Angles in analytic geometry

David Pierce

November 3, 2006; recompiled, November 9, 2007

The words *synthetic* and *analytic* are sometimes used as opposites or complements. The geometry pioneered by Rene Descartes [1] is called **analytic geometry;** by contrast, the geometry of ancient mathematicians like Euclid of Alexandria [2] and Apollonius of Perge [3] is called **synthetic geometry.**

The word *synthetic* comes from the Greek συνθετικός meaning *skilled in putting together* or *constructive.* This Greek adjective derives from the verb συντίθημι *put together, construct.* The word *analytic* is the English form of ἀναλυτικός, which derives from the verb ἀναλύω *undo, set free, dissolve.*

What do these words mean in the context of mathematics? Although we refer to ancient geometry as synthetic, the Ancients evidently recognized both analytic and synthetic methods. Pappus of Alexandria writes:

> Now **analysis** is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution.
>
> But in **synthesis,** proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis. [4, p. 597]

The main point seems to be that synthesis (and synthetic geometry in particular) should start from first principles and build from there; while analysis (and analytic geometry) is a kind of search for principles from which a desired result would follow.

Euclid of Alexandria begins his *Elements* with five principles:

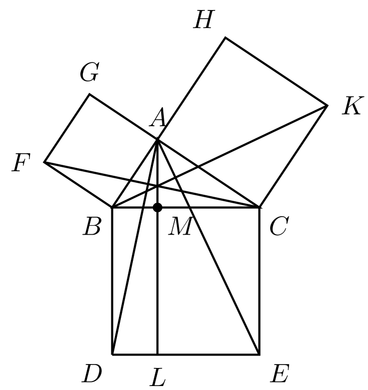
1. any two points can be joined by a [straight] line;
2. any [straight] line can be extended indefinitely;

3. a circle can be drawn with any center and radius;
4. all right angles are equal;
5. if two angles, say ABC and BCD, are together less than two right angles, then lines BA and CD, extended as necessary beyond B and D, must meet.

The 47th proposition that Euclid derives from these principles is commonly known by another name:

Theorem 1 (Pythagoras)**.** *In a right triangle, the square on the hypotenuse is equal to the squares on the legs.*

Proof. The proof is based on the picture at the right, where ABC is a triangle with right angle at A, the squares on the sides are drawn as shown, and AL is perpendicular to BC.

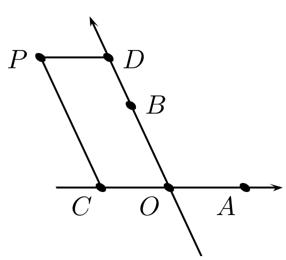


The square $ABFG$ is twice the triangle CBF, which is congruent to DBA, which is half the rectangle $DBML$. So $ABFG$ is equal to $DBML$. Likewise, $ACKH$ is equal to $ECML$. But $DBML$ and $ECML$ are together the square $BCED$. □

One way to *analyze* the Pythagorean Theorem is to understand it as 'really' being about lengths: If the side of ABC opposite angle A has length a, and so forth, and the angle at A is right, then

$$a^2 = b^2 + c^2. \tag{1}$$

We shall see how such an equation can arise when we understand the points A, B, and C as ordered pairs (or triples) of real numbers.



In Euclidean geometry, two distinct lines intersecting at a point determine a plane in the following sense. Let the lines be OA and OB. If C is on OA, and D is on OB, then there is a unique parallelogram $CODP$ for some point P. (The parallelogram is 'degenerate' if C or D is O.) Such points P compose a plane, and every point P in this plane determines a unique such parallelogram. Therefore, instead of working with the points P, we can work with the pairs (x, y), where x is the 'signed' distance of C from O (that is, x is negative if it is on the opposite side of OA from A), and y is the signed distance of D from O. Here we understand signed distances to be just real numbers; so our plane becomes the set $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$ of ordered pairs of real numbers.

So *plane* analytic geometry is about the set $\mathbb{R}^2$; we think of its elements as points. We conceive of $\mathbb{R}^2$ as having **axes,** called X and Y respectively. The X**-axis** consists of points $(x, 0)$; the Y**-axis** consists of points $(0, y)$. Nothing that we have said so far requires these axes to be perpendicular; indeed, it is not yet clear what it would *mean* for the axes to be perpendicular, since these axes are just sets of ordered pairs of numbers. However, Equation (1) is a clue.

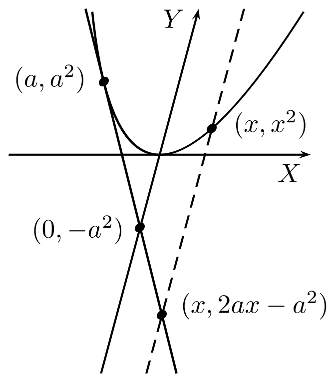
Not everything interesting that we can say about $\mathbb{R}^2$ requires us to conceive of the axes as perpendicular. For example, from the inequality

$$0 \leqslant (x-a)^2 = x^2 - 2ax + a^2,$$

we obtain

$$2ax - a^2 \leqslant x^2.$$

This means that every point on the curve defined by $y = x^2$ is above the point with the same X-coordinate on the line $y = 2ax - a^2$. As the picture shows, this makes visual sense, even if the two axes are not perpendicular.



The same element of $\mathbb{R}^2$ can be written as $\vec{u}$ or (u_1, u_2). Then ${u_1}^2 + {u_2}^2 \geqslant 0$, so $\sqrt{{u_1}^2 + {u_2}^2}$ is a well-defined, non-negative number: let us call this number the **norm** of $\vec{u}$ and denote it by

$$|\vec{u}|.$$

So, by definition, we have the identity

$$|\vec{u}| = \sqrt{{u_1}^2 + {u_2}^2}. \tag{2}$$

The norm is intended to express a notion of *distance:* $|\vec{u}|$ should make sense as the distance between $\vec{u}$ and $\vec{0}$ (that is, $(0,0)$). *Does* it make sense? Well, in Euclidean geometry, $\vec{u}$ is the length of the hypotenuse of a right triangle whose legs have lengths u_1 and u_2. But what is a right triangle in $\mathbb{R}^2$?

We can add elements of $\mathbb{R}^2$ coordinate-wise:

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2). \tag{3}$$

Likewise, we can multiply them by real numbers:

$$a \cdot \vec{u} = (a \cdot u_1, a \cdot u_2). \tag{4}$$

Here a may be called a **scalar;** the elements of $\mathbb{R}^2$ are then called **vectors.** The operations on vectors have various nice properties that follow from the corresponding properties of operations on scalars.

Two vectors are **parallel** if one of them is a scalar multiple of the other: if $a \cdot \vec{u} = \vec{v}$, or $a \cdot \vec{v} = \vec{u}$, then

$$\vec{u} \parallel \vec{v}. \tag{5}$$

Some algebraic consequences of (2) follow almost immediately:

Theorem 2. *For all $\vec{u}$ in $\mathbb{R}^2$ and a in $\mathbb{R}$,*

1. $0 \leqslant |\vec{u}|$;
2. $0 = |\vec{u}| \iff \vec{0} = \vec{u}$;
3. $|a \cdot \vec{u}| = |a| \cdot |\vec{u}|$.

Proof. We observed (1) while defining $|\vec{u}|$. For (2), the direction $\Leftarrow$ follows by computation: $|\vec{0}| = \sqrt{0^2+0^2} = 0$; for the direction $\Rightarrow$, we prove the *contrapositive:* If $\vec{0} \neq \vec{u}$, then one of u_1 and u_2 is not 0; without loss of generality, we may assume $u_1 \neq 0$. Then

$$|\vec{u}|^2 = {u_1}^2 + {u_2}^2 \geqslant {u_1}^2 > 0,$$

so $|\vec{u}| > 0$, and in particular $|\vec{u}| \neq 0$. Finally, for (3), just compute:

$$\begin{aligned} |a \cdot \vec{u}| &= |(a \cdot u_1, a \cdot u_2)| \\ &= \sqrt{(a \cdot u_1)^2 + (a \cdot u_2)^2} \\ &= \sqrt{a^2 \cdot ({u_1}^2 + {u_2}^2)} \\ &= |a| \cdot \sqrt{{u_1}^2 + {u_2}^2}, \end{aligned}$$

which is $|a| \cdot |\vec{u}|$. □

The theorem does not violate any notion of $|\vec{u}|$ as the distance between $\vec{u}$ and $\vec{0}$. For example, if $a \cdot \vec{u} = \vec{v}$, then the distance from $\vec{0}$ to $\vec{v}$ ought to be $|a|$ times the distance to $\vec{u}$; but this is what (3) expresses.

But we should like $|\vec{u}+\vec{v}|$ to make sense as the length of the side of a triangle whose other two sides have lengths $|\vec{u}|$ and $|\vec{v}|$. In particular, we want

$$|\vec{u}+\vec{v}| \leqslant |\vec{u}| + |\vec{v}|. \tag{6}$$

We cannot *assume* that this is true; it is already either true or not, since it is stating a possible property of $\mathbb{R}$. In fact, we shall prove that it is true. Towards doing so, we note first that (since norms are non-negative,) (6) is logically equivalent to

$$\begin{aligned} |\vec{u}+\vec{v}|^2 &\leqslant (|\vec{u}| + |\vec{v}|)^2 \\ &= |\vec{u}|^2 + 2 \cdot |\vec{u}| \cdot |\vec{v}| + |\vec{v}|^2, \end{aligned}$$

which is equivalent to

$$\frac{|\vec{u}+\vec{v}|^2 - |\vec{u}|^2 - |\vec{v}|^2}{2} \leqslant |\vec{u}| \cdot |\vec{v}|. \tag{7}$$

It turns out to be convenient to give the left-hand member of this inequality an abbreviation and a name: it is the **scalar product** or **dot-product** of $\vec{u}$ and $\vec{v}$, and it is denoted

$$\vec{u} \cdot \vec{v}.$$

So, by definition, we have the identity

$$\vec{u} \cdot \vec{v} = \frac{|\vec{u}+\vec{v}|^2 - |\vec{u}|^2 - |\vec{v}|^2}{2}. \tag{8}$$

Presently we shall see an alternative expression for $\vec{u} \cdot \vec{v}$; but let us first note that some basic properties of the scalar product follow directly from (8):

Theorem 3. *For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$:*

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$;

2. $\vec{u} \cdot \vec{0} = 0$;
3. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

Proof. Left to the reader. □

To be able to say much more, we need:
Lemma 4. *For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$,*

$$\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2. \tag{9}$$

Proof. Just compute. □

The lemma allows us to show:
Theorem 5. *For all $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^2$, and a in $\mathbb{R}$,*

1. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$;
2. $\vec{u} \cdot (a \cdot \vec{v}) = a \cdot (\vec{u} \cdot \vec{v})$.

Proof. Computation. □

A special case of (2) is

$$\vec{u} \cdot (-\vec{v}) = -\vec{u} \cdot \vec{v}.$$

Using this, in (8), we can replace $\vec{v}$ with $-\vec{v}$ and rearrange to get

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2 \cdot \vec{u} \cdot \vec{v}. \tag{10}$$

Note the similarity to the Law of Cosines.
Theorem 6 (Cauchy–Schwartz)**.** *For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$,*

$$|\vec{u} \cdot \vec{v}| \leqslant |\vec{u}| \cdot |\vec{v}|, \tag{11}$$

with equality if and only if $\vec{u} \parallel \vec{v}$.

Proof. Let x be a scalar. Now matter how x changes, we must have

$$0 \leqslant |\vec{u} - x \cdot \vec{v}|,$$

equivalently,

$$0 \leqslant |\vec{u} - x \cdot \vec{v}|^2. \tag{12}$$

Now compute:

$$\begin{aligned} |\vec{u} - x \cdot \vec{v}|^2 &= (\vec{u} - x \cdot \vec{v}) \cdot (\vec{u} - x \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2x \cdot \vec{u} \cdot \vec{v} + x^2 \cdot \vec{v} \cdot \vec{v} && \text{[by Theorem 5]} \\ &= |\vec{u}|^2 - 2x \cdot \vec{u} \cdot \vec{v} + x^2 \cdot |\vec{v}|^2 && \text{[by Theorem 3]} \end{aligned}$$

This is a quadratic polynomial in x; it may be written in the more usual fashion as

$$|\vec{v}|^2 \cdot x^2 - 2 \cdot (\vec{u} \cdot \vec{v}) \cdot x + |\vec{u}|^2. \tag{13}$$

By the general theory of such things, the polynomial ax^2+bx+c takes an extreme value at $-b/(2a)$, and this extreme value is $c-b^2/(4a)$; this is a maximum value if $a>0$. In particular, our polynomial (13) has minimum value

$$|\vec{u}|^2 - \frac{(\vec{u}\cdot\vec{v})^2}{|\vec{v}|^2}.$$

This cannot be negative, by (12). That is,

$$0 \leqslant |\vec{u}|^2 - \frac{(\vec{u}\cdot\vec{v})^2}{|\vec{v}|^2},$$

$$\frac{(\vec{u}\cdot\vec{v})^2}{|\vec{v}|^2} \leqslant |\vec{u}|^2,$$

$$(\vec{u}\cdot\vec{v})^2 \leqslant |\vec{u}|^2 \cdot |\vec{v}|^2,$$

and therefore

$$|\vec{u}\cdot\vec{v}| \leqslant |\vec{u}|\cdot|\vec{v}|.$$

Finally, this inequality is an equation if and only if it is possible for $\vec{u}-x\cdot\vec{v}$ to be $\vec{0}$; but this is possible if and only if $\vec{u}$ and $\vec{v}$ are parallel. □

If we accept that there is a function cos on $\mathbb{R}$ that takes on every value in the interval $[-1,1]$, then, by the Cauchy–Schwartz Inequality (11), Theorem, if $\vec{u}$ and $\vec{v}$ are non-zero, there is θ such that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}; \tag{14}$$

in particular, $\cos\theta = 1$ if and only if $\vec{u} \parallel \vec{v}$. Rewriting (14) as

$$|\vec{u}|\cdot|\vec{v}|\cdot\cos\theta = \vec{u}\cdot\vec{v} \tag{15}$$

and substituting into (10), we get a more familiar form of the Law of Cosines.

References

[1] René Descartes. *The Geometry of René Descartes*. Dover Publications, Inc., New York, 1954. Translated from the French and Latin by David Eugene Smith and Marcia L. Latham, with a facsimile of the first edition.

[2] Euclid. *The thirteen books of Euclid's Elements translated from the text of Heiberg. Vol. I: Introduction and Books I, II. Vol. II: Books III–IX. Vol. III: Books X–XIII and Appendix*. Dover Publications Inc., New York, 1956. Translated with introduction and commentary by Thomas L. Heath, 2nd ed.

[3] Apollonius of Perga. *On Conic Sections, Books I–III*. Great Books of the Western World. Encyclopædia Britannica, 1952. Translated by R. Catesby Taliaferro.

[4] Ivor Thomas, editor. *Selections illustrating the history of Greek mathematics. Vol. II. From Aristarchus to Pappus*. Harvard University Press, Cambridge, Mass, 1951. With an English translation by the editor.