On Invariants of Towers of Function Fields over Finite Fields

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Notations

- $F/\mathbb{F}_q$: a function field with full constant field $\mathbb{F}_q$.
- $g(F)$: genus of $F/\mathbb{F}_q$,
- $B_r(F)$: $\#$ places of $F/\mathbb{F}_q$ of degree r,
- $\mathcal{F} = (F_n)_{n\geq 0}$: a sequence of function fields $F_n/\mathbb{F}_q$ with $g(F_n) \to \infty$ as $n \to \infty$.

Definition (M. A. Tsfasman, 1992)

$\mathcal{F}$ is called asymptotically exact if for all $r \geq 1$ the limit

$$\beta_r(\mathcal{F}) := \lim_{n\to\infty} \frac{B_r(F_n)}{g(F_n)}$$

exists.

Generalized Drinfeld-Vladut bound: For any exact sequence $\mathcal{F}$ over $\mathbb{F}_q$, one has

$$\sum_{r=1}^{\infty} \frac{r\beta_r(\mathcal{F})}{q^{r/2}-1} \leq 1. \qquad (1)$$

Aim: to construct exact sequences of function fields with various $\beta_r > 0$ and small

$$\delta := 1\text{-the left hand side of } (1).$$

Problems:

For any given $N \subseteq \mathbb{N}$, find exact sequences of function fields over $\mathbb{F}_q$ with

(1) $N \subseteq \mathcal{P}(\mathcal{F})$, where $\mathcal{P}(\mathcal{F}) := \{r \in \mathbb{N} : \beta_r(\mathcal{F}) > 0\}$.

(2) $\mathcal{P}(\mathcal{F}) = N$.

Lemma (Tsfasman, Vladut (2002)-T. Hasegawa (2007))

Any tower of function fields $\mathcal{F}$ over $\mathbb{F}_q$ is an exact sequence.

In 2007, T. Hasegawa and P. Lebacque proved independently the existence of towers with finitely many prescribed β_r being positive, by using class field theory.

Strategy

We consider a tower $\mathcal{F} = (F_n)_{n\geq 0}$ over $\mathbb{F}_q$ with $\beta_1(\mathcal{F}) > 0$ and construct an appropriate finite separable extension E/F_0 such that $\mathcal{G} := (EF_n)_{n\geq 0}$ defines a tower over $\mathbb{F}_q$. Then we estimate the invariants $\beta_r(\mathcal{G})$ of $\mathcal{G}$.

Definition: Let $\mathcal{F} = (F_n)_{n\geq 0}$ be a tower over $\mathbb{F}_q$.

(a) The genus of $\mathcal{F}/\mathbb{F}_q$ is defined as

$$\gamma(\mathcal{F}) := \lim_{n\to\infty} \frac{g(F_n)}{[F_n : F_0]} > 0$$

(b) Let P be a place of F_0 and for any $r \geq 1$,

$$B_r(P, F_n) := \#\{\text{ places of } F_n/\mathbb{F}_q \text{ of degree } r \text{ lying above } P\}.$$

We define the local invariants of $\mathcal{F}$ at P as

$$\nu_r(P,\mathcal{F}) := \lim_{n\to\infty} \frac{B_r(P,F_n)}{[F_n : F_0]}, \quad \beta_r(P,\mathcal{F}) := \frac{\nu_r(P,\mathcal{F})}{\gamma(\mathcal{F})} \quad \text{and}$$

the global invariants of $\mathcal{F}$ as

$$\nu_r(\mathcal{F}) := \lim_{n\to\infty} \frac{B_r(F_n)}{[F_n : F_0]}, \quad \beta_r(\mathcal{F}) := \lim_{n\to\infty} \frac{B_r(F_n)}{g(F_n)}.$$

Then clearly

$$\nu_r(\mathcal{F}) = \sum_P \nu_r(P,\mathcal{F}) \text{ and } \beta_r(\mathcal{F}) = \sum_P \beta_r(P,\mathcal{F}) = \frac{v_r(\mathcal{F})}{\gamma(\mathcal{F})}.$$

For any $r \geq 1$, we want to estimate

$$\beta_r(\mathcal{G}) = \frac{\nu_r(\mathcal{G})}{\gamma(\mathcal{G})},$$

depending on the invariants of $\mathcal{F}$.

Lemma

Set $m := [E : F_0]$. For the genus $\gamma(\mathcal{G})$ the following holds:

$$m\gamma(\mathcal{F}) \leq \gamma(\mathcal{G}) \leq g(E) - 1 + m(1 - g(F_0) + \gamma(\mathcal{F})).$$

Theorem

Suppose that $\mathcal{G} := (EF_n)_{n\geq 0}$ *is a tower over* $\mathbb{F}_q$*. Set* $E := F_0(y)$, $m := [E : F_0]$, *and consider the set*

$$M := \left\{P \in \mathbb{P}(F_0) : \{1, y, ..., y^{m-1}\} \text{ is an integral basis for } E/F_0 \text{ at } P\right\}.$$

Let $P \in M$ *s.t. for all extensions* Q *of* P *in* E*, the ramification index* $e(Q|P)$ *is coprime to any ramification index of* P *in* $\mathcal{F}$*. Then for any such* Q *and* $r \geq 1$*,*

$$\nu_r(Q, \mathcal{G}) = \frac{f(Q|P)}{r} \sum_{\substack{d\in\mathbb{N} \\ lcm(\deg Q, d) = r}} d \cdot \nu_d(P, \mathcal{F}).$$

Theorem

Let $\mathcal{F}$ be a tower over $\mathbb{F}_q$ with a finite support and let $N \subset \mathbb{N}$ be a finite set. Then there exists a finite separable extension E/F_0 s.t. $\mathcal{G} := (EF_n)_{n\geq 0}$ is a tower over $\mathbb{F}_q$ with
(i) for all $r \in \mathbb{N}$,

$$\nu_r(\mathcal{G}) = \sum_{\substack{f\in N \\ d\in\mathcal{P}(\mathcal{F})}} \frac{f}{r} \sum_{\substack{P\in Supp(\mathcal{F}) \\ lcm(f\deg P, d)=r}} d\cdot\nu_d(P,\mathcal{F})$$

and

$$Supp(\mathcal{G}) = \left\{Q \in \mathbb{P}(E) : Q \cap F_0 \in Supp(\mathcal{F})\right\},$$

$$\mathcal{P}(\mathcal{G}) = \left\{r \in \mathbb{N} : r = lcm(f \deg P, d) \text{ with } f \in N,\ d \in \mathbb{N},\ P \in Supp(\mathcal{F})\right\}.$$

Theorem continued

(ii) If furthermore $\mathcal{F}$ is pure, then for all $r \in \mathbb{N}$,

$$\nu_r(\mathcal{G}) = \sum_{\substack{f \in N,\ d \in \mathcal{P}(\mathcal{F}) \\ fd = r}} \nu_d(\mathcal{F}) \qquad \text{and}$$

$$\mathcal{P}(\mathcal{G}) = \big\{ r \in \mathbb{N} : r = fd \text{ with } f \in N,\ d \in \mathcal{P}(\mathcal{F}) \big\}.$$

Corollary

For any prime power q, one can construct a tower of function fields over $\mathbb{F}_q$ with finitely many precribed invariants β_r being positive, by using explicit extensions.

Corollary

Let $N \subseteq \mathbb{N}$ be a finite set and q be a prime power. Then there exists a recursive (explicit) tower of function fields $\mathcal{F}/\mathbb{F}_q$ such that N is included in the set

$$\mathcal{P}(\mathcal{F}) := \{r \in \mathbb{N} : \beta_r(\mathcal{F}) > 0\}.$$

Moreover, in the following cases there exists a recursive tower $\mathcal{F}/\mathbb{F}_q$ with $\mathcal{P}(\mathcal{F}) = N$:

(i) *q is any prime power and N is a finite set with each $k \in N$ a multiple of r for some r such that q^r is a square,*

(ii) *$q = 2^e$ with $3 \nmid e$ and each element $k \in N$ is a multiple of 3,*

(iii) *$q = p^e$ with $p \geq 3$, $e > 2$ and $2 \nmid e$, and each $k \in N$ is an even integer.*

Example

A. Garcia, H. Stichtenoth: The equation $y^3 = (x+\mu)^3 + 1$ (for $\mathbb{F}_4^* =<\mu>$) defines a tower $\mathcal{F} = (F_n)_{n\geq 0}$, with $F_0 := \mathbb{F}_4(x_0)$, over $\mathbb{F}_4$ with $\beta_1(\mathcal{F}) = 1$.

Let $E := F_0(z)$ where z satisfies

$$z^6 + \mu z^5 + \mu z^4 - z^3 - \mu z^2 - \mu z - 1/x_0 = 0.$$

Then $\mathcal{G} := (EF_n)_{n\geq 0}$ gives a tower over $\mathbb{F}_4$ with

- $\mathcal{P}(\mathcal{F}) = \{1, 2\}$ and $\beta_1(\mathcal{G}) = \frac{2}{3},\ \beta_2(\mathcal{G}) = \frac{1}{6}$,
- $\delta = 1 - \sum_{r=1}^{\infty} \frac{r\beta_r(\mathcal{G})}{4^{r/2}-1} \approx 0.22.$

Problem

Are there any sequences, in particular towers, of function fields over finite fields with infinitely many positive invariants β_r ?

Thank you for your attention.