# CONIC SECTIONS

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## 1. Introduction

These notes are about the plane curves known as *conic sections.* The mathematical presentation is mainly in the 'analytic' style whose origins are sometimes said to be the *Geometry* [7] of René Descartes. However, the features of conic sections presented in § 3 below were apparently known to mathematicians of the eastern Mediterranean in ancient times. Accordingly, § 2 below contains a review what I have been able to find out about the ancient knowledge. I try to give references to the original texts (or translations of them). Meanwhile, I list some relevant approximate dates; the ancient dates are selected from [5, pp. 685 f.]:

−350: Menaechmus on conic sections;
−300: Euclid, *Elements;*
−225: Apollonius, *Conics;*
−212: death of Archimedes;
+320: Pappus, *Mathematical Collections;*
+560: Eutocius, commentaries on Archimedes;
+1637: Descartes, *Geometry.*

The reader of these notes may agree that the conic sections are worthy of study, independently of any application. However, Isaac Newton (1643–1727), for example, could not have developed his theory of gravitation [8] without knowing what the Ancients knew about conic sections.[1]

## 2. Background

2.1. **Definitions.** A **cone** and its associated **conic surface** are determined by the following data:

(1) a circle, called the **base** of the cone;
(2) a point, called the **vertex** of the cone and the conic surface; the vertex must not lie in the plane of the base.

The conic surface consists of the points on the lines that pass through the vertex and the circumference of the base. The cone itself is the solid figure bounded by the surface and the base. See Figure 1.

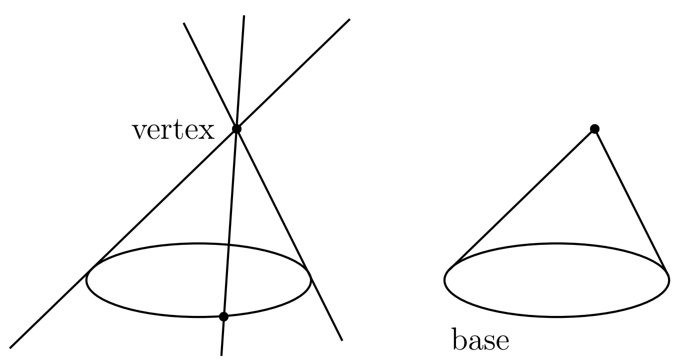


Figure 1. A conic surface and cone

[1]An inverse-square law of gravitation causes planetary orbits to be conic sections. Newton showed this, apparently using such knowledge as can be found in Apollonius. It may be that Newton inferred, from ancient secondary sources, that the ancient scientists themselves were aware of an inverse-square law of gravity [9, § 11.7].

The definitions of cone and conic surface can be found at the beginning of the treatise *On Conic Sections* [1, 2, 3, 11], by Apollonius of Perga.[2] The **axis** of the cone is the line joining the vertex to the center of the base. There is no assumption that the axis is perpendicular to the base; if it is, then the cone is **right;** otherwise, the cone is **oblique.**

A **conic section** is the intersection of a plane with a conic surface. The discovery of conic sections (as objects worthy of study) is generally[3] attributed to Apollonius's predecessor Menaechmus. However, there are three kinds of conic sections: the **ellipse,** the **parabola,** and the **hyperbola.** According to Eutocius [11, pp. 276–281], Apollonius was the first mathematician to show that each kind of conic section can be obtained from *every* conic surface. Indeed, the names of the three kinds of conic sections appear [11, p. 283 f., n. *a*] to be due to Apollonius as well. The names are meaningful in Greek and reflect the different geometric properties of the sections, in a way shown in § 3.

2.2. **Motivation.** Menaechmus used conic sections to solve the problem of **duplicating the cube.** Suppose a cube is given, with volume $V$; how can a cube be constructed with volume $2V$? We can give a symbolic answer: If the side of the original cube has length $s$, then the new cube must have side of length $s\sqrt[3]{2}$. But how can a side of that length be *constructed?*

The corresponding problem for squares can be solved as follows. Suppose $AB$ is a diameter of a circle, and $C$ is on $AB$, and $D$ is on the circumference of the circle, and $CD \perp AB$. Then the square on $CD$ is equal in area to the rectangle whose sides are $AC$ and $BC$. More symbolically, if lengths are as in Figure 2, then

$$\frac{a}{x} = \frac{x}{b}, \qquad \text{or} \qquad ab = x^2, \tag{1}$$

so that

$$\frac{x^2}{a^2} = \frac{b}{a}. \tag{2}$$

In particular, if $b/a = 2$, then a square with side of length $x$ has area twice that of a square with side of length $a$.

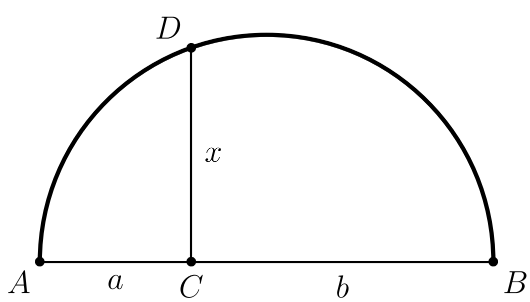


FIGURE 2. The method of finding a mean proportional

Suppose instead we have

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}. \tag{3}$$

Then

$$\frac{x^3}{a^3} = \frac{x}{a} \cdot \frac{y}{x} \cdot \frac{b}{y} = \frac{b}{a}. \tag{4}$$

[2]Perga or Perge was near what is now Antalya; its remains are well worth a visit.
[3]See for example [11, p. 283 f., n. *a*] or [6, p. 1].

If $b/a = 2$, then a cube with side of length $x$ has volume twice that of a cube with side of length $a$. In any case, the several lengths can be arranged as in Figure 3. There, angle $ACB$ is right, and $BCD$ and $ACE$ are diameters of the indicated circles.

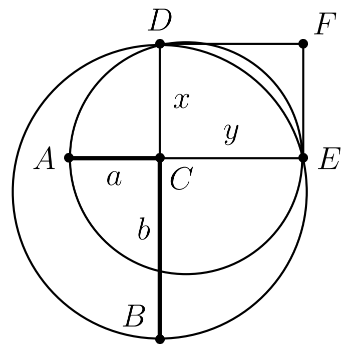


FIGURE 3. Two mean proportionals

The problem is, How can $D$ and $E$ be chosen on the extensions of $BC$ and $AC$ so that the circles intersect as in Figure 3? The solution of Menaechmus (along with many other solutions) is given in the commentary [4, pp. 288-290] by Eutocius on the second volume *On the Sphere and the Cylinder* by Archimedes. In Figure 3, if $CDFE$ is a rectangle, then $F$ determines $x$ and $y$. But by Equations (3), rearranged, $x$ and $y$ must satisfy two equations,

$$ay = x^2, \qquad bx = y^2. \tag{5}$$

Each of these equations determines a curve, and $F$ is the intersection of the two curves. The curves turn out to be conic sections, namely parabolas. Points on the curve given by $ay = x^2$ can be plotted as in Figure 4.

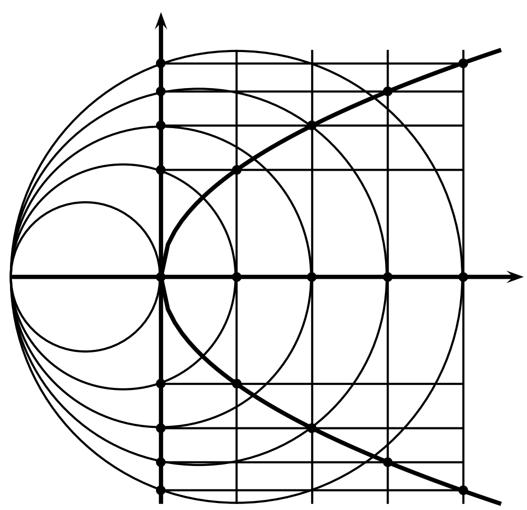

FIGURE 4. Construction of points of the parabola

If one imagines that the circles in Figure 4 are not all in the same plane, but serve as parallel bases of cones bounded by the same conic surface, then one may be able to see how the curve arises as a section of that surface. However, an alternative approach to the conic sections was given by Pappus of Alexandria [10, p. 492–503]; it may have been due originally to Euclid of Alexandria, although his works on conic sections are lost. We can take the alternative approach as follows.

## 3. Equations

3.1. **Focus and directrix.** A **conic section** $\zeta$ is determined by the following data:

(1) a line $d$, called the **directrix** of $\zeta$;
(2) a point $F$ (not on $d$), called the **focus** of $\zeta$;
(3) a positive real number (or distance) $e$, called the **eccentricity** of $\zeta$.

Then $\zeta$ comprises the points $P$ (in the plane of $d$ and $F$) such that

$$|PF| = e \cdot |Pd|. \tag{6}$$

Some examples are in Figure 5, with the same directrix and focus, but various eccentricities. The examples are drawn (by computer) by means of (25) below. (See also Figure 6.)

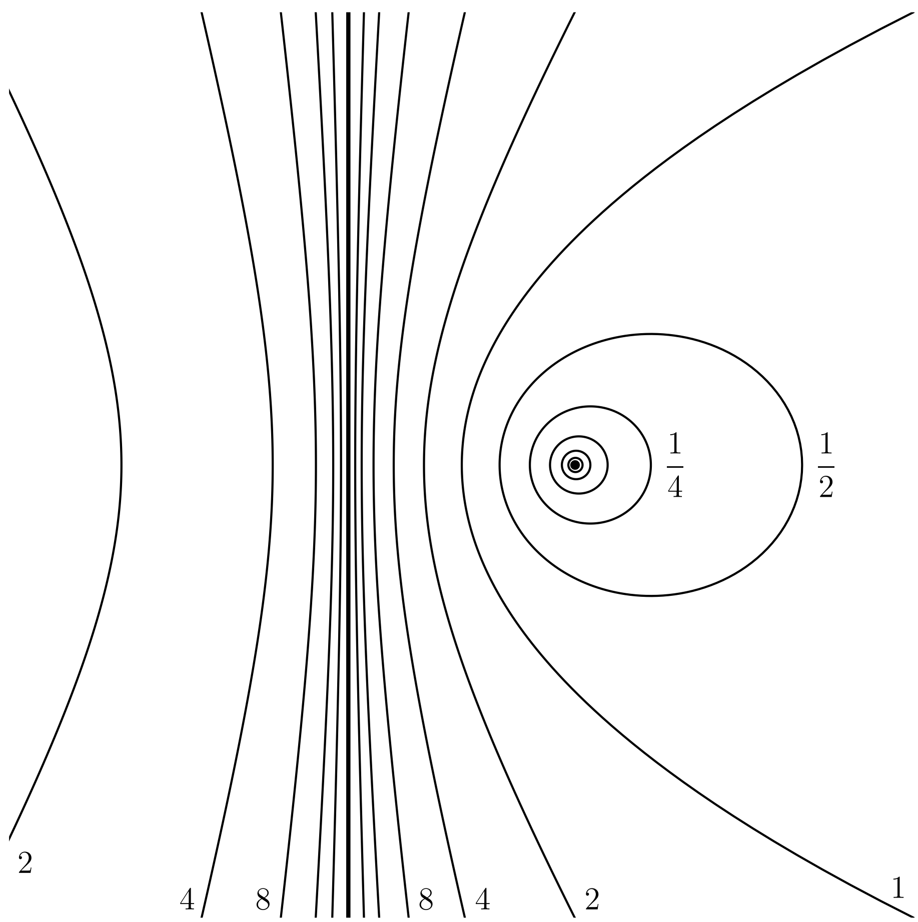


Figure 5. Conic sections of different eccentricities

Suppose we assign a rectangular coordinate system to the plane of $\zeta$ in which $F$ has the coordinates $(h, k)$, and $d$ is defined by

$$Ax + By + C = 0 \tag{7}$$

(where $A \neq 0$ or $B \neq 0$). Then $\zeta$ is defined by

$$\sqrt{(x-h)^2+(y-k)^2} = e \cdot \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}, \tag{8}$$

hence also by

$$(x-h)^2+(y-k)^2 = e^2 \cdot \frac{(Ax+By+C)^2}{A^2+B^2}. \tag{9}$$

This equation is not very useful for showing the shape of $\zeta$. By choosing the rectangular coordinate system appropriately, we can ensure

$$(h,k)=(0,0), \quad B=0, \quad A=1, \quad C>0. \tag{10}$$

Then $C$ is the distance between the focus and the directrix, and (9) becomes

$$x^2+y^2=e^2(x+C)^2. \tag{11}$$

3.2. **The polar equation.** Equation (11) is nicer than (9), but is still not the most useful rectangular equation for $\zeta$. However, (11) becomes more useful when converted to polar form. Recall the conversion-equations:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta; \end{cases} \qquad \begin{cases} r^2 = x^2+y^2, \\ \tan\theta = \dfrac{y}{x}. \end{cases} \tag{12}$$

So the polar form of (11) is

$$r^2 = e^2(r\cos\theta + C)^2, \tag{13}$$

which is equivalent to

$$\pm r = e(r\cos\theta + C). \tag{14}$$

The plus-or-minus sign here is needed, unless we know that $r$ always has the sign of $r\cos\theta + C$, or always has the opposite sign. It does not.

However, note well that the same point can have different polar coordinates; in particular, the same point has polar coordinates $(r,\theta)$ and $(-r,\theta+\pi)$. We shall use this fact frequently. The equation

$$-r = e(r\cos\theta + C) \tag{15}$$

is equivalent to

$$-r = e(-r\cos(\theta+\pi) + C). \tag{16}$$

Hence, if $(s,\varphi)$ satisfies (15), then $(-s,\varphi+\pi)$ satisfies

$$r = e(r\cos\theta + C). \tag{17}$$

So we can take either (15) or (17) as the polar equation for $\zeta$. We can also derive (14) directly from the original definition of $\zeta$; see Figure 6.

We can rewrite (17) as

$$r = er\cos\theta + eC, \tag{18}$$

$$r - er\cos\theta = eC, \tag{19}$$

$$r(1-e\cos\theta) = eC. \tag{20}$$

Since $eC \neq 0$, the factor $1 - e\cos\theta$ will never be 0, so we can divide by it, obtaining

$$r = \frac{eC}{1-e\cos\theta}. \tag{21}$$

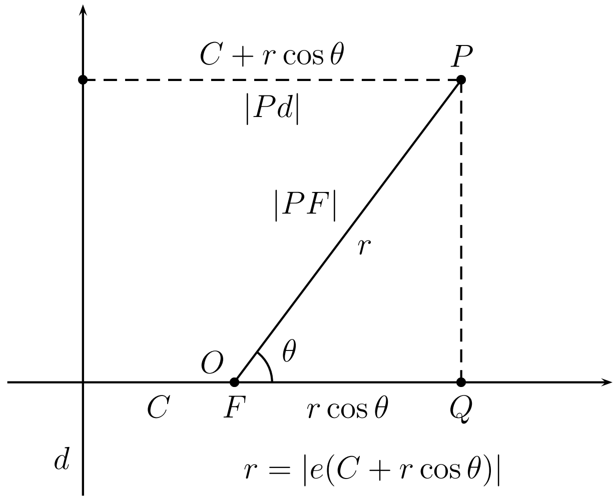


FIGURE 6. Derivation of the polar equation of a conic section

If we rewrite (15) the same way, we get

$$r = \frac{eC}{-1 - e\cos\theta}. \tag{22}$$

Again, either (21) or (22) by itself defines $\zeta$.

The line through the focus and parallel to the directrix is defined by $\theta = \pi/2$. By (21) (or from the original definition of $\zeta$), this line meets $\zeta$ in two points, $L_0$ and $L_1$, whose coordinates are $(eC, \pi/2)$ and $(eC, -\pi/2)$. It will be convenient to denote the distance $|L_0L_1|$ by $2\ell$: this means defining

$$\ell = eC. \tag{23}$$

Then (18), (21) and (22) can be rewritten as

$$r = er\cos\theta + \ell, \tag{24}$$

$$r = \frac{\ell}{1 - e\cos\theta} \tag{25}$$

$$r = \frac{\ell}{-1 - e\cos\theta}. \tag{26}$$

3.3. **Lines through the focus.** By (25), *each* line $\theta = \varphi$ through the origin meets $\zeta$ in two points, namely

$$\left(\frac{\ell}{1 - e\cos\varphi}, \varphi\right) \quad \text{and} \quad \left(\frac{\ell}{1 + e\cos\varphi}, \varphi + \pi\right), \tag{27}$$

unless $e\cos\varphi = \pm 1$. There are three possibilities, corresponding the three kinds of conic sections:

(1) If $0 < e < 1$, then $|e\cos\theta|$ is never 1, so every line through the origin meets $\zeta$ at two points, *and* these points are on opposite sides of the origin; $\zeta$ is an **ellipse.** See Figure 7.
(2) If $e = 1$, then every line through the origin meets $\zeta$ at two points, which are are on opposite sides of the origin, *unless* the line is $\theta = 0$: This line meets $\zeta$ only at $(\ell/2, \pi)$, halfway between the focus and the directrix. Now $\zeta$ is a **parabola.** See Figure 8.

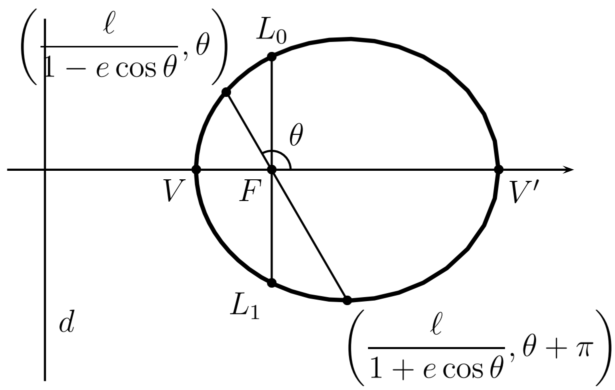


FIGURE 7. The ellipse

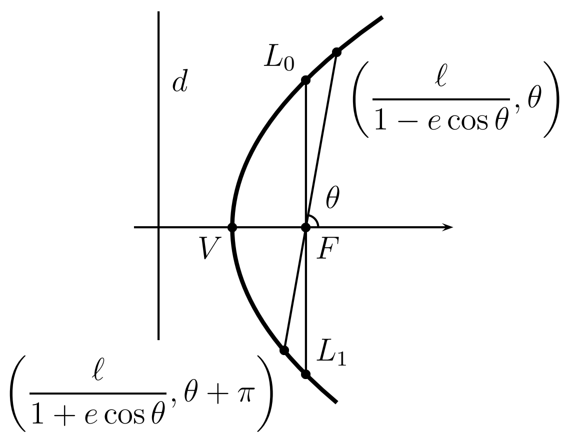


FIGURE 8. The parabola

(3) Suppose $e > 1$. then $\cos\alpha = 1/e$ for some $\alpha$ such that $0 < \alpha < \pi/2$. If $\alpha < \varphi < 2\pi - \alpha$, then the line $\theta = \varphi$ meets $\zeta$ at two points, on opposite sides of the origin, as in the ellipse and parabola. If $-\alpha < \varphi < \alpha$, then the line $\theta = \varphi$ meets $\zeta$ at two points, on the *same* side of the origin. Each of the lines $\theta = \alpha$ and $\theta = -\alpha$ meets $\zeta$ once, at $(\ell/2, \pi + \alpha)$ or $(\ell/2, \pi - \alpha)$. Here $\zeta$ is an **hyperbola.** It is really *two* curves:
   - $\zeta_0$, given by (25), where $\alpha < \theta < 2\pi - \alpha$;
   - $\zeta_1$, given by (25), where $-\alpha < \theta < \alpha$; or by (26), where $\pi - \alpha < \theta < \pi + \alpha$.

   See Figure 9.

3.4. **Distances.** The line through the focus $F$ perpendicular to the directrix $d$ is the **axis** of $\zeta$. Then $\zeta$ is symmetric about its axis, because of the original definition, or by (25). A point of $\zeta$ that lies on the axis is a **vertex** of $\zeta$. Again, there are three cases:

(1) Say $0 < e < 1$, so $\zeta$ is an ellipse. Then $\zeta$ has a vertex $V$, with coordinates $(\ell/(1+e), \pi)$, and a vertex $V'$, given by $(\ell/(1-e), 0)$. Since

$$0 < 1 - e \leqslant 1 - e\cos\theta \leqslant 1 + e, \tag{28}$$

we have

$$\frac{\ell}{1+e} \leqslant \frac{\ell}{1 - e\cos\theta} \leqslant \frac{\ell}{1-e}. \tag{29}$$

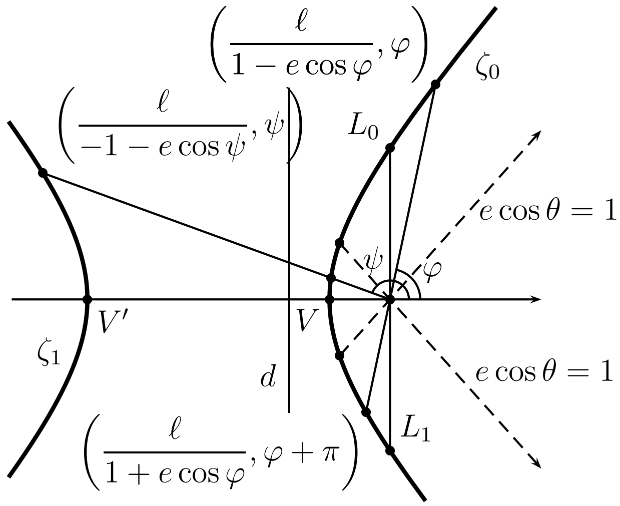


FIGURE 9. The hyperbola

By (25) then, $V$ is the point of $\zeta$ that is closest to the focus, and $V'$ is the point furthest from $F$. Also,

$$|VV'| = \frac{\ell}{1+e} + \frac{\ell}{1-e} = \frac{2\ell}{1-e^2}. \tag{30}$$

See Figure 10.

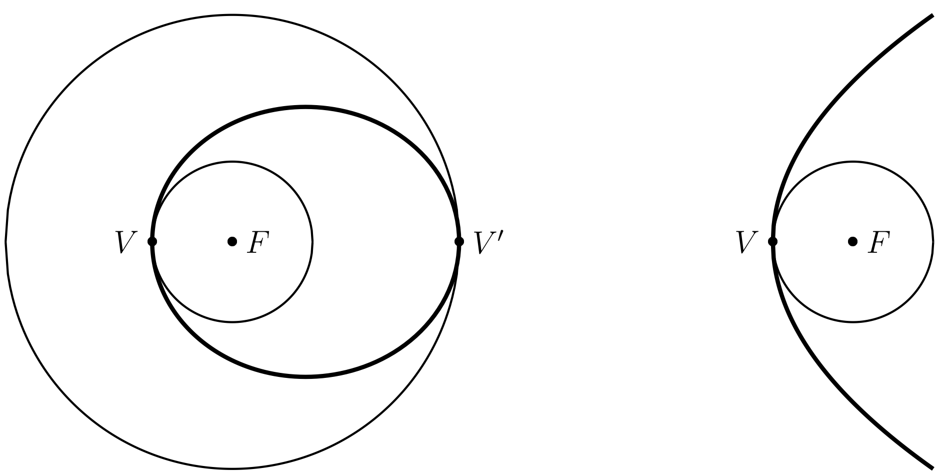


FIGURE 10. Extreme points in the ellipse and parabola

(2) Say $e = 1$, so $\zeta$ is a parabola. Then it has a unique vertex, $V$, with coordinates $(\ell/2, \pi)$. As in the case of the ellipse, so in the parabola, $V$ is the point of $\zeta$ closest to the focus; but there is no furthest point. Again, see Figure 10.

(3) Say $e > 1$, so $\zeta$ is an hyperbola. Then it has two vertices, $V$ and $V'$, with coordinates $(\ell/(e+1), \pi)$ and $(\ell/(e-1), \pi)$ respectively. As before, suppose $\cos\alpha =$

$1/e$, where $0 < \alpha < \pi/2$. If $-\alpha < \theta < \alpha$, then

$$\frac{1}{e} < \cos\theta \leqslant 1, \tag{31}$$

$$1 < e\cos\theta \leqslant e, \tag{32}$$

$$0 < e\cos\theta - 1 \leqslant e - 1, \tag{33}$$

$$0 < \frac{\ell}{e-1} \leqslant \frac{\ell}{e\cos\theta - 1}; \tag{34}$$

so $V'$ is the point of $\zeta_1$ closest to the focus. If $\alpha < \theta < 2\pi - \alpha$, then

$$-1 \leqslant \cos\theta < \frac{1}{e}, \tag{35}$$

$$-e \leqslant e\cos\theta < 1, \tag{36}$$

$$-1 < -e\cos\theta \leqslant e, \tag{37}$$

$$0 < 1 - e\cos\theta \leqslant e + 1, \tag{38}$$

$$\frac{\ell}{e+1} \leqslant \frac{\ell}{1 - e\cos\theta}; \tag{39}$$

so $V$ is the point of $\zeta_0$ closest to the focus. Finally,

$$|VV'| = \frac{\ell}{e-1} - \frac{\ell}{e+1} = \frac{2\ell}{e^2 - 1}. \tag{40}$$

See Figure 11.

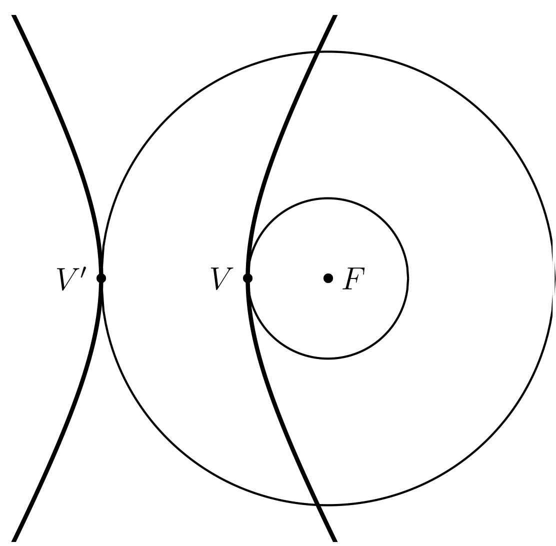


FIGURE 11. Extreme points in the hyperbola

In both the ellipse and the hyperbola then, the distance between the two vertices is $2\ell/|e^2 - 1|$; this may also be denoted by $2a$, so that

$$a = \frac{\ell}{|e^2 - 1|}. \tag{41}$$

3.5. **Areas.** Let $P$ be an arbitrary point with coordinates $(r, \theta)$ on $\zeta$, and let the foot of the perpendicular from $P$ to the axis of $\zeta$ be $Q$ (as in Figure 6). Then $Q$ has coordinates $(r\cos\theta, 0)$. We consider the position of $Q$ with respect to the vertices:

(1) If $0 < e < 1$, then by (29) and (24)

$$\frac{\ell}{1+e} \leqslant r \leqslant \frac{\ell}{1-e}, \tag{42}$$

$$\frac{\ell}{1+e} \leqslant er\cos\theta + \ell \leqslant \frac{\ell}{1-e}, \tag{43}$$

$$-\frac{\ell e}{1+e} \leqslant er\cos\theta \leqslant \frac{\ell e}{1-e}, \tag{44}$$

$$-\frac{\ell}{1+e} \leqslant r\cos\theta \leqslant \frac{\ell}{1-e}; \tag{45}$$

so $Q$ is between $V$ and $V'$, and

$$|VQ| = r\cos\theta + \frac{\ell}{1+e}, \tag{46}$$

$$|V'Q| = \frac{\ell}{1-e} - r\cos\theta. \tag{47}$$

(2) If $e = 1$, then

$$\frac{\ell}{2} \leqslant r = r\cos\theta + \ell, \tag{48}$$

$$-\frac{\ell}{2} \leqslant r\cos\theta, \tag{49}$$

$$|VQ| = r\cos\theta + \frac{\ell}{2}. \tag{50}$$

(3) If $e > 1$, then there are two cases:
(a) if $P$ is on $\zeta_0$, then

$$\frac{\ell}{1+e} \leqslant r = er\cos\theta + \ell, \tag{51}$$

$$-\frac{\ell e}{1+e} \leqslant er\cos\theta, \tag{52}$$

$$-\frac{\ell}{1+e} \leqslant r\cos\theta, \tag{53}$$

$$|VQ| = r\cos\theta + \frac{\ell}{e+1}, \tag{54}$$

$$|V'Q| = r\cos\theta + \frac{\ell}{e-1}; \tag{55}$$

(b) if $P$ is on $\zeta_1$, then

$$\frac{\ell}{e-1} \leqslant r = -(er\cos\theta + \ell), \tag{56}$$

$$\frac{\ell e}{e-1} \leqslant -er\cos\theta, \tag{57}$$

$$\frac{\ell}{e-1} \leqslant -r\cos\theta, \tag{58}$$

$$|VQ| = -\left(r\cos\theta + \frac{\ell}{e+1}\right), \tag{59}$$

$$|V'Q| = -\left(r\cos\theta + \frac{\ell}{e-1}\right). \tag{60}$$

In either case, $Q$ is *not* between $V$ and $V'$.

Now we can compute:

$$\begin{aligned}
|PQ|^2 &= r^2\sin^2\theta && (61)\\
&= r^2 - r^2\cos^2\theta && (62)\\
&= (er\cos\theta+\ell)^2 - r^2\cos^2\theta && (63)\\
&= (r[e+1]\cos\theta+\ell)(r[e-1]\cos\theta+\ell). && (64)
\end{aligned}$$

There are two cases:

- If $e = 1$, then this equation becomes

$$|PQ|^2 = (2r\cos\theta + \ell)\cdot\ell = 2\ell\cdot|VQ|. \tag{65}$$

- If $e \neq 1$, then

$$\begin{aligned}
|PQ|^2 &= (e^2-1)\left(r\cos\theta+\frac{\ell}{e+1}\right)\left(r\cos\theta+\frac{\ell}{e-1}\right) && (66)\\
&= |e^2-1|\cdot|VQ|\cdot|V'Q| && (67)\\
&= 2\ell\cdot\frac{|V'Q|}{|VV'|}\cdot|VQ|. && (68)
\end{aligned}$$

Let $VR$ be drawn perpendicular to the axis of $\zeta$ so that $|VR| = 2\ell$. This line segment is called the **latus rectum** of $\zeta$. This is the term commonly used in English, although it is the *Latin* translation of the original Greek found in Apollonius; however, the literal English translation, 'upright side,' is used in [2]. Then the square with side $PQ$

- is the area of the rectangle with sides $VQ$ and $VR$, if $\zeta$ is a parabola;
- falls short of this area, if $\zeta$ is an ellipse;
- exceeds this area, if $\zeta$ is an hyperbola.

This is what is suggested by the Greek names of the curves. See Figure 12.

3.6. **The rectangular equations.** For the parabola, choose a rectangular coordinate system in which $V$ is the origin and the $X$-axis is the axis of $\zeta$. Then (65) becomes

$$y^2 = 2\ell x. \tag{69}$$

This is the standard rectangular equation for a parabola. The focus is at $(\ell/2, 0)$, and the directrix is given by $x + \ell/2 = 0$.

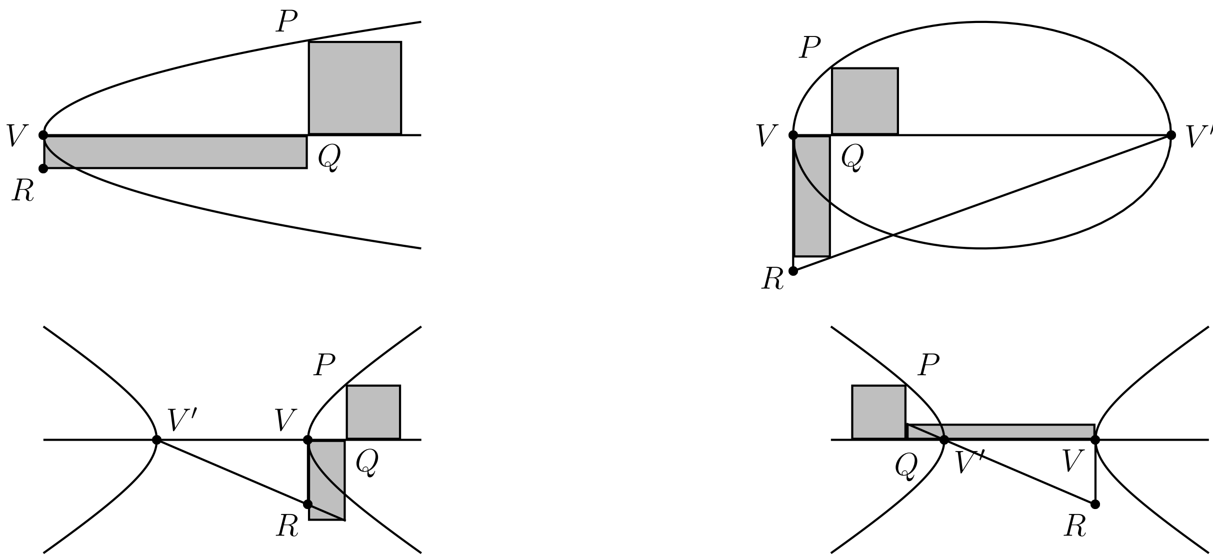


FIGURE 12. Conic sections as determined by equations of areas

For the ellipse and the hyperbola, let the origin of a rectangular coordinate system be the midpoint $O$ of $VV'$: this is the **center** of the conic section. Let the $X$-axis contain the vertices. Then the vertices will have coordinates $(\pm a, 0)$. By (67), the curve is symmetric about the new $Y$-axis. In particular, the curve has, not just one focus, but two foci; hence it has, not just one directrix, but two directrices, one for each focus. The curve is now given by

$$y^2 = |e^2 - 1| \cdot |x - a| \cdot |x + a| = |e^2 - 1| \cdot |x^2 - a^2|. \tag{70}$$

Moreover, by the previous subsection, in the ellipse, $e^2 - 1$ and $x^2 - a^2$ are both negative; in the hyperbola, positive. Hence (70) can be written

$$y^2 = (e^2 - 1)(x^2 - a^2), \tag{71}$$

$$\frac{y^2}{a^2(e^2 - 1)} = \frac{x^2}{a^2} - 1, \tag{72}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \tag{73}$$

Recalling (41), we can write

$$\frac{x^2}{a^2} \pm \frac{y^2}{a\ell} = 1, \tag{74}$$

where the upper sign is for the ellipse, and the lower is for the hyperbola. We may let $b$ be the positive number such that

$$b^2 = a\ell, \tag{75}$$

so that (74) becomes

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1. \tag{76}$$

The $Y$-intercepts of the ellipse are $(0, \pm b)$; the hyperbola has no $Y$-intercepts. By (41) and (75),

$$e = \sqrt{1 \mp \frac{b^2}{a^2}}; \tag{77}$$

where again the upper sign is for the ellipse. Also,

$$|FO| = a \mp \frac{\ell}{1+e} = a - \frac{a(1-e^2)}{1+e} = a - a(1-e) = ae; \tag{78}$$

so the foci are at $(\pm ae, 0)$. Likewise,

$$|dO| = ae \pm \frac{\ell}{e} = ae + \frac{a(1-e^2)}{e} = \frac{a}{e}; \tag{79}$$

so the directrices are given by $x \pm a/e = 0$.

Finally, the hyperbola given by (76) does not meet the two lines given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{80}$$

These lines—given also by $ay \pm bx = 0$—are the **asymptotes** of the hyperbola. Their slopes are $\pm b/a$. In general, a line through $O$ meets the hyperbola if and only if the slope of the line is less than $b/a$ in absolute value. Indeed, the equations

$$\begin{cases} y = mx, \\ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \end{cases} \tag{81}$$

if solved simultaneously, yield

$$\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1, \tag{82}$$

$$\frac{b^2}{a^2} - m^2 = \frac{b^2}{x^2}, \tag{83}$$

$$\frac{b^2}{a^2} - m^2 > 0, \tag{84}$$

$$m^2 < \frac{b^2}{a^2}, \tag{85}$$

$$|m| < \frac{b}{a}; \tag{86}$$

and if the last inequality holds, then there *is* a simultaneous solution, obtainable from (83) and then (81).

The two hyperbolas $x^2/a^2 - y^2/b^2 = \pm 1$ have the same asymptotes. Also, their foci are at the same distance from the center, namely $\sqrt{a^2 + b^2}$. Such hyperbolas are **conjugate.** The ellipse $x^2/a^2 + y^2/b^2 = 1$ is tangent to them at their vertices. See Figure 13.

The segment joining the two vertices of an ellipse is the **major axis** of the ellipse; the **minor axis** passes through the center, but is perpendicular to the major axis.

A **circle** can be described as an ellipse of eccentricity 0. Strictly, however, a circle is not a conic section by the definition given in § 3.1. The circle does not have a directrix. However, the circle is a kind of 'limit' of the ellipses with the same focus and latus rectum, as the directrix moves indefinitely far away (which means the eccentricity tends to 0). See Figure 14.

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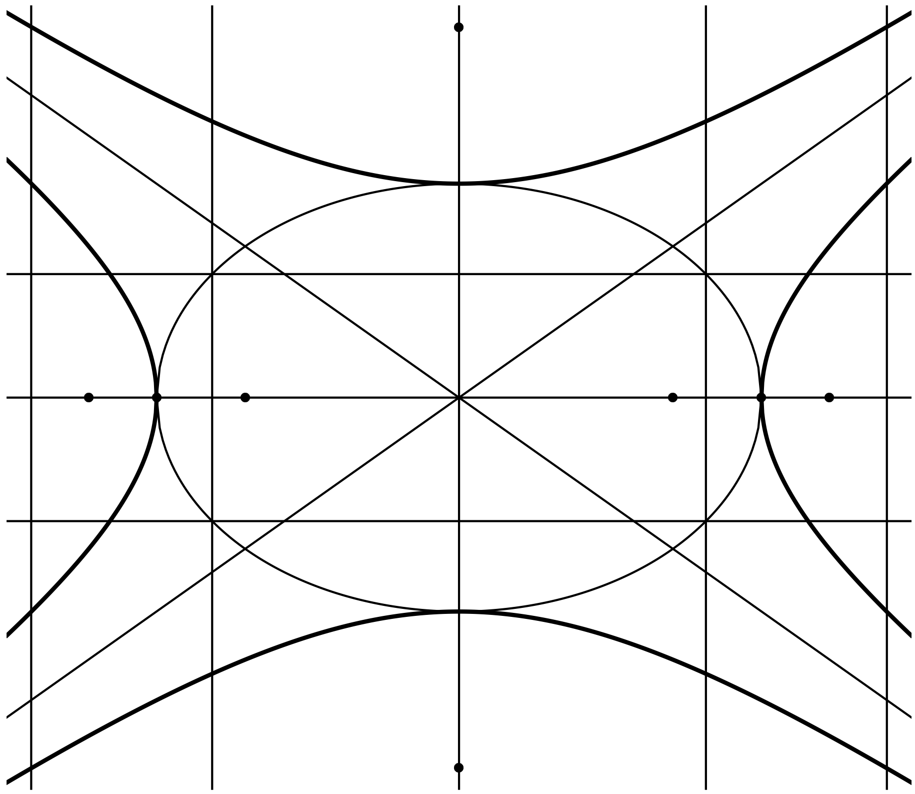

FIGURE 13. Conjugate hyperbolas


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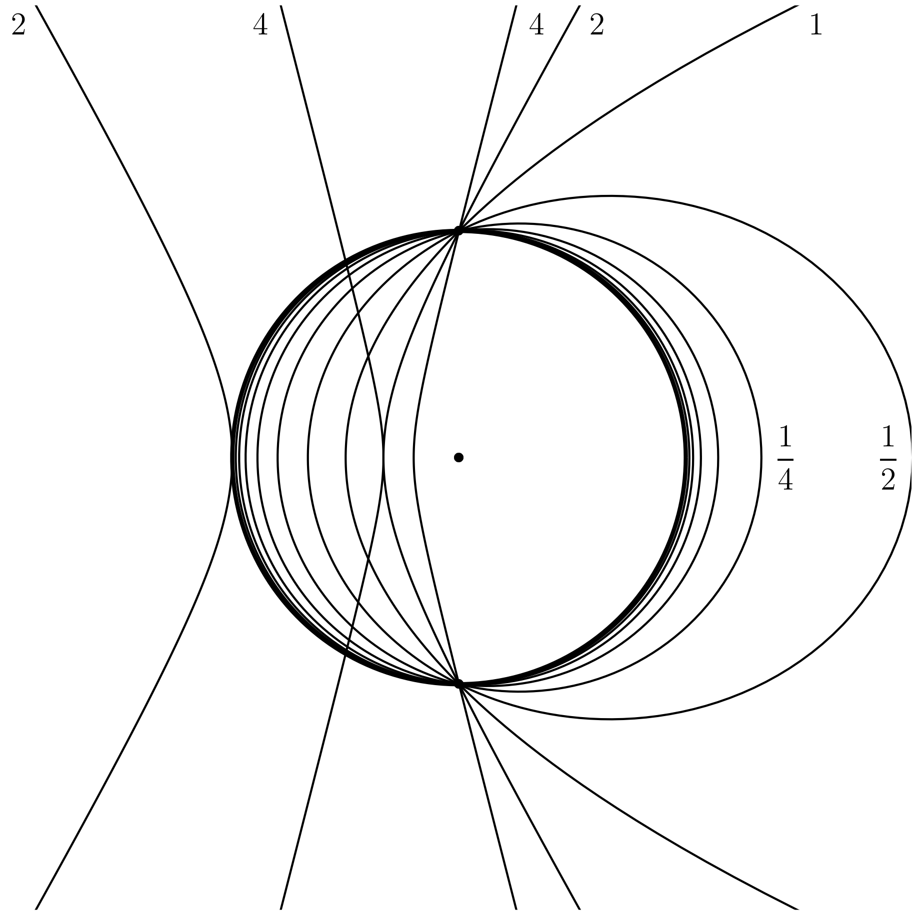


Figure 14. The circle as a limit of conics

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