MAT482: Cebirde Seçme Konular II 12 Mart 2012
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Alıştırmalar II

1. Let $T : \mathbb{R}[x]_2 \to \mathbb{R}^{2\times 2}$ be the linear transformation defined by

$$T(p(x)) = \begin{pmatrix} p(2) - p(1) - p(0) & 0 \\ p(1) - p(2) & p(0) \end{pmatrix}$$

for $p(x) \in \mathbb{R}[x]_2$.
(a) Find a basis for $\mathrm{Ker}(T)$.
(b) Find a basis for $\mathrm{Im}(T)$. Justify.

2. Let U and W be subspaces of $\mathbb{R}^5$ such that

$$U = \langle (1,1,0,1,1), (0,2,2,0,0), (1,-1,-3,1,1), (1,1,1,1,1) \rangle$$

and

$$W = (-1,1,2,-1,-1), (2,0,-2,2,2), (1,1,0,0,0) \rangle.$$

(a) Find a basis for $U + W$.
(b) Find the dimension of $U \cap W$. Justify your steps.
(c) Find a basis for $U/\langle (1,-1,-3,1,1), (1,1,1,1,1) \rangle$.

3. Let $\varphi : V \to W$ be linear. Prove that for every $w \in W$, $\varphi^{-1}[w]$ has 0, 1 or infinitely many elements. Illustrate each case by an example.

4. Show that there is no linear transformation $\varphi : V \to W$ such that $\varphi^{-1}[v_1]$ has one element and $\varphi^{-1}[v_2]$ has infinitely many elements, for some $v_1, v_2 \in V$.

5. Let $T : V \to V$ be linear, U a subspace of V such that $T(U) \subseteq U$ and $V = U \oplus \mathrm{Im}(T)$. Then
(a) Show that $U \subseteq \mathrm{Ker}(T)$.
(b) Conclude that $\mathrm{Ker}(T) = U$.
(c) Give an example of such a transformation T.

6. State whether the following statements are **true or false**. If true, give a brief proof; if false, write a counter-example. Below V and W are finite-dimensional vector spaces over $\mathbb{R}$.

   (a) If $V = A \oplus B = A \oplus C$ for some subspaces A, B, C of V, then $B = C$.

   (b) There exits a linear transformation $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{Ker}(\alpha) = \mathrm{Im}(\alpha)$.

   (c) There exits a linear transformation $\beta : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{Ker}(\beta) = \mathrm{Im}(\beta)$.

(d) If two linear transformations $V \to W$ have the same kernel and image, then they are identical on V.

(e) If two linear transformations $V \to W$ agree on a basis of V, then they are identical on V.

(f) There exist infinitely many distinct isomorphisms from $\mathbb{R}[x]_{261}$ onto $\mathbb{R}^{262}$.

(g) If $|S| = \dim V$, then S is linearly independent iff S spans V.

(h) Let $\mathbf{v}$ and $\mathbf{w}$ be two linearly independent column vectors (matrices) in $\mathbb{R}^{2\times 1}$, and let A be an invertible 2×2 matrix. Then the vectors $A\mathbf{v}$ and $A\mathbf{w}$ are linearly independent.

(i) If $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, and $w = a_1v_1 + a_2v_2 + \cdots + a_kv_k$. Then

$$\{v_1, v_2, \cdots, v_{k-1}, w, v_{k+1}, \cdots, v_n\}$$

is a basis for V iff $a_k \neq 0$

(j) If $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, then $\{v_1 + v_2, v_2 + v_3, \cdots, v_{n-1} + v_n, v_n - v_1\}$ is a basis for V, for all $n \geqslant 2$.

(k) If $\{v_1, v_2, \cdots, v_n\}$ is a basis for V, then $\{v_1 + v_2, v_2 + v_3, \cdots, v_{n-1} + v_n, v_n + v_1\}$ is a basis for V, for all $n \geqslant 2$.

(l) If $\{v_1, v_2, \cdots, v_n\}$ generates V then each $v \in V$ is a unique linear combination of the vectors in this set.

(m) Any subset of $V = \mathbb{R}[x]_n$ which has exactly $n+1$ polynomials of different degrees is a basis of V.

(n) If a vector $v \in \mathbb{R}^n$ has no zero entires in its coordinate matrix with respect to the standard basis of $\mathbb{R}^n$, then v has no zero entires in its coordinate matrix with respect to any basis of $\mathbb{R}^n$.

(o) If $\dim(V) < \dim(W)$, then there is no surjective linear transformation from V into W.