Simple Polyadic Groups

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A simple notation

During this presentation, we use the following notations:

1. Any sequence of the form $x_i, x_{i+1}, \ldots, x_j$ will be denoted by

$$x_i^j$$

2. The notation $\overset{(t)}{x}$ will denote the sequence $x, x, \ldots, x$ (t times). So if G is a set and $f : G^n \to G$ is a function, we can denote the element $f(x_1, x_2, \ldots, x_n)$ by $f(x_1^n)$.

A polyadic group is . . .

a non-empty set G together with an n-ary operation $f : G^n \to G$ such that

1. The operation f is associative, i.e.

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

where $1 \leq i, j \leq n$, and $x_1, \ldots, x_{2n-1} \in G$.

2. For fixed $a_1, a_2, \ldots, a_n, b \in G$ and all $i \in \{1, \ldots, n\}$, the following equations have unique solutions for x;

$$f(a_1^{i-1}, x, a_{i+1}^n) = b.$$

We denote the polyadic group by (G, f). More precisely, we call (G, f) an n-ary group.

Examples of polyadic groups

Suppose $(G, \circ)$ is an ordinary group and define

$$f(x_1^n) = x_1 \circ x_2 \circ \cdots \circ x_n.$$

Then (G, f) is polyadic group which is called of *reduced* type. We write $(G, f) = der^n(G, \circ)$.

Example ...

Suppose $(G, \circ)$ is an ordinary group and $b \in Z(G)$. Define

$$f(x_1^n) = x_1 \circ x_2 \cdots \circ x_n \circ b.$$

Then (G, f) is polyadic group which is called b-derived polyadic group from G and it is denoted by $der_b^n(G, \circ)$.

Example ...

Suppose $G = S_m \setminus A_m$, (the set of all odd permutations of degree m). Then by the ternary operation

$$f(x_1, x_2, x_3) = x_1 x_2 x_3$$

the set G is a ternary group.

Example ...

Suppose ω is a primitive $n-1$-th root of unity in a field K. Let

$$G = \{x \in GL_m(K) : det\ x = \omega\}.$$

Then G is an n-ary group by the operation

$$f(x_1^n) = x_1 x_2 \cdots x_n.$$

Identity in polyadic groups

An n-ary group (G, f) is of reduced type iff it contains an element e (called an n*-ary identity*) such that

$$f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x$$

holds for all $x \in G$ and $i = 1, \ldots, n$.

Skew element

From the definition of an n-ary group (G, f), we can directly see that for every $x \in G$, there exists only one $z \in G$ satisfying the equation

$$f(\overset{(n-1)}{x}, z) = x.$$

This element is called *skew* to x and is denoted by $\overline{x}$.

Retracts of polyadic groups

Let (G, f) be an n-ary group and $a \in G$ be a fixed element. Define a binary operation on G by

$$x * y = f(x, \overset{(n-2)}{a}, y).$$

It is proved that $(G, *)$ is an ordinary group, which we call the **retract** of G over a.
The notation for retract: $Ret_a(G, f)$, or simply by $Ret_a(G)$.
Retracts of a polyadic group are isomorphic.

The identity and inverse

The identity of the group $Ret_a(G)$ is $\overline{a}$. The inverse element to x has the form

$$x^{-1} = f(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a}).$$

Recovering a polyadic group from its retracts

Any n-ary group can be uniquely described by its retract and some automorphism of this retract.

Theorem

Let (G, f) be an n-ary group. Then
1. on G one can define an operation $\cdot$ such that $(G, \cdot)$ is a group,
2. there exist an automorphism θ of $(G, \cdot)$ and $b \in G$, such that $\theta(b) = b$,
3. $\theta^{n-1}(x) = bxb^{-1}$, for every $x \in G$,
4. $f(x_1^n) = x_1\theta(x_2)\theta^2(x_3)\cdots\theta^{n-1}(x_n)b$, for all $x_1, \ldots, x_n \in G$.

Remark

According to this theorem, we use the notation $der_{\theta,b}(G,\cdot)$ for (G,f) and we say that (G,f) is (θ,b)-derived from the group $(G,\cdot)$.
The binary group $(G,\cdot)$ is in fact $Ret_a(G,f)$.
We will assume that $(G,f)=der_{\theta,b}(G,\cdot)$.

Normal subgroups

An n-ary subgroup H of a polyadic group (G, f) is called *normal* if

$$f(\overline{x}, \overset{(n-3)}{x}, h, x) \in H$$

for all $h \in H$ and $x \in G$.

GTS

If every normal subgroup of (G, f) is singleton or equal to G, then we say that (G, f) is *group theoretically simple* or it is GTS for short. If $H = G$ is the only normal subgroup of (G, f), then we say it is *strongly simple in the group theoretic sense* or GTS^* for short.

UAS

An equivalence relation R over G is said to be a *congruence*, if

1. $\forall i: \; x_i R y_i \Rightarrow f(x_1^n) R f(y_1^n)$,
2. $xRy \Rightarrow \overline{x} R \overline{y}$.

We say that (G, f) is *universal algebraically simple* or UAS for short, if the only congruence is the *equality* and $G \times G$.

Quotients are reduced

Theorem

Suppose $H \trianglelefteq (G, f)$ *and define* $R = \sim_H$ *by*

$$x \sim_H y \Leftrightarrow \exists h_1, \ldots, h_{n-1} \in H : y = f(x, h_1^{n-1}).$$

Then R *is a congruence and if we let* $xH = [x]_R$*, (the equivalence class of* x*), then the set* $G/H = \{xH : x \in G\}$ *is an* n*-ary group with the operation*

$$f_H(x_1H, \ldots, x_nH) = f(x_1^n)H.$$

Further we have

$$(G/H, f_H) = der(ret_H(G/H, f_H)),$$

UAS is also GTS

Theorem

Every UAS is also GTS. But the converse is not true!

Facts about congruences

$Cong(G, f)$ is the set of all congruences of (G, f). This set is a lattice under the operations of intersection and product (composition). We also denote by $Eq(G)$ the set of all equivalence relations of G.

Theorem

$R \in Cong(G, f)$ iff $R \in Eq(G)$ and R is a θ-invariant subgroup of $G \times G$.

Corollary

We have $Cong(G, f) = \{R \leq_\theta G \times G : \Delta \subseteq R\}$.

UAS

Theorem

(G, f) is UAS iff the only normal θ-invariant subgroups of $(G, \cdot)$ are trivial subgroups.

Structure of normals

For $u \in G$, define a new binary operation on G by $x * y = xu^{-1}y$. Then $(G, *)$ is an isomorphic copy of $(G, \cdot)$

Theorem

We have $H \trianglelefteq (G, f)$ iff there exists an element $u \in H$ such that

1. *H is a ψ_u-invariant normal subgroup of G_u,*
2. *for all $x \in G$, we have $\theta^{-1}(x^{-1}u)x \in H$.*

GTS

Theorem

A polyadic group (G, f) is GTS^* iff whenever K is a θ-invariant normal subgroup of $(G, \cdot)$ with θ_K inner, then $K = G$.

Example

Example

Let $(G, \cdot)$ be a non-abelian simple group and θ be an automorphism of order $n-1$. Then $der_\theta(G, \cdot)$ is a UAS n-ary group.

The number of non-isomorphic polyadic groups of the form $der_\theta(G, \cdot)$ is the same as the number of conjugacy classes of $Out(G)$, the group of outer automorphisms of $(G, \cdot)$

Example

Example

Suppose p is a prime and $G = \mathbb{Z}_p \times \mathbb{Z}_p$. Let $q(t) = t^2 + at + b$ be an irreducible polynomial over the field $\mathbb{Z}_p$ and choose a matrix $A \in GL_2(p)$ with the characteristic polynomial $q(t)$. Let $A^{n-1} = I$ and define an automorphism $\theta : G \to G$ by $\theta(X) = AX$. Clearly, θ has no non-trivial invariant subgroup, since $q(t)$ is irreducible. So, $der_\theta(G, \cdot)$ is a UAS n-ary group. Note that, we have

$$f(X_1^n) = X_1 + AX_2 + \cdots + A^{n-2}X_{n-1} + X_n.$$

Example

Example

Let H be a non-abelian simple group with an outer automorphism θ. Let $\theta^{n-1} = id$ and $G = H \times H$. Then θ extends to G by $\theta(x, y) = (\theta(x), \theta(y))$. The subgroups $K_1 = H \times 1$ and $K_2 = 1 \times H$ are the only θ-invariant normal subgroups of G. Clearly $\theta_{K_i} : G/K_i \to G/K_i$ is not inner as we supposed θ an outer automorphism. Therefore $der_\theta(G, \cdot)$ is a GTS polyadic group but it is not UAS.

Thank You!

Thanks to

The Participants ....For Listening...

and

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