Analysis lecture notes

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I am preparing these notes while teaching a class in mathematical analysis (METU Math 271, year 2002/3, fall semester). The current table of contents is at the end. Mistakes and inconsistencies are possible here (as in all human productions). I say some things about $\mathbb{R}$ which are true—and are later needed—for arbitrary (ordered) fields.

Our main purpose is to understand $\mathbb{R}$.

1 Fields: a definition

A **commutative** (or **abelian**) **group** is a structure

$$(G, *, \hat{\ }, e)$$

—where $*$ is a binary, $\hat{\ }$ a unary, and e a nullary operation—satisfying:

- $\forall x\ \forall y\ x * y = y * x$ [**commutativity**],
- $\forall x\ \forall y\ \forall z\ x * (y * z) = x * y * x$ [**associativity**],
- $\forall x\ e * x = x$ [**identity element**],
- $\forall x\ x * \hat{x} = e$ [**inverses**].

A **field** is a structure

$$(F, +, -, \cdot, 0, 1)$$

where $F \setminus \{0\}$ is also equipped with $^{-1}$ such that

- $(F, +, -, 0)$ and $(F \setminus \{0\}, \cdot, ^{-1}, 1)$ are commutative groups, and
- $\forall x\ \forall y\ \forall z\ x \cdot (y \cdot z) = x \cdot y + x \cdot z$ [**distributivity**].

Exercise 1. Prove that a field satisfies:

$$\forall x\ \forall y\ (xy = 0 \to x = 0 \lor y = 0)\ [\textbf{no zero-divisors}].$$

The set $\mathbb{R}$ of real numbers is a field (when equipped with the usual operations).

2 Functions into a field

We shall be interested in functions from arbitrary sets into $\mathbb{R}$. Say S is a set, and $f, g : S \to \mathbb{R}$ are functions. Then we can use the field-operations to form new functions:

- $f + g : x \mapsto f(x) + g(x)$,
- &c.

Exercise 2. Let A be the set of all functions from S to $\mathbb{R}$. Which properties of fields does A have? For example, does it have no zero-divisors?

3 Polynomial and rational functions

Let X be a variable; we define

$$\mathbb{R}[X]$$

to be the set of polynomials in X with coefficients in $\mathbb{R}$. So a typical element of $\mathbb{R}[X]$ is

$$a_0 + a_1X + a_2X^2 + \cdots + a_nX^n,$$

that is,

$$\sum_{k=0}^{n} a_k X^k,$$

where $a_k \in \mathbb{R}$. Any member of $\mathbb{R}[X]$ determines a function from $\mathbb{R}$ to itself. In particular, X is the identity-function on $\mathbb{R}$. Then $\mathbb{R}[X]$ is the smallest set of functions from $\mathbb{R}$ to itself that contains the identity-function and is closed under the field-operations induced from $\mathbb{R}$.

Note that the identity-function is not the multiplicative identity 1 or the additive identity 0. Also, in this definition, multiplicative inversion is not a field-operation.

Say $f \in \mathbb{R}[X]$. What can we say about the set

$$\{x \in \mathbb{R} : f(x) = 0\}?$$

It is a consequence of the field-axioms that this set is finite (and is no bigger than the *degree* of f). Let $1/f$ be the function $x \mapsto f(x)^{-1}$. Then $1/f$ is defined at all but finitely many points of $\mathbb{R}$.

The function $1/f$ is the multiplicative inverse of f (not the functional inverse, f^{-1}). We can form the function g/f whenever $g \in \mathbb{R}[X]$. The set of all such functions is

$$\mathbb{R}(X),$$

the *field* of **rational** functions in X over $\mathbb{R}$.

4 Subfields

A field $(F, +, -, \cdot, 0, 1)$ is a **sub-field** of a field $(G, \oplus, \ominus, \odot, 0', 1')$ if $F \subseteq G$, and

- $x \oplus y = x + y$ for all x and y in F,
- &c.

Then $\mathbb{Q}$ is a sub-field of $\mathbb{R}$; in fact, $\mathbb{Q}$ is the smallest sub-field of $\mathbb{R}$. Also, $\mathbb{R}$ is a sub-field of $\mathbb{C}$; but $\mathbb{Z}/p\mathbb{Z}$ is not a sub-field of $\mathbb{Q}$ (or $\mathbb{R}$, or $\mathbb{C}$).

Each element a of $\mathbb{R}$ can be considered as the function

$$x \mapsto a : \mathbb{R} \to \mathbb{R}.$$

In this way, $\mathbb{R}$ is a sub-field of $\mathbb{R}(X)$. Also, $\mathbb{R}(X)$ is the smallest field that contains the identity-function and includes $\mathbb{R}$ as a sub-field.

5 Ordered fields

An **ordered** field is a field with a binary relation $\leqslant$ satisfying:

- $\forall x\ x \leqslant x$,
- $\forall x\ \forall y\ (x \leqslant y \land y \leqslant x \to x = y)$,
- $\forall x\ \forall y\ \forall z\ (x \leqslant y \land y \leqslant z \to x \leqslant z)$,
- $\forall x\ \forall y\ (x \leqslant y \lor y \leqslant x)$,
- $\forall x\ \forall y\ \forall z\ (x \leqslant y \to x + z \leqslant y + z)$,
- $\forall x\ \forall y\ \forall z\ (x \leqslant y \land 0 \leqslant z \to xz \leqslant yz)$.

Then $\mathbb{R}$ is an ordered field.

In an alternative definition, a field is ordered if it has a subset P that is closed under $+$ and $\cdot$ such that the whole field is the disjoint union

$$\{x : -x \in P\} \sqcup \{0\} \sqcup P.$$

Then we write $x < y$ just in case $y - x \in P$. The set P is the set of **positive** elements of the field.

Exercise 3. Prove the equivalence of the definitions.

6 Convexity and bounds

A set A in space is called convex if, for all points P and Q in A, the line segment joining P and Q lies within A.

A subset A of an ordered field K is **convex** if K satisfies

$$\forall x\ \forall y\ \forall z\ (x \in A \land y \in A \land x \leqslant z \leqslant y \to z \in A).$$

An **upper bound** for A is an element b of K such that

$$\forall x\ (x \in A \to x \leqslant b).$$

Likewise, **lower bound**. A set is **bounded** if it has an upper and a lower bound.

Exercise 4. An ordered field has exactly one convex subset that has no upper or lower bound. What is it?

An ordered field has **intervals** of nine kinds:

$(-\infty, b)$	$(-\infty, b]$	$(-\infty, \infty)$
(a, b)	$(a, b]$	(a, ∞)
$[a, b)$	$[a, b]$	$[a, \infty)$

where $a < b$. Here $(-\infty, b) = \{x : x < b\}$, &c; a square bracket means the corresponding end-point is contained in the interval.

Exercise 5. Every interval is convex.

7 Completeness

An ordered field is **complete** if every subset with an upper bound has a least upper bound, also called a **supremum**. If it exists, the supremum of A is denoted

$$\sup A.$$

We postulate that $\mathbb{R}$ is complete.

An **infimum** is a greatest lower bound. The infimum of A, if it exists, is denoted

$$\inf A.$$

Exercise 6. In $\mathbb{R}$, every convex set with at least two members is an interval. (This is not true in $\mathbb{Q}$.)

Exercise 7. Here, P is the set of positive elements of $\mathbb{R}$.

(1) If $ab \in P$, what can you conclude about a and b?

(2) Write P as an interval.

(3) Prove that P is closed under $x \mapsto x^{-1}$.

(4) Let $P[X]$ be the set of [non-zero] polynomials in X whose [non-zero] coefficients are from P. If $f \in P[X]$, show that P is closed under $x \mapsto f(x)$.

(5) If $f \in \mathbb{R}[X]$, and P is closed under $x \mapsto f(x)$, can you conclude that $f \in P[X]$?

Exercise 8. Prove that, in a complete ordered field, all sets with lower bounds have infima.

8 Triangle inequalities

On any ordered field, we define the **absolute-value** function

$$x \mapsto |x|$$

by the rule

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Then

$$\forall x \ \forall y \ (|x| \leqslant y \leftrightarrow -y \leqslant x \leqslant y).$$

Hence the **triangle-inequality**:

$$|x + y| \leqslant |x| + |y|$$

and the variants

$$||x| - |y|| \leqslant |x - y| \leqslant |x| + |y| .$$

Exercise 9. Prove the triangle-inequality (and variants).

9 The natural numbers

The set $\mathbb{N}$ of natural numbers can be defined as the smallest subset of $\mathbb{R}$ that contains 0 and that contains $x+1$ whenever it contains x. (Any intersection of subsets with this property continues to have this property; so $\mathbb{N}$ is the intersection of all subsets of $\mathbb{R}$ with this property.)

Therefore, proof by **induction** works in $\mathbb{N}$: if $A \subseteq \mathbb{N}$, and $0 \in A$, and $x+1 \in A$ whenever $x \in A$, then $A = \mathbb{N}$.

Exercise 10. Prove that $\mathbb{N}$ is **well-ordered**, that is, every non-empty subset of $\mathbb{N}$ has a least element.

The definition of $\mathbb{N}$ works for any ordered field. (Why not a non-ordered field?)

10 Archimedean ordered fields

An ordered field is called **Archimedean** if $\mathbb{N}$ has no upper bound in the field.

Theorem. *$\mathbb{R}$ is Archimedean.*

Proof. Suppose $\mathbb{N}$ has an upper bound. Then $\mathbb{N}$ has a supremum, say a. Then $a-1$ is not an upper bound of $\mathbb{N}$, so some n in $\mathbb{N}$ satisfies

$$a - 1 < n.$$

Hence $a < n+1$, but $n+1 \in \mathbb{N}$, contradicting that a is an upper bound for $\mathbb{N}$. □

Exercise 11. Make $\mathbb{R}(X)$ into a non-Archimedean ordered field such that

$$a \leqslant X$$

for all a in $\mathbb{R}$.

11 The integers

We can define the set $\mathbb{Z}$ of **integers** to be the set

$$\{x \in \mathbb{R} : -x \in \mathbb{N}\} \cup \mathbb{N}.$$

Note that 0 is in both sets. The set $\mathbb{Z}$ is closed under $+$ and $-$ and $\cdot$.

Because $\mathbb{R}$ is Archimedean, we can define the **greatest-integer** function

$$x \mapsto \lfloor x \rfloor$$

by the rule

$$\lfloor x \rfloor = \min\{n \in \mathbb{Z} : x < n+1\}.$$

Then $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$.

12 The rational numbers

The set $\mathbb{Q}$ of **rational** numbers is

$$\{x \in \mathbb{R} : xy \in \mathbb{Z} \text{ for some positive integer } y\}.$$

This is a field. Rational numbers exist in any ordered field.

Lemma. *In an Archimedean ordered field, if*

$$0 \leqslant x \leqslant r$$

for all positive rational numbers r, then $x = 0$.

Proof. Under the assumption, we have $0 \leqslant 1/n$ for all positive integers n. If $x \neq 0$, then $1/x$ is an upper bound for $\mathbb{N}$. □

13 Open and closed sets

A subset A of $\mathbb{R}$ is called **open** if, for every x in A, there is a positive real number ε such that

$$(x - \varepsilon, x + \varepsilon) \subseteq A.$$

A subset of $\mathbb{R}$ is **closed** if its complement is open.

Exercise 12. The intervals $(-\infty, \infty)$, $(-\infty, b)$, (a, b) and (a, ∞) are open. The intervals $(-\infty, \infty)$, $(-\infty, b]$, $[a, b]$ and $[a, \infty)$ are closed. The set $\{a\}$ is closed.

Lemma. *If A is a closed subset of $\mathbb{R}$ with an upper bound, then*

$$\sup A \in A.$$

Proof. Suppose b is an upper bound of A, but $b \notin A$. Since A is closed, the set $\mathbb{R} \setminus A$ is open. Therefore

$$(b - \varepsilon, b + \varepsilon) \subseteq \mathbb{R} \setminus A$$

for some positive ε. Hence $b - \varepsilon$ is an upper bound for A. (Why?

$$A \subseteq (-\infty, b],$$

but also

$$A \subseteq \mathbb{R} \setminus (b - \varepsilon, b + \varepsilon),$$

so

$$A \subseteq (-\infty, b] \setminus (b - \varepsilon, b + \varepsilon) = (-\infty, b - \varepsilon],$$

which means $b - \varepsilon$ is an upper bound for A.) Thus b is not $\sup A$. That is, no upper bound of A that is not in A is $\sup A$; but $\sup A$ is an upper bound of A; therefore $\sup A$ is in A. □

14 Sequences of sets

A **sequence** is a function on $\mathbb{N}$. A function $n \mapsto a_n : \mathbb{N} \to S$ may also be written $(a_n : n \in \mathbb{N})$ or $(a_n)_n$ or just (a_n).

Theorem. *Suppose* $(F_n : n \in \mathbb{N})$ *is a sequence of non-empty bounded closed subsets of* $\mathbb{R}$ *such that*

$$F_{n+1} \subseteq F_n$$

for all n *in* $\mathbb{N}$. *Then*

$$\bigcap_{n\in\mathbb{N}} F_n \neq \varnothing.$$

Proof. Since F_n is bounded and empty, $\sup F_n$ exists; and $\sup F_n \in F_n$ by the Lemma, since F_n is closed. Then the set

$$\{\sup F_n : n \in \mathbb{N}\}$$

is bounded below by any lower bound of F_0. Hence this set has an infimum, say L. I claim that $L \in F_n$ for each n in $\mathbb{N}$. To prove this, since F_n is closed, it is enough to show that

$$F_n \cap (L - \varepsilon, L + \varepsilon) \neq \varnothing$$

whenever $\varepsilon > 0$. But by definition of L, for each positive ε, there is m in $\mathbb{N}$ such that

$$L \leqslant \sup F_m < L + \varepsilon.$$

Let $k = \max\{m, n\}$; then

$$L \leqslant \sup F_k \leqslant \sup F_m < L + \varepsilon$$

and $\sup F_k \in F_k \subseteq F_n$. Therefore $\sup F_k \in F_n \cap (L - \varepsilon, L + \varepsilon)$. □

The sets F_n of the Theorem can be called **nested**.

Example. Let $F_n = \{x \in \mathbb{R} : |x^2 - 2| \leqslant 1/n\}$. Then each F_n is closed, and $F_{n+1} \subseteq F_n$, so

$$\bigcap_{n\in\mathbb{N}} F_n \neq \varnothing.$$

In fact, the intersection is $\{\pm\sqrt{2}\}$.

Exercise 13. The Theorem puts four conditions on the sequence of nested sets F_n: each F_n must

(1) be non-empty,

(2) be closed,

(3) have an upper bound, and

(4) have a lower bound.

Show that each of these conditions is necessary (that is, if any one of them is removed, then the theorem cannot be proved).

Note that the theorem is true in any complete ordered field. The next theorem gives an alternative definition for *complete*.

Theorem. *Suppose* K *is an Archimedean ordered field, and suppose that*

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \varnothing$$

whenever $([a_n, b_n] : n \in \mathbb{N})$ *is a sequence of nested closed intervals of* K. *Then* K *is a complete ordered field.*

Proof. Let C be a subset of K with element a and upper bound b. We shall recursively define a nested sequence $([a_n, b_n] : n \in \mathbb{N})$ in the following way. First,

$$[a_0, b_0] = [a, b].$$

Then $[a_0, b_0]$ contains a member of C and an upper bound of C. Suppose $[a_k, b_k]$ has been defined and contains a point of C and an upper bound of C. Then one of the intervals

$$\left[a_k, \frac{a_k + b_k}{2}\right], \qquad \left[\frac{a_k + b_k}{2}, b_k\right]$$

has the same property (why?). Let $[a_{k+1}, b_{k+1}]$ be this interval. If both intervals have this property, then we can just define

$$[a_{k+1}, b_{k+1}] = \left[a_k, \frac{a_k + b_k}{2}\right].$$

By assumption, there is some c in $\bigcap_{n \in \mathbb{N}} [a_n, b_n]$. Then c is an upper bound of C (why?), and no upper bound of C is less than c (why?). Therefore $c = \sup C$. □

Exercise 14. Supply the missing details in the proof of the last theorem.

15 Different treatments of the real numbers

We have treated $\mathbb{R}$ *axiomatically.* That is, we have written down

- axioms for *fields*,
- additional axioms for *ordered* fields, and finally
- the *completeness* axiom for fields.

Then we have declared that $\mathbb{R}$ is a complete ordered field.

The alternative approach is *constructive.* Here, one starts with $\mathbb{N}$—as given by axioms, namely

- $0 \neq n + 1$ for any n in $\mathbb{N}$,
- $\forall x\, \forall y\, (x + 1 = y + 1 \to x = y)$, and
- for all subsets A of $\mathbb{N}$, if $0 \in A$, and $n + 1 \in A$ whenever $n \in A$, then $A = \mathbb{N}$.

Then, using $\mathbb{N}$, one defines $\mathbb{Z}$ and the Archimedean ordered field $\mathbb{Q}$. Finally, it is possible to define an ordered field-structure on the set of all *convex open subsets* of $\mathbb{Q}$ that have upper but not lower bounds. The result is a complete ordered field, and $\mathbb{R}$ can be *defined* to be this field.

Again, we are taking the axiomatic approach to $\mathbb{R}$, and not the constructive approach. But we shall see presently why the constructive approach works.

A subset A of an ordered field K is called **dense** in K if every interval of K contains an element of A.

Lemma. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. It is enough to show that if $a, b \in \mathbb{R}$ and $a < b$, then $a < c < b$ for some c in $\mathbb{Q}$. But since $a < b$, we have $0 < b - a$. From an earlier lemma, since $\mathbb{R}$ is Archimedean, we conclude that some positive rational number r satisfies $r < b - a$. We may assume (why?) that $r = 2/n$ for some natural number n. Then $nb - na > 2$. Therefore

$$na < \lfloor nb \rfloor - 1 < \lfloor nb \rfloor \leqslant nb$$

(why?). Finally, $a < (\lfloor nb \rfloor - 1)/n < b$, and $(\lfloor nb \rfloor - 1)/n \in \mathbb{Q}$. □

Theorem. *There is a one-to-one correspondence between:*

- *real numbers, and*
- *convex open subsets of* $\mathbb{Q}$ *that are bounded above, but not below.*

Proof. The correspondence is given by the function

$$x \mapsto \{r \in \mathbb{Q} : r < x\}$$

(why?). □

Exercise 15. Supply the details in the proof of the last theorem.

(If A is a convex subset of $\mathbb{Q}$ with an upper but not a lower bound, then the pair $(A, \mathbb{Q} \setminus A)$ is called a *cut*. It was Dedekind who first defined the real numbers in terms of cuts.)

16 Topology

Topology is the study of open and closed sets as such.

Theorem. $\mathbb{R}$ *and* $\varnothing$ *are open subsets of* $\mathbb{R}$*. If* A *and* B *are open subsets of* $\mathbb{R}$*, then so is* $A \cap B$*. If* $\{U_i : i \in I\}$ *is a family of open subsets of* $\mathbb{R}$*, then the union*

$$\bigcup_{i \in I} U_i$$

is open.

Proof. It is trivial that $\mathbb{R}$ and $\varnothing$ are open. If A and B are open, and $x \in A \cap B$, then $x \in A$ and $x \in B$, so there are positive reals ε_A and ε_B such that

$$(x - \varepsilon_S, x + \varepsilon_S) \subseteq S$$

when $S \in \{A, B\}$. Let $\varepsilon = \min\{\varepsilon_A, \varepsilon_B\}$; then

$$(x - \varepsilon, x + \varepsilon) \subseteq A \cap B.$$

Thus $A \cap B$ is open. Finally, if U_i is open for *each* i in the index-set I, and $x \in \bigcup_{i \in I} U_i$, then $x \in U_i$ for *some* i in I, so $(x - \varepsilon, x + \varepsilon) \subseteq U_i$, and therefore

$$(x - \varepsilon, x + \varepsilon) \subseteq \bigcup_{i \in I} U_i.$$

Therefore $\bigcup_{i \in I} U_i$ is open. □

Exercise 16. Prove that:

- $\mathbb{R}$ and $\varnothing$ are closed subsets of $\mathbb{R}$;

- if A and B are closed subsets of $\mathbb{R}$, then so is $A \cup B$;
- if $\{F_i : i \in I\}$ is a family of open subsets of $\mathbb{R}$, then the union
$$\bigcap_{i \in I} F_i$$
is open.

An arbitrary *intersection* of open sets may not be open:

Example. $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n+1}, \frac{1}{n+1}\right) = \{0\}.$

Exercise 17. Show that the only subsets of $\mathbb{R}$ that are both open and closed are $\varnothing$ and $\mathbb{R}$ itself.

(The exercise shows that $\mathbb{R}$ is *connected*.)

17 The Cantor set

By definition, a non-empty open subset of $\mathbb{R}$ includes an open interval. Does a non-empty closed set include a closed interval?

No. We can obtain a closed set that has no subsets that are intervals in the following way. Start with a closed interval. Remove an open interval, so two closed intervals remain. Remove an open interval from each of these, and continue. In the end, no interval can remain.

More precisely, we can proceed thus. If a and b are integers, we write
$$a \equiv b \pmod{3}$$
if $a - b$ is a multiple of 3. For n in $\mathbb{N}$, let U_n be the set
$$\{x \in \mathbb{R} : \lfloor 3^n x \rfloor < 3^n x \wedge \lfloor 3^n x \rfloor \equiv 1 \pmod{3}\}.$$
Then let F be the set
$$[0,1] \setminus \bigcup_{n \in \mathbb{N}} U_n.$$
This set F is called the **Cantor set**.

Exercise 18. The Cantor set is closed, but none of its subsets is an interval.

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