MATH 303 EXAMINATION SOLUTIONS (DRAFT)

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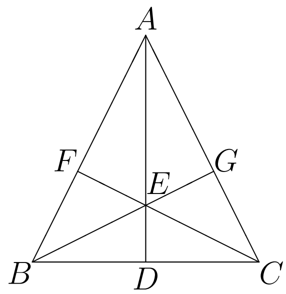
Note: There are no new diagrams in these solutions, simply because creating them electronically is too time-consuming.

Problem 1. What is wrong with the following proof that all triangles are isosceles?—:

1. Let a triangle be given, namely ABC.
2. Let BC be bisected at D.
3. Let a straight line, DE, be drawn at right angles to BC.
4. Let also the straight line AE bisect the angle BAC.
5. Let the straight lines BE and CE be drawn.
6. $BE = CE$.
7. Let the straight line EF be drawn perpendicular to AB.
8. Let the straight line EG be drawn perpendicular to AC.
9. $AF = AG$ and $EF = EG$.
10. $BF = CG$.
11. $AF + FB = AG + GC$.
12. $AF + FB = AB$ and $AG + GC = AC$.
13. $AB = AC$; in particular, ABC is isosceles.



Solution. Step 12 is not justified. In fact, if $AB > AC$, then $AF + FB = AB$, but $AC + GC = AG$.

Remark. 1. The diagram is misleading; but (contrary to what some people seemed to think) the proof never assumes that AED or BEG or CEF is a straight line.

2. Step 4 may *appear* unjustified; however, steps 2, 3, and 4 together say simply that the bisector of angle BAC and the perpendicular bisector of BC meet at E. This style of writing can be seen for example in Euclid's Proposition I.44.

3. The proof does wrongly assume that E lies within the triangle; but the proof can easily be adjusted to the case where E lies outside the triangle. Euclid usually does not bother to consider all possible cases: we noted this for example in Proposition I.7. The real problem is the assumption that either both F and G lie on the triangle, or both lie below the triangle.

Problem 2. Write English translations of the following words:

(a) θεώρημα, (b) πρόβλημα, (c) ἀνάλυσις, (d) συνθέσις, (e) πολύγωνον, (f) τρίγωνον.

Solution. Theorem, problem, analysis, synthesis, polygon, triangle.

Remark. 1. A *transliteration* of the words into English (or Latin) letters would be *theorêma, problêma, analysis, synthesis, polygônon, trigônon,* but this is not what was asked.

Date: Friday, November 6, 2009.

2. The first two words on the list have been discussed in class; these (along with the next two) are also discussed in some notes that I put on the web.

3. The last two words on the list derive from *γωνία angle,* which is apparently related to *γόνυ*; this word shares its meaning, and an Indo-European ancestor, with the English *knee.* (Here is a point where English spelling is useful; if *knee* were spelled phonetically, then its relation with *γόνυ* could not be seen.)

4. As a translation of *τρίγωνον*, I do find the word *trigon* in the Oxford English Dictionary; but the more usual word is of course *triangle.*

Problem 3. Write the letters of the Greek alphabet in the standard order. Write only the capital letters *or* only the minuscule letters.

Solution.

$$A\ B\ \varGamma\ \varDelta\ E\ Z\ H\ \varTheta\ I\ K\ \varLambda\ M\ N\ \varXi\ O\ \varPi\ P\ \varSigma\ T\ \varUpsilon\ \varPhi\ X\ \varPsi\ \varOmega$$

or

$$\alpha\ \beta\ \gamma\ \delta\ \epsilon\ \zeta\ \eta\ \theta\ \iota\ \kappa\ \lambda\ \mu\ \nu\ \xi\ o\ \pi\ \rho\ \sigma\ \tau\ \upsilon\ \phi\ \chi\ \psi\ \omega.$$

Problem 4. Proclus writes:

> Every problem and every theorem that is furnished with all its parts should contain the following elements:
> (1) an *enunciation* (πρότασις),
> (2) an *exposition* (or *setting out:* ἔκθεσις),
> (3) a *specification* (or *definition of goal:* διορισμός),
> (4) a *construction* (κατασκευή),
> (5) a *proof* (ἀπόδειξις), and
> (6) a *conclusion* (συμπέρασμα).

Below is the enunciation (in Heath's translation) of Proposition I.6 of Euclid's *Elements.* Supply the remaining parts (in your own words, which may or may not be Euclid's).

> *If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.*

Solution. 1. (As above, namely:) If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

2. Let ABC be a triangle in which angles ABC and ACB are equal.

3. We shall show that $AB = AC$.

4. On BA, extended if necessary, let BD be cut off equal to CA.

5. Then triangle DBC is equal to ACB, and therefore D must coincide with A. Consequently, $BA = CA$.

6. Thus we have shown that, if in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Remark. Euclid's proof is a *reductio ad absurdum,* that is, a proof by contradiction. In particular, Euclid first assumes $AB \neq AC$ and *then* finds D. In this case, to which of Proclus's six parts does the hypothesis $AB \neq AC$ belong? I don't know whether Proclus considers this question.

Problem 5. *Without* using Euclid's method of 'application', prove Proposition I.8 of the *Elements,* whose enunciation is, *If two triangles have two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal whch are contained by the equal straight lines.*

Solution. Suppose ABC and DEF are triangles such that $AB = DE$, $BC = EF$, and $AC = DF$. We shall show that angles ABC and DEF are equal. To this end, let AG be dropped perpendicular to BC, extended if necessary [by I.12]. On EF, extended if necessary, cut off EH equal to BG [by I.3]. Erect HK perpendicular to EF [by I.11] and equal to AG [by I.3 again]. Then $EK = AB$ and angles KEF and ABC are equal [by I.4], and similarly, since $HF = GC$, we have $FK = CA$. Hence $EK = ED$ and $FK = FD$. Therefore K and D coincide [by I.7], and in particular, angles DEF and ABC are equal.

Now, we have used two propositions [namely I.11 and 12] that Euclid proves by means of I.8. However, alternative proofs are as follows.

If A does not lie on the straight line BC, then by drawing a circle with center A that cuts the line, we may assume B and C have been chosen so that $AB = AC$. Draw an equilateral triangle BCD (on the opposite side of BC from A) [by I.1]. Draw the straight line AD, which cuts BC at a point E. Then angles BAD and CAD are equal [by I.5 and 4], and therefore angles AEB and AEC are equal [again by I.4], so the latter angles are right. Therefore AE has been dropped perpendicular to AB.

If A does lie on BC, we may still assume $AB = AC$. Draw an equilateral triangle BCD and straight line AD. Then angles BAD and CAD are equal [by I.5 and 4], so they are right. Thus AD has been erected perpendicular to BC.

Remark. 1. It is not necessary to name the propositions used.

2. Some people argued by contradiction that if (in the notation above) angle ABC is greater than DEF, then BC must be greater than EF. This is Proposition I.24; but I.24 relies on I.23, which in turn relies on I.8. It is not clear to me that there is a way to prove I.24 without first proving I.8.

3. One person suggested an interesting argument that I understand as follows. If angle ABC is greater than DEF, then inside the former angle, there must be an angle ABG equal to DEF. We may then assume $BG = BC = EF$. But then $GA = FA$ [by I.4], so we have violated I.7, which is absurd; therefore $ABC = DEF$. Now, if this argument is valid, then what is the point of I.3? If straight line AB is greater than straight line C, why does Euclid not declare that there must be a part of AB, namely AE, that is equal to C? Why does Euclid feel the need to *construct* AE?

Problem 6. In triangle ABC, suppose BC is bisected at D, and straight line AD is drawn. Assuming AB is greater than AC, prove that angle BAD is less than DAC.

Solution. Extend AD to E so that $DE = DA$. Then angles DEC and DAB are equal, and $CE = BA$ [by I.4]. But then angle CAE is greater than CEA [by I.18], so $CAD > DAB$.

Remark. I think the argument just given is the best of several variants that were found by different people. The argument I had thought originally of was more complicated: Since angle BDA must be greater than ADC, inside angle BDA we can construct angle ADE equal to ADC, with $DE = DC$. Then BE is parallel to AD [why?], so E lies outside triangle ABD. Therefore angle BAD is less than EAD; but the latter is equal to DAC.

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