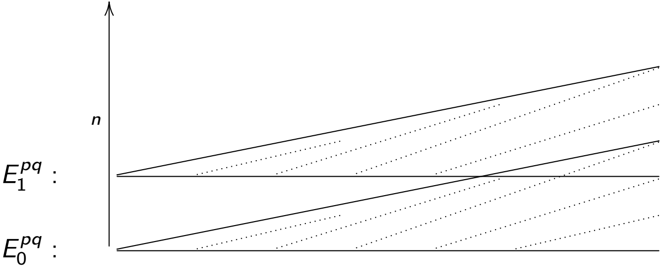
E_0^{pq} :



Definition

A *spectral sequence* in an abelian category $\mathcal{C}$ is a family $\{E_r^{pq}, d_r^{pq}\}_{p,q,r\in\mathbb{Z},r>a}$, where the E_r^{pq} are objects of $\mathcal{C}$ and $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ are morphisms with the following relations:

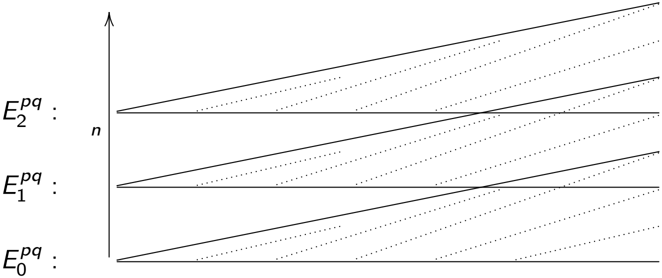
$$d_r^{pq} \circ d_r^{p-r,q+r-1} = 0 \quad \text{and} \quad E_{r+1}^{pq} = \ker(d_r^{pq})/\mathrm{Im}(d_r^{p-r,q+r-1}).$$



Definition

A *spectral sequence* in an abelian category $\mathcal{C}$ is a family $\{E_r^{pq}, d_r^{pq}\}_{p,q,r\in\mathbb{Z},r>a}$, where the E_r^{pq} are objects of $\mathcal{C}$ and $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ are morphisms with the following relations:

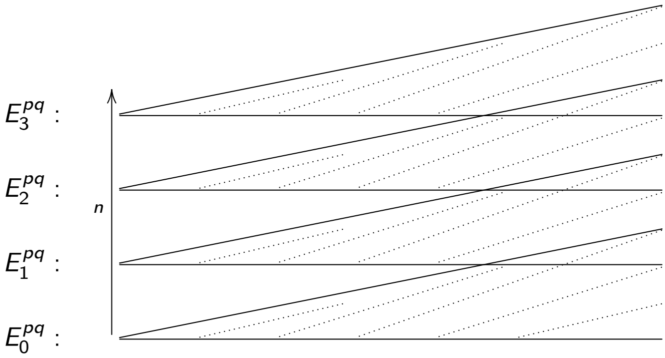
$$d_r^{pq} \circ d_r^{p-r,q+r-1} = 0 \quad \text{and} \quad E_{r+1}^{pq} = \ker(d_r^{pq})/\mathrm{Im}(d_r^{p-r,q+r-1}).$$

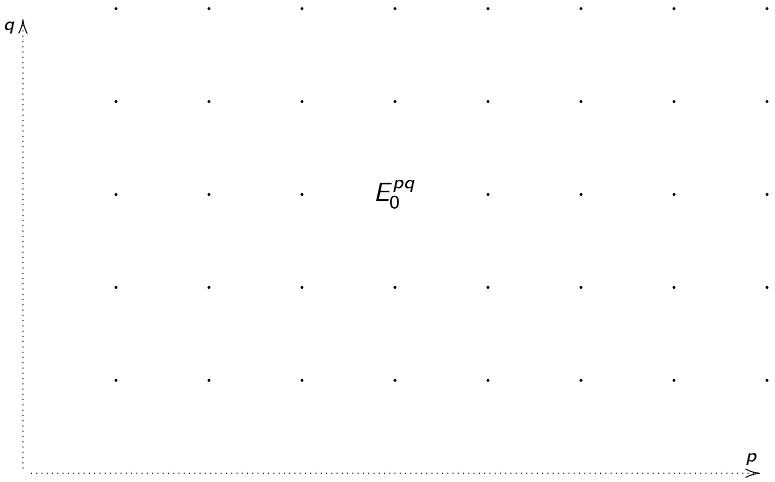


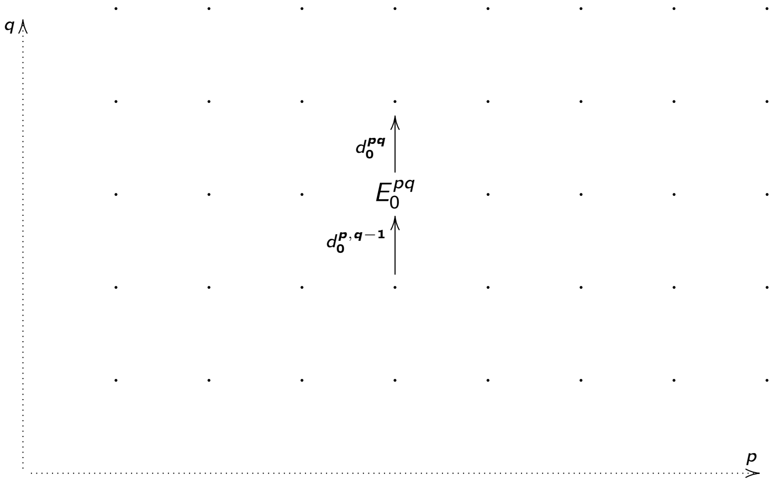
Definition

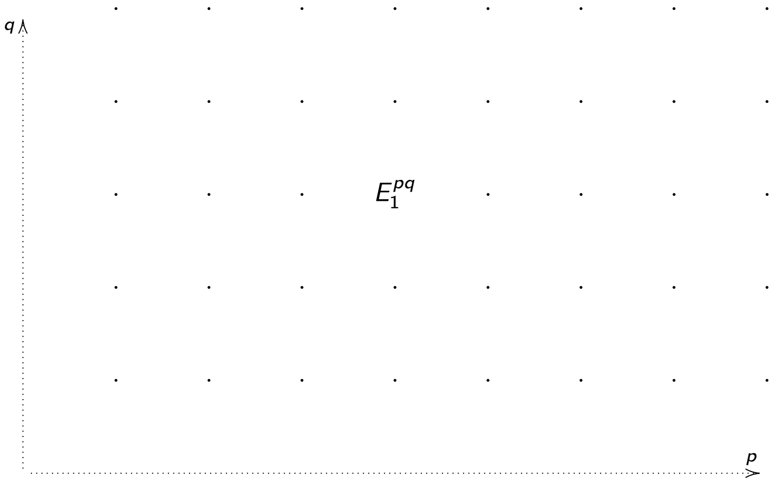
A *spectral sequence* in an abelian category $\mathcal{C}$ is a family $\{E_r^{pq}, d_r^{pq}\}_{p,q,r\in\mathbb{Z},r>a}$, where the E_r^{pq} are objects of $\mathcal{C}$ and $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ are morphisms with the following relations:

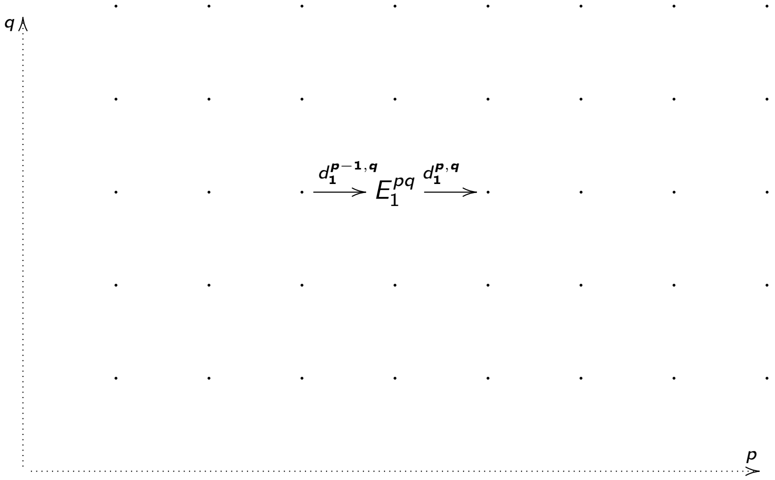
$$d_r^{pq} \circ d_r^{p-r,q+r-1} = 0 \quad \text{and} \quad E_{r+1}^{pq} = \ker(d_r^{pq})/\mathrm{Im}(d_r^{p-r,q+r-1}).$$

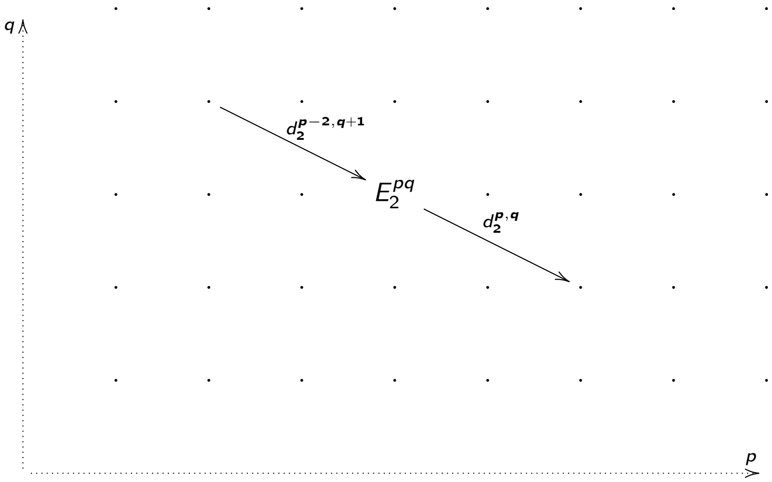



q
E_0^{pq}
p

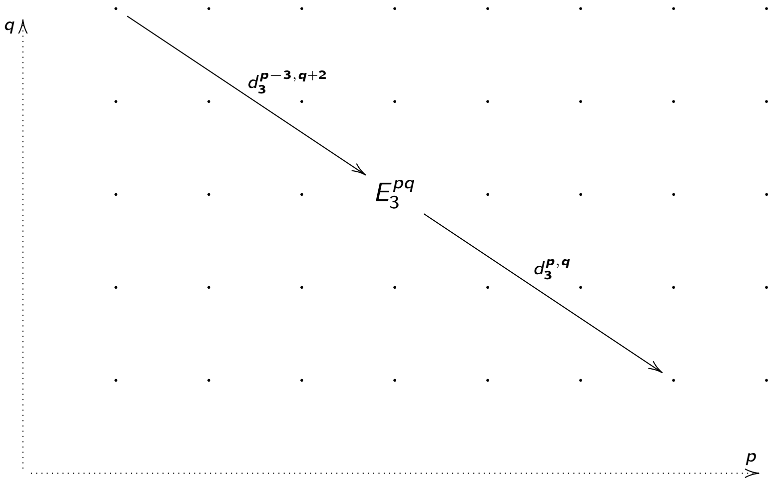

q
p
d_0^{pq}
E_0^{pq}
$d_0^{p,q-1}$

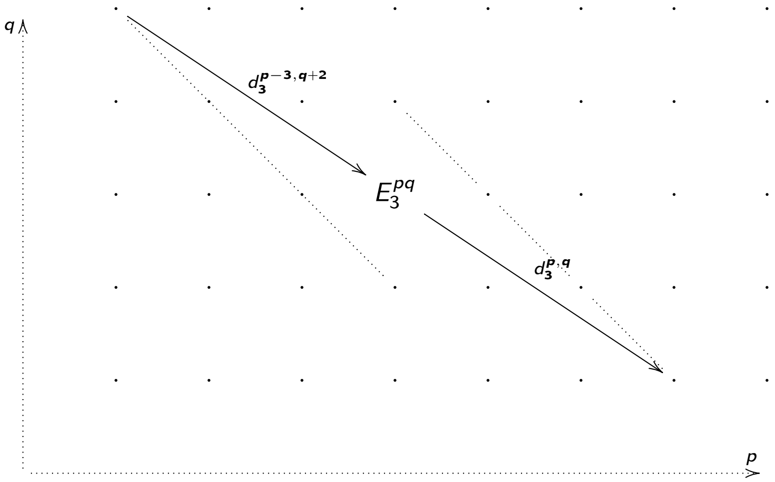

q
p
E_1^{pq}


q
p
$\xrightarrow{d_1^{p-1,q}} E_1^{pq} \xrightarrow{d_1^{p,q}}$



q
p
$d_2^{p-2,q+1}$
E_2^{pq}
$d_2^{p,q}$


q
$d_3^{p-3,q+2}$
E_3^{pq}
$d_3^{p,q}$
p


q
$d_3^{p-3,q+2}$
E_3^{pq}
$d_3^{p,q}$
p

Theorem

Let G be a profinite group and H an open normal subgroup of G. Denote by π the quotient G/H. Then, for any discrete G-module A, there exists a spectral sequence such that the second level has the form:

$$E_2^{pq} = \hat{\mathrm{H}}^p(\pi, \mathrm{H}^q(H, A)).$$

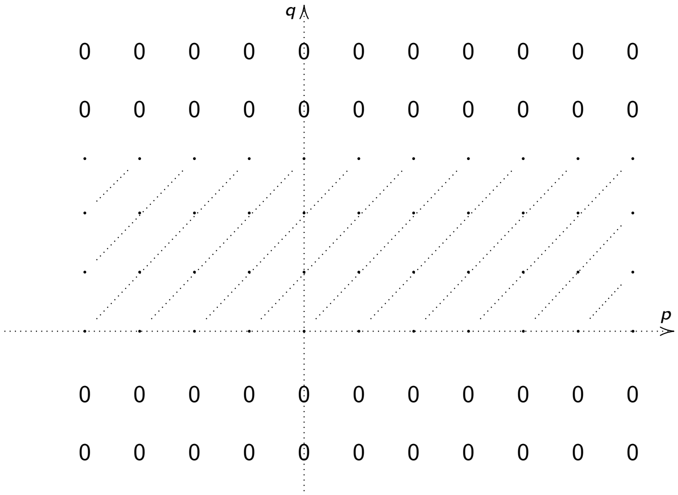
We denote this spectral sequence by $\hat{E}_r^{pq}$ and we call it the Tate cohomology spectral sequence. Moreover, if G has finite cohomological dimension, we have that

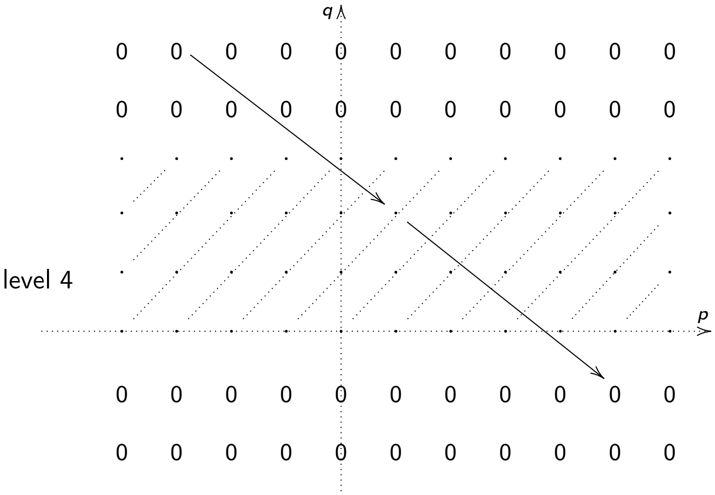
$$\hat{E}_r^{pq} \Rightarrow 0.$$

Note that $\mathrm{cd}(G) \leq n$ implies that $\mathrm{cd}(H) \leq n$,hence

$$\hat{E}_2^{pq} = \hat{\mathrm{H}}^p(\pi, \mathrm{H}^q(H, A)) = \hat{\mathrm{H}}^p(\pi, 0) = 0 \quad \text{for } q > n.$$

Moreover, $\hat{E}_r^{pq} = 0$ for every $r > 2$ and for every $q > n$.


q
p
0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0



We can say that $E_n^{pq} = E_4^{pq}$ for every p, q, and $n > 4$.

Example (Main)

Let E/F be a finite extension of local fields.

- G Denote the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ of F by G_F,
- H denote the absolute Galois group of E by G_E,
- π and the finite Galois group $\mathrm{Gal}(E/F)$ by π.

For the module A, take the multiplicative group $\bar{F}^\times$.

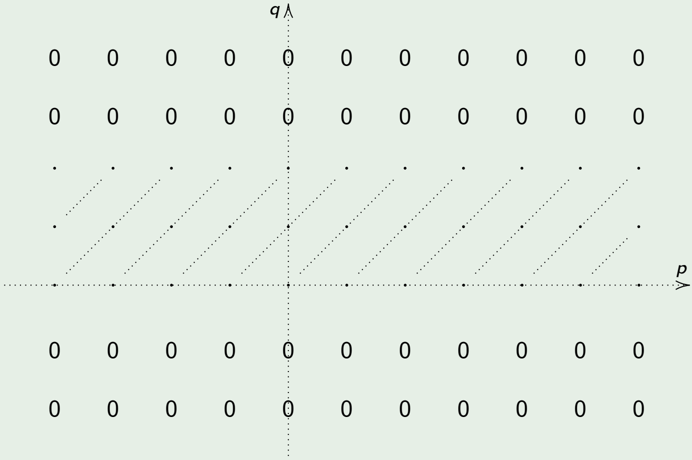
- By Hilbert 90, we have $\mathrm{H}^1(G_E, \bar{F}^\times) = 0$.
- It can be proved that both G_F and its subgroup G_E have cohomological dimension and strict cohomological dimension equal to 2, thus $\mathrm{H}^q(G_E, \bar{F}^\times)$ vanishes for $q \geq 3$.

Hence:

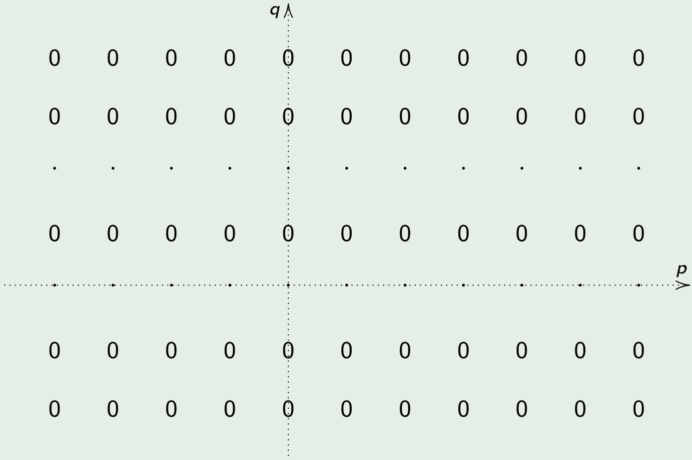
$$\hat{E}_2^{pq} = \hat{\mathrm{H}}^p(\pi, \mathrm{H}^q(G_E, \bar{F}^\times)) = 0 \quad \text{if } q \neq 0, 2$$

- for $q = 0$, we have $\hat{E}_2^{p0} = \hat{\mathrm{H}}^p(\pi, \mathrm{H}^0(G_E, \bar{F}^\times)) = \hat{\mathrm{H}}^p(\pi, E^\times)$;
- for $q = 2$, we have $\hat{E}_2^{p2} = \hat{\mathrm{H}}^p(\pi, \mathrm{H}^2(G_E, \bar{F}^\times)) = \hat{\mathrm{H}}^p(\pi, \mathbb{Q}/\mathbb{Z})$.

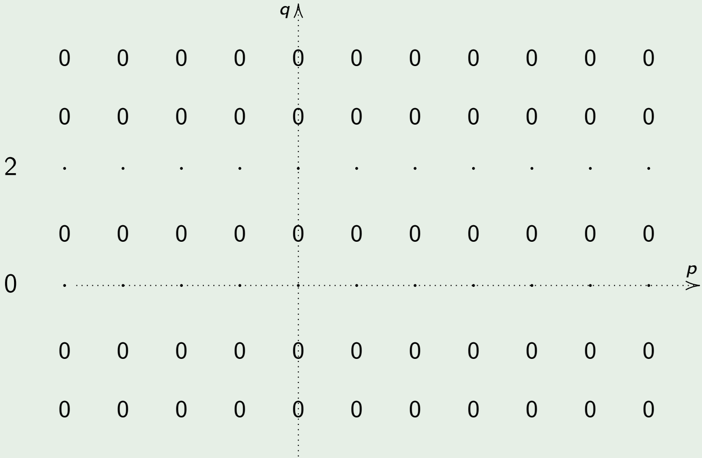
Example (Main - cont.)



Example (Main - cont.)



Example (Main - cont.)



$\hat{E}_2^{p0} = \hat{\mathrm{H}}^p(\pi, E^\times)$ and $\hat{E}_2^{p2} = \hat{\mathrm{H}}^p(\pi, \mathbb{Q}/\mathbb{Z})$.

Example (Main - cont.)

Remarks:

- $d_2^{pq} = 0$ for every p and q, hence $\hat{E}_3^{pq} = \hat{E}_2^{pq}$,
- from the convergence to 0 of the spectral sequence, it follows that $d_3^{pq} : \hat{E}_3^{p-1,2} \to \hat{E}_3^{p0}$ is an isomorphism for every p and q,
- $\hat{\mathrm{H}}^{p-1}(\pi, \mathbb{Q}/\mathbb{Z}) \simeq \hat{\mathrm{H}}^{p}(\pi, \mathbb{Z})$, for every p.

To sum up, we obtained a family of isomorphisms

$$\hat{\mathrm{H}}^{p}(\pi, \mathbb{Z}) \simeq \hat{\mathrm{H}}^{p-1}(\pi, \mathbb{Q}/\mathbb{Z}) \xrightarrow{d_3^{p-1,2}} \hat{\mathrm{H}}^{p+2}(\pi, E^\times).$$

- taking $p = -2$, we obtain

$$\pi^{\mathrm{ab}} \simeq \hat{\mathrm{H}}^{-2}(\pi, \mathbb{Z}) \simeq \hat{\mathrm{H}}^{0}(\pi, E^\times) \simeq F^\times / N(E^\times),$$

- taking $p = 0$, we obtain

$$\mathbb{Z}/n\mathbb{Z} \simeq \hat{\mathrm{H}}^{0}(\pi, \mathbb{Z}) \simeq \hat{\mathrm{H}}^{2}(\pi, E^\times) \simeq \mathrm{Br}(E/F).$$

Example (Main - cont.)

On the other hand, let $c_{E/F}$ be the fundamental class of the extension E/F, which is a particular generator of the cyclic group $\mathrm{H}^2(\pi, E^\times)$.A well-known result of class field theory (see section XIII.4 of *Corps Locaux, J.P. Serre*) states that the morphism

$$\hat{\mathrm{H}}^p(\pi, \mathbb{Z}) \xrightarrow{\cup c_{E/F}} \hat{\mathrm{H}}^{p+2}(\pi, E^\times) \tag{1}$$

is an isomorphism for every p.
With D. Vauclair of the "Université de Caen - Basse Normandie", we proved that the isomorphisms constructed coincide with the ones induced by the fundamental class $c_{E/F}$.

Thank you for your attention!
(and sorry for the headache)