Gröbner-Shirshov Bases for Affine Weyl Group Type $\widetilde{A_n}$

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Outline

Introduction

Gröbner and Gröbner-Shirshov bases theories are generating increasing interest because of its usefulness in providing computational tools and in giving algebraical structures which are applicable to a wide range of problems in mathematics, science, engineering, and computer science. In general, Gröbner-Shirshov bases theory is a powerful tool to solve the following classical problems.

- **(i)** normal form;
- **(ii)** word problem;
- **(iii)** rewriting system;
- **(iv)** embedding theorems;
- **(v)** extentions;
- **(vi)** growth function;
- **(vii)** Hilbert series; etc.

Introduction

In particular, the technique of Gröbner-Shirshov bases is proved to be very useful in the study of presentations of associative algebras, Lie algebras, semigroups, groups, etc. by generators and defining relations, see, for example, the book [4] written by L. A. Bokut and G. Kukin, survey papers [[5], [6]] written by L. A. Bokut and P. Kolesnikov, and [7] written by L. A. Bokut and Yuqun Chen.

My thesis deals with Gröbner-Shirshov bases theory for affine Weyl groups. Gröbner-Shirshov bases and normal form of the elements were already found for the Coxeter groups of type A_n; B_n and D_n in [1]. The Gröbner-Shirshov bases of the other finite Coxeter groups are given in [10] and [16].

Monomial Ordering

Suppose S is a linearly ordered set and $\Bbbk$ is a field. Let S^* be the free monoid generated by S. The elements of S^* are called words. The empty word is the identity which is denoted by 1. Let $\Bbbk < S >$ be free associative algebra over $\Bbbk$ defined by

$$\Bbbk < S > \quad = \quad \{\sum_{i=1}^{m} c_{\alpha_i} w_i, \quad c_{\alpha_i} \in \Bbbk \quad and \quad w_i \in S^*\}.$$

Definition

A well ordering $<$ on S^* is called monomial ordering if it agrees with left and right multiplications by words:

$$u > v \quad \Rightarrow \quad w_1 u w_2 > w_1 v w_2 \quad \forall w_1, w_2 \in S^*.$$

For a word $w \in S^*$, we denote the length of w by $|w|$.

Leading Word & Monic

Definition

Let $f = \alpha\bar{f} + \sum \alpha_i u_i \in \mathbb{k} < S >$ where $\alpha, \alpha_i \in \mathbb{k}$, $\bar{f} \in S^*$ and $u_i < \bar{f}$ for each i. Then we call $\bar{f}$ the leading word of f.

Definition

If leading word $\bar{f}$ of f has a coefficient 1, then f is called monic.

Composition of Intersection & Composition of Including

Definition

For two monic polynomials f and g in $\Bbbk < S >$ and a word w, the composition of intersection is defined by

$$(f, g)_w = fb - ag \quad if \quad w = \bar{f}b = a\bar{g}, \quad |\bar{f}| + |\bar{g}| > |w|$$

Definition

For two monic polynomial f and g in $\Bbbk < S >$ and a word w, the composition of including is defined by

$$(f, g)_w = f - agb \quad if \quad w = \bar{f} = a\bar{g}b$$

Trivial Relative

Definition

A composition $(f, g)_w$ is called trivial relative to some $R \subset \mathbb{k} < S >$ if $(f, g)_w = \sum \alpha_i a_i t_i b_i$ where $\alpha_i \in \mathbb{k}$, $t_i \in R$, $a_i, b_i \in S^*$ and $a_i t_i b_i < w$. In particular, if $f \mapsto (f, g)_w = f - agb = 0$ of R, then $(f, g)_w$ is trivial relative to R.

Gröbner-Shirshov Basis

Definition

A subset R of $\Bbbk < S >$ is called a Gröbner-Shirshov basis if any composition of polynomials from R is trivial relative to R.

Buhberger-Shirshov Algorithm

Definition

If a subset R of $\mathbb{k} < S >$ is not a Gröbner-Shirshov basis then one can add to R all nontrivial compositions of polynomials of R, and continue this process until get a Gröbner-Shirshov basis. This procedure is named as Buchberger Shirshov algorithm. Unfortunately this process may continue infinitely many steps.

Gröbner-Shirshov Basis for $\widetilde{A_n}$

Definition

The affine Weyl group $\widetilde{A_n}$ has a presentation with generators $\{r_0, r_1, \ldots, r_n\}$ and defining relations
$r_i r_i = 1 \quad 0 \leq i \leq n$
$r_i r_j = r_j r_i \quad 0 \leq i < j - 1 < n \text{ and } (i, j) \neq (0, n)$
$r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1} \quad 0 \leq i \leq n - 1$
$r_0 r_n r_0 = r_n r_0 r_n$.

Let us define

$$r_{ij} = \begin{cases} r_i r_{i+1} \dots r_j, & i < j; \\ r_i r_{i-1} \dots r_j, & i > j; \\ r_i, & i = j; \\ 1, & i = 1, j = 0; \\ 1, & i = n, j = n+1. \end{cases}$$

Identifying each relation $u = v$ by $u - v$, we define the following polynomials

$f_1^{(i)} = r_i r_i - 1 \quad 0 \leq i \leq n$

$f_2^{(i,j)} = r_i r_j - r_j r_i \quad 0 \leq i < j - 1 < n \text{ and } (i, j) \neq (0, n)$

$f_3^{(i)} = r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1} \quad 0 \leq i \leq n - 1$

$f_4 = r_0 r_n r_0 - r_n r_0 r_n$

Lemma

Let $R = \{f_1, f_2, f_3, f_4\}$. A Gröbner-Shirshov Basis of $\widetilde{A_n}$ with respect to deglex order with $r_0 > r_1 > \cdots > r_n$ contains the following polynomials.

$$g_1^{(i,j)} = r_{ij}r_i - r_{i+1}r_{ij} \qquad 0 \leq i < j-1 < n \text{ with } (i,j) \neq (0,n)$$

$$g_2 = r_{0n}r_0r_n - r_1r_{0n}r_0$$

$$g_3^{(j,k)} = r_0r_{nk}r_j - r_jr_0r_{nk} \qquad 2 \leq j < k-1 < n$$

$$g_4^{(j)} = r_0r_{nj}r_{j+1} - r_jr_0r_{nj} \qquad 2 \leq j < n$$

$$g_5^{(k)} = r_0r_{nk}r_0 - r_nr_0r_{nk} \qquad 2 \leq k < n$$

Lemma

$$g_6^{(k,l)} = r_0 r_{nk} r_{1l} r_{0l} - r_n r_0 r_{nk} r_{1l} r_{0,l-1} \qquad 1 \le l < n,\ 2 \le k \le n$$

$$g_7^{(k,l)} = r_0 r_{nk} r_{1l} r_0 r_{nk} - r_1 r_0 r_{nk} r_{1l} r_0 r_{n,k+1} \qquad 1 \le l < k-1 < n$$

$$g_8^{(k,l)} = r_0 r_{nk} r_{1l} r_0 r_{n,k-1} - r_1 r_0 r_{nk} r_{1l} r_0 r_{nk} \quad 3 \le k \le n, \quad k-1 \le l \le n$$

$$g_9^{(j,k,l)} = r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1l} - r_n r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1,l-1}$$
$$2 \le k \le n-1, \quad k+1 \le j \le n, \quad 1 \le l \le j-2$$

$$g_{10}^{(j,k,l)} = r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1,l+1} - r_n r_0 r_{nk} r_{1l} r_0 r_{nj} r_{1l}$$
$$2 \le k \le n, \quad k \le j \le n, \quad j-1 \le l \le n-1$$

Let R' be a set of the polynomials given in previous lemma together with defining polynomials. At this point we cannot claim that R' is a Gröbner-Shirshov basis for $\widetilde{A_n}$ because Bucberger-Shirshov algorithm may contain infinitely many steps. However using famous Composition-Diamond Lemma, we can prove that R' is a Gröbner-Shirshov basis for $\widetilde{A_n}$.

Composition-Diamond Lemma for associative algebras

Lemma

Let $\Bbbk$ be a field, $A = \Bbbk\langle S|R\rangle = \Bbbk\langle S\rangle/Id(R)$ and $<$ a monomial ordering on S^*, where $Id(R)$ is the ideal of $k\langle S\rangle$ generated by R. Then the following statements are equivalent:

(i) *R is a Gröbner-Shirshov basis.*

(ii) *$f \in Id(R) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in R$ and $a, b \in S^*$.*

(iii) *The set of R-reduced words*

$$Red(R) = \{w \in S^* | w \neq a\bar{s}b, a, b \in S^*, s \in R\}$$

is a $\Bbbk$-linear basis for the algebra $A = \Bbbk\langle S|R\rangle$.

Our strategy is to obtain the set $Red(R')$ as an explicit classes of words. After that, we compute the number of all reduced words with respect to these classes by means of a generating function. This generating function turns out to be same with the generating function which gives the number of elements in each length of the affine Weyl group $\widetilde{A}_n$. Therefore, by the Composition-Diamond Lemma the functions in R' form Gröbner-Shirshov basis for the affine Weyl group $\widetilde{A}_n$.

Lemma

Any reduced word not containing r_0 is in the form

$$r = (r_{nj_n})^{\alpha_n}(r_{n-1,j_{n-1}})^{\alpha_{n-1}}, \ldots, (r_{2j_2})^{\alpha_2}(r_{1j_1})^{\alpha_1}$$

where $i \leq j_i \leq n$ and $\alpha_i \in \{0, 1\}$.

Lemma

The following words are reduced.
(i) $w = r_0 r_{nk} r_{1l}$ $2 \leq k \leq n+1$, $0 \leq l \leq n$
(ii)

$$(r_0 r_{nk} r_{1l})(r_0 r_{np} r_{1q}) = \begin{cases} (k < p) \wedge (l > q), & \text{if} \quad q-p < l-k < -1 \\ (k \leq p) \wedge (l > q), & \text{if} \quad (q-p < -1) \wedge (l-k \geq -1) \\ (k \leq p) \wedge (l \geq q), & \text{if} \quad l-k > q-p \geq -1 \end{cases}$$

where $r_0 r_{n,n+1} r_{1l} = r_{0l}$ *and* $r_0 r_{nk} r_{10} = r_0 r_{nk}$.

Definition

Define

$$u = (a_n)^{m_n}(a_{n-1})^{m_{n-1}} \dots (a_1)^{m_1}$$

where $m_i \geq 0$ for $i = 1, \dots, n$. Here, if $a_i = r_0 r_{nk} r_{1l}$, then we have two possibilities for a_{i-1}.
(i) $a_{i-1} = r_0 r_{n,k+1} r_{1l}$
(ii) $a_{i-1} = r_0 r_{nk} r_{1,l-1}$.
We call a_i's the components of the word u.

Notice that the number of possible u's is 2^{n-1}.

Definition

Define

$$v^{(b_t)} = b_t(b_{t-1})^{\alpha_{t-1}} \cdots (b_s)^{\alpha_s}$$

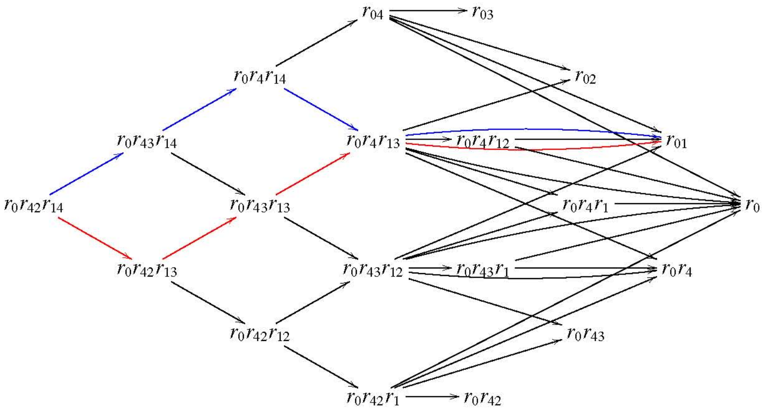
where $\alpha_i \in \{0, 1\}$. Here, if $b_i = r_0 r_{np_i} r_{1q_i}$, then $b_{i-1} = r_0 r_{np_{i-1}} r_{1q_{i-1}}$ for $p_i < p_{i-1}$ and $q_i > q_{i-1}$.
Furthermore, if $p_i = n+1$ or $q_i = 0$, then $\alpha_j = 0$ for $j = s, \ldots, i-1$.
For the convenience, we define $1 = r_0 r_{n,\infty} r_{1,-1}$ and $v^{(1)} = 1$.

Notice that the number of possible v is $n!$.

Proposition

The word u, $v^{(b_t)}$ and their combinations $w = uv^{(b_t)}$ are reduced words if $a_1 = r_0 r_{nk} r_{1l}$ and $b_t = r_0 r_{np} r_{1q}$ with $p \geq k$ and $q < l$ in w.

Reduced Words for $n = 4$



$$w_1 = (r_0r_{42}r_{14})^{m_1}(r_0r_{42}r_{13})^{m_2}(r_0r_{43}r_{13})^{m_3}(r_0r_4r_{13})^{m_4}r_{01}$$
$$w_2 = (r_0r_{42}r_{14})^{m_1}(r_0r_{43}r_{14})^{m_2}(r_0r_4r_{14})^{m_3}(r_0r_4r_{13})^{m_4}r_{01}.$$

$$w_1 = (r_0r_{42}r_{14})^{m_1}(r_0r_{42}r_{13})^{m_2}(r_0r_{43}r_{13})^{m_3}(r_0r_4r_{13})^{m_4}r_{01}$$

and

$$w_2 = (r_0r_{42}r_{14})^{m_1}(r_0r_{43}r_{14})^{m_2}(r_0r_4r_{14})^{m_3}(r_0r_4r_{13})^{m_4}r_{01}.$$

If we take $m_2 = m_3 = 0$ and $m_1 = m_4 = 1$ in both w_1 and w_2, then the subword $(r_0r_{42}r_{14})(r_0r_4r_{13})r_{01}$ is written twice. To avoid this situation, we force $m_2 \geq 1$ in w_1.

Arranged Word

In general to avoid repetition we defined arranged words an marked components as follows,

Definition

Let $w = uv^{(b_t)}$ be a reduced word where $b_t = r_0 r_{np} r_{1q}$ and

$$u = (a_n)^{(m_n)} (a_{n-1})^{(m_{n-1})} \cdots (a_1)^{(m_1)}.$$

For $i = 2, \ldots, n-1$, a_i is called marked component if

$$\begin{tikzcd} a_{i+1} = r_0 r_{nk} r_{1,l+1} \arrow[dr] & & a_{i-1} = r_0 r_{n,k+1} r_{1l} \\ & a_i = r_0 r_{nk} r_{1l} \arrow[ur] & \end{tikzcd}$$

Arranged Word

Definition

a_1 is called marked component if

$$\begin{tikzcd}
a_2 = r_0 r_{nk} r_{1,l+1} \arrow[dr] & & b_t = r_0 r_{np} r_{1q} \\
& a_1 = r_0 r_{nk} r_{1l} \arrow[ur] &
\end{tikzcd}$$

where $p > k$. If we let the power of marked components starts from 1 , then w is called arranged word.

Theorem

If all reduced words $w = uv^{(b_t)}$ are arranged, then each subword is written uniquely.

Counting Reduced Words

The number of the elements in an arranged word $w = uv^{(b_t)}$ given by the generating function

$$\frac{x^{\alpha}}{(1-x^{2n})(1-x^{2n-1})\cdots(1-x^{n+1})}$$

where α is the length of the word $\widehat{w}$ which is correspondence to marked components of w. In order to count all reduced words starting with r_0 we have to find the number of the words $\widehat{w}$ whose length is α for any power α. To do this, we will find a correspondence between these words and some special partitions of integers.

Let n be a positive integer. Any partition $m = d_1 + d_2 + \ldots + d_k$ where $k \leq n$ can be identify by the $n-$tuple $(d_1, d_2, \ldots, d_k, 0, 0, \ldots, 0)$.
We can also represent each word

- $r_0 r_{nk} r_{1l} \leftrightarrow (k, 1, \ldots, 1, 0, \ldots, 0)$
- $r_{nk} \leftrightarrow (k, 0, \ldots, 0)$
- $r_{0l} \leftrightarrow (1, 1 \ldots, 1, 0, \ldots, 0)$

where the number of 1 is equal to l for $2 \leq k \leq n-1, 1 \leq l \leq n$.

Basic Partition

Definition

Let n be a positive integer. The n-tuples $(k, 1 \ldots, 1, 0, \ldots, 0)$ where the number of 1's is l for $1 \leq k \leq n$, $1 \leq l \leq n-1$ are called basic partitions.
The basic partition $(k_1, 1 \ldots, 1, 0 \ldots, 0)$ is said to be connected to the basic partition $(k_2, 1, \ldots, 1, 0, \ldots, 0)$ if $k_1 > k_2$ and the number of 1's in the first one is greater than number of 1's in the second one. Hence a chain of connected partition $a_1 a_2 \ldots a_m$ corresponds to a word $\widehat{w}$.

Let $a_1 a_2 \ldots a_m$ be a chain of connected partitions where $a_i = (k_i, \underbrace{1, \ldots, 1}_{l_i}, 0, \ldots, 0)$. Hence $k_i > k_j$ and $l_i > l_j$ for $1 \leq i < j \leq m$.
Define

$$\bigoplus_{i=1}^{m} a_i = \sum_{i=1}^{m} \sigma^{i-1}(a_i)$$

where $\sigma(p_1, p_2, \ldots, p_{n-1}, p_n) = (p_n, p_1, p_2, \ldots, p_{n-1})$.

Using the sum defined in previous slide, we found one to one correspondence between sequences of connected partitions and the partitions in which there are at most n parts and in which no parts is larger than n. Hence

Theorem

There is one to one correspondence between words $\widehat{w}$ and the partitions in which there are at most n parts and in which no parts is larger than n.

q-Binomials

The number of partitions in which there are at most r parts and in which no parts is larger than $m - r$ is given by the q-binomials.

Definition

Let m and r be positive integers. The q-binomial is defined by

$$\binom{m}{r}_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}$$

In particular $\left(\begin{array}{c} 2n \\ n \end{array} \right)_q$ gives the number of partitions in which there are at most n parts and in which no parts are larger than n

The number of reduced words starting with r_0

$$\frac{\binom{2n}{n}_x}{(1-x^{2n})(1-x^{2n-1})\cdots(1-x^{n+1})} = \frac{\frac{(1-x^{2n})(1-x^{2n-1})\cdots(1-x^{n+1})}{(1-x)(1-x^2)\cdots(1-x^n)}}{(1-x^{2n})(1-x^{2n-1})\cdots(1-x^{n+1})}$$
$$= \frac{1}{(1-x)(1-x^2)\cdots(1-x^n)}.$$

The number of reduced words not including r_0

$$(1+x)(1+x+x^2)\cdots(1+x+\cdots+x^n).$$

Hence, the number of all reduced words

$$\frac{(1+x)(1+x+x^2)\cdots(1+x+\cdots+x^n)}{(1-x)(1-x^2)\cdots(1-x^n)}$$

which is well known Poincaré polynomial of the affine Weyl group $\widetilde{A}_n$. (see [19]). Therefore these are all reduced words of $\widetilde{A}_n$.

Main Result

Theorem

Then the reduced Gröbner-Shirshov basis of the affine Weyl group $\widetilde{A}_n$ is the set R'. Moreover all the reduced words are the form rw where r is a reduced word not including r_0 and w is a arranged word.

Further

We published this article whose subject is "Gröbner-Shirshov Bases and Reduced Words for Affine Weyl Group Type $\widetilde{A_n}$ at arXiv.com. We submitted this article to a good Algebraic Journal. Now, we are studying about the Gröbner-Shirshov Bases for Affine Weyl Groups Type $\widetilde{B_n}$ and $\widetilde{D_n}$.

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