Products of homogeneous subspaces in free Lie algebras

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Antalya Algebra Days at Çeșcme, May 2012

This is joint work with Nil Mansuroğlu.

Lie algebras

Take any associative algebra A over a field K, define a new binary operation on A by setting

$$[a, b] = ab - ba \qquad (a, b \in A),$$

the Lie bracket, you get a **Lie algebra**. That is an algebra over K with a bilinear binary operation $[,]$ satisfying

$$[a, a] = 0$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

for all $a, b, c \in A$.

Now take a set $X = \{x_1, \ldots, x_r\}$, take the algebra of non-commutative polynomials $A = A[X]$, turn it into a Lie algebra. Then take the Lie subalgebra generated by X. **This is the free Lie algebra $L = L(X)$ on X.**

The elements of $L(X)$ are called *Lie elements* or *Lie polynomials*.

Homogenous components

Every non-commutative polynomial is a linear combination of homogeneous polynomials, i.e. linear combinations of monomials of a given degree. In the same way any Lie polynomial is a linear combination of homogeneous Lie polynomials. If A_n denotes the space of all homogeneous polynomials of degree n, then $A = \bigoplus_{n \geqslant 0} A_n$.

$$\dim A_n = r^n.$$

If L_n denotes the space of all homogeneous Lie polynomials of degree n, then $L = \bigoplus_{n \geqslant 0} L_n$.

$$\dim L_n = f(n, r) = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d}.$$

(Witt 1937)

Problem: $\dim[L_m, L_n] = ?$, $m \geqslant n$.

Easy in some special cases:

- If $m/2 < n < m$, then $\dim[L_m, L_n] = \dim L_m \dim L_n$.
- If $m = n$, then $\dim[L_m, L_m] = \binom{\dim L_m}{2}$.

Partial result (Sundaram 1993): If $m > n$ and $n \nmid m$, then

$$\dim[L_m, L_n] = \dim L_m \dim L_n.$$

Solution

Theorem (Vaughan-Lee and RS, 2009)

If $m > n$ and $n \nmid m$, then

$$\dim([L_m, L_n]) = \dim L_m \dim L_n,$$

and if $m = sn$ with $s \geq 1$, then

$$\dim([L_m, L_n]) = (\dim L_m - f(s, \dim L_n)) \dim L_n + f(s+1, \dim L_n).$$

Shirshov's Lemma

The main ingredient of the proof is **Shirshov's Lemma**, a very powerful tool in the theory of free Lie algebras. The celebrated **Shirshov-Witt Theorem** asserts that *any subalgebra of a free Lie algebra is itself free*. Shirshov's Lemma was the main ingredient in Shirshov's original proof of that result. A set of elements in L is called *independent*, if it is a free generating set for the subalgebra it generates. A set of homogeneous elements in L is called *reduced*, if none of its elements belonges to the subalgebra generated by the remaining elements.

Shirshov's Lemma (1953) - a special case:

Any reduced set of homogeneous elements in L is independent.

Products of three subspaces

Given that we know $\dim[L_m, L_n]$ for all $m, n \geqslant 1$, the next question that arises naturally is:

$$\dim[[L_m, L_n], L_k] = ?$$

Surprise: In contrast to products of two homogeneous components, the dimension of a product of three homogeneous components may depend on the field K. In fact, this happens for

$$[[L_2, L_2], L_1].$$

This is an immediate consequence of an old result by Yu.V. Kuz'min on free centre–by-metabelian Lie rings.

Free centre-by-metabelian Lie rings

Let $\mathfrak{L} = \mathfrak{L}(X)$ denote the free Lie ring on $X = \{x_1, \dots x_r\}$. The free centre-by-metabelian Lie ring $\mathfrak{G} = \mathfrak{G}(X)$ is the quotient

$$\mathfrak{G} = \mathfrak{L}/[\mathfrak{L}'', \mathfrak{L}]$$

where $\mathfrak{L}''$ is the second derived ring of $\mathfrak{L}$. In a celebrated paper of 1977 Kuz'min studied the underlying abelian group of $\mathfrak{G}$.

Theorem (Kuz'min, 1977)

If $r \geqslant 5$, then the degree 5 homogeneous component of the second derived ring $\mathfrak{G}''$ is a direct sum of a free abelian group and an elementary abelian 2-group of rank $\binom{r}{5}$.

However, for the degree 5 homogeneous component of $\mathfrak{G}''$ there is an isomorphism

$$\mathfrak{G}'' \cap \mathfrak{L}_5 \cong [\mathfrak{L}_3, \mathfrak{L}_2]/[[\mathfrak{L}_2, \mathfrak{L}_2], \mathfrak{L}_1].$$

Then Kuz'min's result and some additional argument gives:

Proposition (Mansuroğlu and RS, 2012)

Let L be a free Lie algebra of rank r over a field K. Then

$$\dim[[L_2, L_2], L_1] = \begin{cases} \dim[L_2, L_2] \dim L_1, & \text{if char } K \neq 2; \\ \dim[L_2, L_2] \dim L_1 - \binom{r}{5}, & \text{if char } K = 2, \end{cases}$$

with the convention that $\binom{r}{5} = 0$ for $r < 5$.

$\dim[[L_m, L_n], L_k]$

Our main results are dimension formula for product of three homogeneous components in the free Lie algebra L. The main technical tool is a generalization of the result on the dimension for products of two homogeneous components to products of two arbitrary homogeneous subspaces.

Lemma

Let U and V be subspaces of L such that $U \subseteq L_m$, $V \subseteq L_n$ with $m \geqslant n \geqslant 1$. Then

$$\dim[U, V] = \dim[U \cap L(V), V] + (\dim U - \dim(U \cap L(V))) \dim V.$$

Finally, here is our main result on $[[L_m, L_n], L_k]$

$\dim[[L_m, L_n], L_k]$

Theorem (Mansuroğlu and RS, 2012)

Let m, n and k be positive integers with $m \geqslant n$.

(i) *If $m + n > k$ and $k \nmid m$ or $k \nmid n$, or $k \geqslant m + n$ and $(m + n) \nmid k$, then* $\dim[L_m, L_n, L_k] = \dim[L_m, L_n] \dim L_k$,

(ii) *if $m + n > k$ and $m = sk$ and $n = tk$ with $s, t \geqslant 1$, then*

$$\dim[L_m, L_n, L_k] = \dim[L_s(L_k), L_t(L_k), L_k] + (\dim[L_m, L_n] - \dim[L_s(L_k), L_t(L_k)]) \dim L_k,$$

(iii) *if $k \geqslant m + n$ and $k = p(m + n)$ with $p \geqslant 1$, then*

$$\dim[L_m, L_n, L_k] = \dim L_{p+1}([L_n, L_m]) + (\dim L_k - \dim(L_p([L_m, L_n]))) \dim[L_m, L_n].$$

Teşekkür ederim.