MATH 304, 2009/10, SECOND EXAMINATION *SOLUTIONS*

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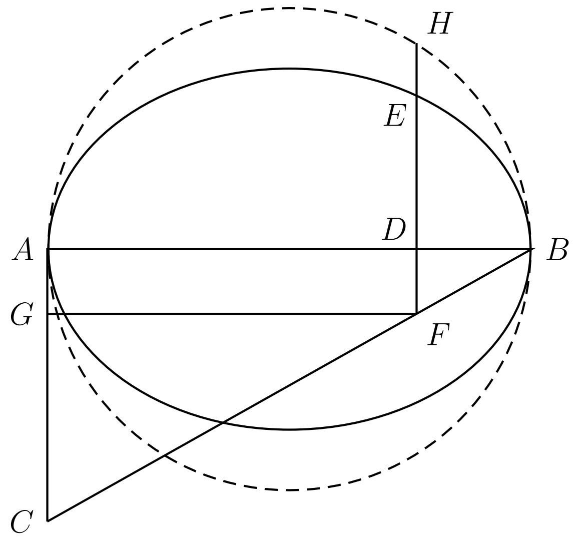
Problem 1. *The ellipse AEB is determined as follows. Triangle ABC is given, the angle at A being right. If a point D is chosen at random on AB, and DE is erected at right angles to AB, then E lies on the ellipse if (and only if) the square on DE is equal to the rectangle $ADFG$ (which is formed by letting ED, extended as necessary, meet BC at F). Let also the circle AHB with diameter AB be given.*

Find h (in terms of the given straight lines) such that h is to AB as the ellipse is to the circle. Prove that your answer is correct, using Newton's lemmas as needed.



Remark. The ellipse appears to result from contracting the circle in one direction. If this is so, then by Newton's Lemma 4, the ratio of ellipse to circle is the factor of contraction, which should be DE/DH. So one should find this ratio and check that it is indeed independent of the choice of D.

Two students solved this problem perfectly. Five others used without proof a rule for the area of an ellipse; but we do not officially have such a rule, and in fact the point of this problem is to establish this rule.

Solution. By construction and the similarity of the triangles BDF and BAC,

$$DE^2 = ADFG = AD \times DF = AD \times DB \times \frac{AC}{AB}.$$

In the circle,

$$DH^2 = AD \times DB.$$

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Let h be a mean proportional of AB and AC, so

$$h^2 = AB \times AC, \qquad \frac{AC}{AB} = \frac{h^2}{AB^2}.$$

Then

$$\frac{DE^2}{DH^2} = \frac{AC}{AB}, \qquad \frac{DE}{DH} = \frac{h}{AB}.$$

If we inscribe series of parallelograms in the ellipse and circle, all of the same breadth, then corresponding parallelograms will be to each other as DE to DH, that is, h to AB. Therefore this is the ratio of the ellipse to the circle [by Newton's Lemma 4].

Problem 2. *We have used without proof Propositions I.33 and 49 of the* Conics *of Apollonius. This problem is an opportunity to prove those propositions, using the techniques of Descartes and Newton as appropriate.*

A straight line ℓ (not shown), a curved line ABE, and a straight line AC are given such that, whenever a point B is chosen at random on ABE, and straight line BC is dropped perpendicular to AC, then the square on BC is equal to the rectangle bounded by ℓ and AC. So ABE is a parabola with axis AC.

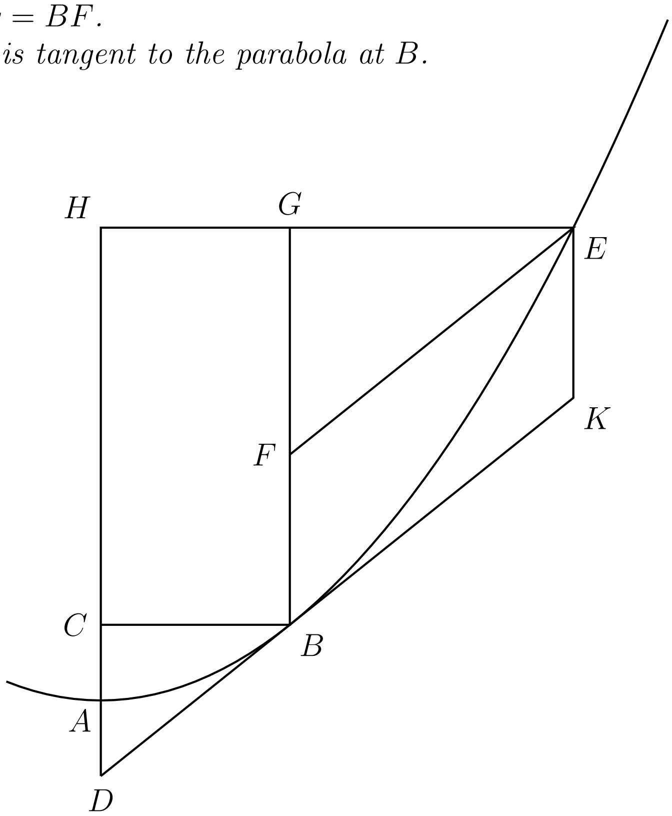
Let B now be fixed; so we may write $BC = a$ and $AC = b$. Extend CA to D so that $AD = AC$. Draw straight line DBK, and let $c = BD$.

Let a point E be chosen at random on the parabola ABE. Draw straight lines BF parallel to AC, and EF parallel to BD.

(a). *Show that the parabola ABE must indeed lie all on one side of DBK.*

(b). *Show that the square on EF varies as BF, and find m (in terms of a, b, and c only) such that $m \times BF$ is equal to the square on EF. For your computations, let $x = EF$ and $y = BF$.*

(c). *Explain why BD is tangent to the parabola at B.*



Remark. One approach to (a) is showing that E lies above K. The height of E above D is the length of DH; by similarity of triangles, the height of K above D is $2b/a$ times EH. The point of using DH and EH is that we know how their lengths are related. Two students solved this problem perfectly; one other was partially successful.

In (b), we want to find x^2/y in terms of fixed magnitudes. We have one equation, $EH^2 = \ell \times AH$, and we can write this in terms of x and y (and fixed magnitudes) by using the similar triangles BCD and EGF. Three students solved this problem completely; two others got halfway there.

For (c), one student showed that DB is the only straight line passing through B and meeting AD that meets the parabola exactly once. A number of students observed that DB does meet the parabola just once; but this is not enough to establish that DB is a tangent. Note also that BG also meets the parabola exactly once, but is not a tangent.

Solution. (a). Assuming KE is parallel to AC, drop a perpendicular KL to AC. We want to show $DH \geqslant DL$ or $AH \geqslant AL$. We have

$$AH = \frac{EH^2}{\ell}, \qquad DL = LK \times \frac{2b}{a} = EH \times \frac{2b}{a},$$

so

$$\ell \times (DH - DL) = EH^2 + b\ell - EH \times \frac{2b\ell}{a} = EH^2 + a^2 - EH \times 2a = (EH - a)^2,$$

which is positive when E is not B; so $DH > DL$.

(b). We have $EH^2 = \ell \times AH$. Since $EG = ax/c$ and $GF = 2bx/c$, this means

$$\Bigl(a + \frac{ax}{c}\Bigr)^2 = \ell\Bigl(y + \frac{2bx}{c} + b\Bigr),$$
$$a^2 + \frac{2a^2x}{c} + \frac{a^2x^2}{c^2} = \ell y + \frac{2b\ell x}{c} + b\ell,$$

and since $a^2 = \ell b$, we have

$$\frac{a^2x^2}{c^2} = \ell y, \qquad x^2 = \frac{c^2}{a^2}\ell y, \qquad m = \frac{c^2}{b}.$$

(c). In the figure, as E approaches B, EK varies as BK^2. Therefore EK/BK varies as BK, so the angle EBK ultimately vanishes.

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