Alternatives for pseudofinite groups.

Abderezak Ould Houcine, Françoise Point

UMons

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Background

A group G (respectively a field K) is *pseudofinite* if it is elementary equivalent to an ultraproduct of finite groups (respectively of finite fields).

Equivalently, G is pseudofinite if G is a model of the theory of the class of finite groups (respectively of finite fields); (i.e. any sentence true in G is also true in some finite group).

Note that here a pseudofinite structure may be finite.

Examples of pseudofinite groups

1. General linear groups over pseudofinite fields ($GL_n(K)$, where K is a pseudofinite field).
   (Infinite pseudofinite fields have been characterized algebraically by J. Ax).
2. Any pseudofinite infinite simple group is isomorphic to a Chevalley group (of twisted or untwisted type) over a pseudofinite field (Felgner, Wilson, Ryten); pseudofinite definably simple groups (P. Urgulu).
3. Pseudofinite groups with a theory satisfying various model-theoretic assumptions like stability, supersimplicity or the non independence property (NIP) have been studied (Macpherson, Tent, Elwes, Jaligot, Ryten)
4. G. Sabbagh and A. Khélif investigated finitely generated pseudofinite groups.

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Alternatives

Tits alternative: a linear group, i.e. a subgroup of some $GL(n, K)$, with K a field, either contains a free nonabelian group F_2 or is soluble-by-(locally finite).

Outline of the talk

1. First, we relate the notion of being pseudofinite with other approximability properties of a class of groups.
2. Transfer of definability properties in classes of finite groups.
3. Properties of finitely generated pseudofinite groups.
4. An $\aleph_0$-saturated pseudofinite group either contains M_2, the free subsemigroup of rank 2 or is nilpotent-by-(uniformly locally finite) (and so is supramenable).
5. An $\aleph_0$-saturated pseudo-(finite of (weakly) bounded Prüfer rank) group either contains F_2 or is nilpotent-by-abelian-by-uniformly locally finite (and so uniformly amenable.
6. Pseudofinite groups of bounded c-dimension (E. Khukhro).

Approximability

Notation: Given a class $\mathcal{C}$ of $\mathcal{L}$-structures, we will denote by $Th(\mathcal{C})$ (respectively by $Th_\forall(\mathcal{C})$) the set of sentences (respectively universal sentences) true in all elements of $\mathcal{C}$.

Given a set I, an ultrafilter $\mathcal{U}$ over I and a set of $\mathcal{L}$-structures $(C_i)_{i\in I}$, we denote by $\prod_I^{\mathcal{U}} C_i$ the ultraproduct of the family $(C_i)_{i\in I}$ relative to $\mathcal{U}$.

Approximability

Definition: Let $\mathcal{C}$ be a class of groups.

• A group G is called approximable by $\mathcal{C}$ (or locally $\mathcal{C}$ or locally embeddable into $\mathcal{C}$) if for any finite subset $F \subseteq G$, there exists a group $G_F \in \mathcal{C}$ and an *injective* map $\xi_F : F \to G_F$ such that $\forall g, h \in F$, if $gh \in F$, then $\xi_F(gh) = \xi_F(g)\xi_F(h)$.
When $\mathcal{C}$ is a class of finite groups, then G is called *LEF* (A.Vershik and E. Gordon).

• A group G is called residually-$\mathcal{C}$, if for any nontrivial element $g \in G$, there exists a homomorphism $\varphi : G \to C \in \mathcal{C}$ such that $\varphi(g) \neq 1$.

• A group G is called fully residually-$\mathcal{C}$, if for any finite subset S of nontrivial elements of G, there exists a homomorphism $\varphi : G \to C \in \mathcal{C}$ such that $1 \notin \varphi(S)$.

• A group G is called pseudo-$\mathcal{C}$ if G satisfies $Th(\mathcal{C})=\bigcap_{C\in\mathcal{C}} Th(C)$.

Approximability

Proposition Let G be a group and $\mathcal{C}$ a class of groups. The following properties are equivalent.

1. The group G is approximable by $\mathcal{C}$.
2. G embeds in an ultraproduct of elements of $\mathcal{C}$.
3. G satisfies $Th_\forall(\mathcal{C})$.
4. Every finitely generated subgroup of G is approximable by $\mathcal{C}$.
5. For every finitely generated subgroup L of G, there exists a sequence of finitely generated residually-$\mathcal{C}$ groups $(L_n)_{n\in\mathbb{N}}$ and a sequence of homomorphisms $(\varphi_n : L_n \to L_{n+1})_{n\in\mathbb{N}}$ such the following properties holds:

   (i) L is the direct limit, $L = \varinjlim L_n$, of the system $\varphi_{n,m} : L_n \to L_m$, $m \geq n$, where $\varphi_{n,m} = \varphi_m \circ \varphi_{m-1} \cdots \circ \varphi_n$.
   (ii) For any $n \geq 0$, for any finite subset S of L_n, if $1 \notin \psi_n(S)$, where $\psi_n : L_n \to L$ is the natural map, there exists a homomorphism $\varphi : L_n \to C \in \mathcal{C}$ such that $1 \notin \varphi(S)$.

Approximability–Examples

1. Let $\mathcal{C}$ be the class of finite groups. A locally residually finite group is locally $\mathcal{C}$ (Vershik, Gordon). There are groups which are not residually finite and which are approximable by $\mathcal{C}$, for instance, there are finitely generated amenable LEF groups which are not residually finite (de Cornulier). There are residually finite groups which are not pseudofinite, for instance the free group F_2.
2. Let $\mathcal{C}$ be the class of free non abelian groups. Let G be a non abelian group. Then, if G is fully residually-$\mathcal{C}$ (or equivalently ω-residually free or a limit group), then G is approximable by $\mathcal{C}$ (Chiswell). Conversely if G is approximable by $\mathcal{C}$, then G is locally fully residually-$\mathcal{C}$. The same property holds also in hyperbolic groups (Sela, Weidmann) and more generally in equationally noetherian groups (Ould Houcine).

Approximability–Examples (continued)

(3) Let V be a possibly infinite-dimensional vector space over a field K. Denote by $GL(V,K)$ the group of automorphisms of V.
Let $g \in GL(V,K)$, then g has finite residue if the subspace $C_V(g) := \{v \in V : g.v = v\}$ has finite-co-dimension.
A subgroup G of $GL(V,K)$ is called a *finitary* (infinite-dimensional) linear group, if all its elements have finite residue.
A subgroup G of $\prod_{i\in I}^{\mathcal{U}} GL(n_i, K_i)$, where K_i is a field, is *of bounded residue* if for all $g \in G$, where $g := [g_i]_{\mathcal{U}}$,
$res(g) := inf\{n \in \mathbb{N} : \{i \in I : res(g_i) \leq n\} \in \mathcal{U}\}$ is finite.
E. Zakhryamin has shown that any finitary (infinite-dimensional) linear group G is isomorphic to a subgroup of bounded residue of some ultraproduct of finite linear groups.
In particular letting $\mathcal{C} := \{GL(n,k)$, where k is a finite field and $n \in \mathbb{N}\}$, any finitary (infinite-dimensional) linear group G is approximable by $\mathcal{C}$.

Definability– Easy Lemmas–Wilson's result on radical

Lemma Let G be a pseudofinite group. Any definable subgroup or any quotient by a definable normal subgroup is pseudofinite.

Let G be a finite group and let $rad(G)$ be the soluble radical, that is the largest normal soluble subgroup of G.
Theorem (J. Wilson) There exists a formula: $\phi_R(x)$, such that in any finite group G, $rad(G)$ is definable by ϕ_R.

Lemma: If G is a pseudofinite group then $G/\phi_R(G)$ is a pseudofinite semi-simple group.

Lemma: Let G be an $\aleph_0$-saturated group. Then either G contains F_2, or G satisfies a nontrivial identity (in two variables). In the last case, either G contains M_2, or G satisfies a finite disjunction of positive nontrivial identities in two variables.

Definability of verbal subgroups in classes of finite groups.

Notation: Let G^n be the verbal subgroup of G generated by the set of all g^n with $g \in G$, $n \in \mathbb{N}$. The width of this subgroup is the maximal number (if finite) of n^{th}-powers necessary to write an element of G^n.

Theorem (N. Nikolov, D. Segal) There exists a function $d \to c(d)$, such that if G is a d-generated finite group and H is a normal subgroup of G, then every element of $[G, H]$ is a product of at most $c(d)$ commutators of the form $[h, g]$, $h \in H$ and $g \in G$. In a finite group G generated by d elements, the verbal subgroup G^n is of finite width bounded by a function $b(d, n)$.

Restricted Burnside problem.

Positive solution of the restricted Burnside problem: (E. Zemanov)
Given k, d, there are only finitely many finite groups generated by k elements of exponent d.

Recall that a group is said to be *uniformly locally finite* if for any $n \geq 0$, there exists $\alpha(n)$ such that any n-generated subgroup of G has cardinality bounded by $\alpha(n)$.

Lemma A pseudofinite group of finite exponent is uniformly locally finite.

Corollary A group G approximable by a class $\mathcal{C}$ of finite groups of bounded exponent is uniformly locally finite.

Definability of verbal subgroups in pseudofinite groups

Lemma Suppose that there exists an infinite set $U \subseteq \mathbb{N}$ such that for any $n \in U$, the finite group G_n involves A_n. Then for any non-principal ultrafilter $\mathcal{U}$ containing U, $G := \prod_{\mathbb{N}}^{\mathcal{U}} G_n$ contains F_2.

Proposition Let L be a pseudo-(d-generated finite groups). Then,
(1) For any definable subgroup H of L, the subgroup $[H, L]$ is definable. In particular the terms of the descending central series of L are 0-definable and of finite width.
(2) The verbal subgroups L^n, $n \in \mathbb{N}^*$, are 0-definable of finite width and of finite index.

Finitely generated pseudofinite groups

Proposition Let G be a finitely generated pseudofinite group and suppose that G satisfies one of the following conditions.

1. G is of finite exponent, or
2. (Khélif) G is soluble, or
3. G is soluble-by-(finite exponent), or
4. G is pseudo-(finite linear of degree n in characteristic zero), or
5. G is simple.

Then such a group G is finite.

Applications

A group G is *CSA* if for any maximal abelian group A and any $g \in G - A$, $A^g \cap A = \{1\}$.

- A finite CSA group is abelian.

Corollary: There are no nontrivial torsion-free hyperbolic pseudofinite groups

Proof: A torsion-free hyperbolic group is a CSA-group and thus if it were pseudofinite then it would be abelian and there are no infinite abelian finitely generated pseudofinite groups (Sabbagh).

The free monoid and supra-amenability.

Theorem Let G be an $\aleph_0$-saturated pseudofinite group. Then either G contains a free subsemigroup of rank 2 or G is nilpotent-by-(uniformly locally finite).

Definition

• Let G be a group and S a finite generating set of G. We let $\gamma_S(n)$ to be the cardinal of the ball of radius n in G (for the word distance with respect to $S \cup S^{-1}$), namely $|B^G_{S\cup S^{-1}}(n)|$.

• A group G is said to be *exponentially bounded* if for any finite subset $S \subseteq G$, and any $b > 1$, there is some $n_0 \in \mathbb{N}$ such that $\gamma_S(n) < b^n$ whenever $n > n_0$.

• A group G is supramenable if for any $A \subset G$, there is a finitely additive measure μ on $\mathcal{P}(G)$ invariant by right translation such that $\mu(A) = 1$.

Alternative: $\not\supset M_2$/ supra-amenability

Corollary Let G be an $\aleph_0$-saturated pseudofinite group. Then the following properties are equivalent.

(1) G is superamenable.

(2) G has no free subsemigroup of rank 2.

(3) G is nilpotent-by-(uniformly locally finite).

(4) G is nilpotent-by-(locally finite).

(5) Every finitely generated subgroup of G is nilpotent-by-finite.

(6) G is exponentially bounded.

Already known: (6) $\Rightarrow$ (1), (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (3), (5) $\Rightarrow$ (6).

Milnor identities.

For $a, b \in G$, we let $H_{a,b} =< a^{b^n} | n \in \mathbb{Z} >$ and $H'_{a,b}$ its derived subgroup.

A nontrivial word $t(x, y)$ in x, y is a *N-Milnor word* of *degree* $\leq \ell$ if it can be put in the form $yx^{m_1}y^{-1}...y^{\ell}x^{m_\ell}y^{-\ell}.u = 1$, where $u \in H'_{x,y}$, $\ell \geq 1$, $gcd(m_1, ..., m_\ell) = 1$ (some of the m_i's are allowed to take the value 0) and $\sum_{i=1}^{\ell} |m_i| \leq N$.
A group G is locally N-Milnor if for every $a, b \in G$ there is a nontrivial N-Milnor word $t(x, y)$ such that $t(a, b) = 1$.

(Rosenblatt) Let $G \not\supset M_2$, where M_2 is the free subsemigroup of rank 2. Then for any $a, b \in G$, the subgroup $H_{a,b}$ is finitely generated, and G is locally 1-Milnor.

To a Milnor word $t(x, y) := yx^{m_1}y^{-1}...y^{\ell}x^{m_\ell}y^{-\ell}.u$, $u \in H'_{x,y}$, one associates a polynomial $q_t[X] = \sum_{i=1}^{\ell} m_i.X^i \in \mathbb{Z}[X]$.

Milnor identities and finite groups.

Theorem: (Traustason) Given a finite number of Milnor words $t_i,\ i \in I$ and their associated polynomials q_{t_i}, $i \in I$, there exist positive integers $c(q)$ and $e(q)$ only depending on $q := \prod_{i \in I} q_{t_i}$, such that a finite group G satisfying $\bigvee_{i \in I} t_i = 1$, is nilpotent of class $\leq c(q)$-by-exponent dividing $e(q)$.

$(2) \Rightarrow (3)$

Free subgroups, amenability

A group G is amenable if there is a finitely additive measure μ on $\mathcal{P}(G)$ invariant by right translation such that $\mu(G) = 1$, equivalently, for every finite subset A of G and every $0 < \epsilon < 1$ there is a finite subset E of G with $|E.A| < (1+\epsilon)|E|$ (Folner).

Let $\sigma_{p,n,f}$ be the following sentence with $(p, n) \in \mathbb{N}^2$ and $f : \mathbb{N}^2 \to \mathbb{N}$: $\forall a_1 \cdots \forall a_n \exists y_1 \cdots \exists y_{f(p,n)}$

$$p.|\{a_i.y_j : 1 \leq i \leq n; 1 \leq j \leq f(p,n)\}| < (p+1).f(p,n).$$

A group G is uniformly amenable if there exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that $G \models \sigma_{p,n,f}$ for any $(p, n) \in \mathbb{N}^2$

(Keller, Wysoczanski) An $\aleph_0$-saturated group is amenable if and only if it is uniformly amenable.

Free subgroups, amenability

Theorem The following properties are equivalent.

1. Every $\aleph_0$-saturated pseudofinite group either contains F_2 or it is amenable.
2. Every ultraproduct of finite groups either contains a free nonabelian group or it is amenable.
3. Every finitely generated residually finite group satisfying a nontrivial identity is amenable.
4. Every finitely generated residually finite group satisfying a nontrivial identity is uniformly amenable.

Free subgroups, alternatives

A function is said to be r-bounded if it is bounded in terms of r only.
A class $\mathcal{C}$ of finite groups is *of r-bounded rank* if for each element $G \in \mathcal{C}$, every finitely generated subgroup of G can be generated by r elements.

Theorem: (S. Black) Let G be an $\aleph_0$-saturated pseudo-(finite of bounded Prüfer rank) group. Then either G contains F_2 or G is nilpotent-by-abelian-by-finite.

One uses a result of Shalev to reduce to finite soluble groups and then a result of Segal on residually finite soluble groups.

Corollary: An $\aleph_0$-saturated pseudo-(finite of bounded Prüfer rank) group either contains a nonabelian free group or is uniformly amenable.

Weakly r-bounded

A class $\mathcal{C}$ of finite groups is *weakly of r-bounded rank* if for each element $G \in \mathcal{C}$, the index of the sockel of $G/rad(G)$ is r-bounded and $rad(G)$ has r-bounded rank.

Theorem Let G be an $\aleph_0$-saturated pseudo-(finite weakly of bounded rank) group. Then either G contains F_2 or G is nilpotent-by-abelian-by-(uniformly locally finite).

One uses in addition, the result of Jones that a non trivial variety of groups only contains finitely many finite simple groups.

Centralizer dimension

A group G has *finite c-dimension* if there is a bound on the chains of centralizers.
A class $\mathcal{C}$ of finite groups has *bounded c-dimension* if there is $d \in \mathbb{N}$ such that for each element $G \in \mathcal{C}$, the c-dimensions of $rad(G)$ and of the sockel of $G/rad(G)$ are d-bounded.
(Note that a class of finite groups of bounded Prüfer rank is of bounded c-dimension.)

Proposition Let $\mathcal{C}$ be a class of finite groups of bounded c-dimension and suppose G is a pseudo-$\mathcal{C}$ group satisfying a nontrivial identity. Then G is soluble-by-(uniformly locally finite).
We use a result of E. Khukhro on groups with finite c-dimension.

Corollary Let G be an $\aleph_0$-saturated pseudo-(finite of bounded c-dimension) group. Then either G contains F_2 or G is soluble-by-(uniformly locally finite).