MATH 304 FINAL EXAMINATION

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Write your solutions on separate sheets; you may keep *the* problem sheets. There are two numbered problems (with several lettered parts each) and a bonus. *İyi çalışmalar; kolay gelsin.*

Problem 1. This problem is about the cubic equations

$$x^3 + 3x^2 = 6x + 17, \tag{$*$}$$

$$t^3 = 9t + 9. \tag{$\dagger$}$$

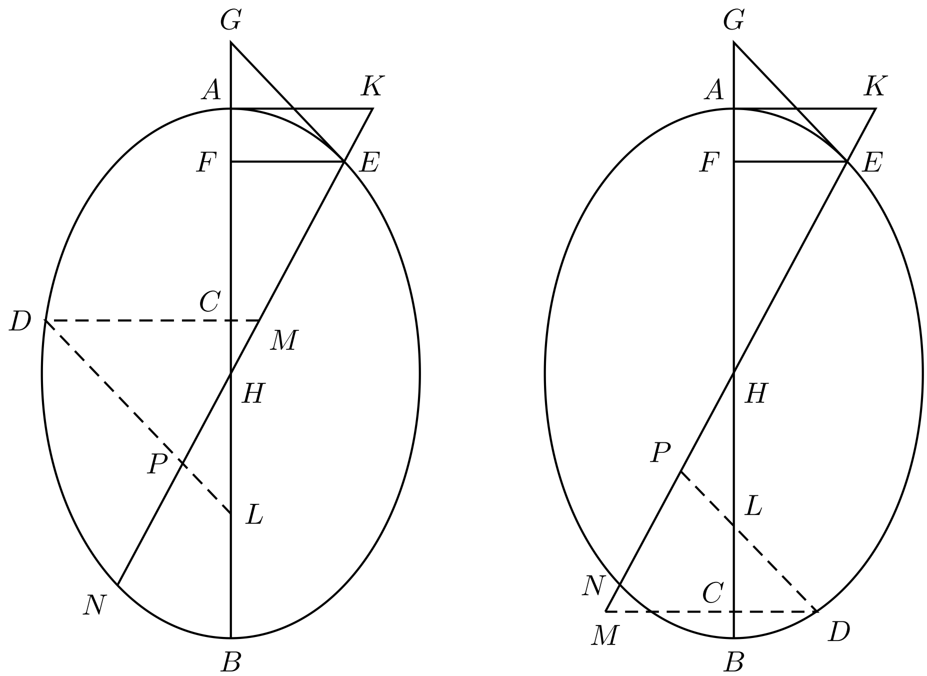
A. Explain the relation between the solutions of $(*)$ and $(\dagger)$.

B. For one of $(*)$ and $(\dagger)$, find a solution geometrically, by intersecting conic sections (as Omar Khayyam does).

C. Find *three* solutions in the same way (some might be negative).

D. Find a solution of $(*)$ or $(\dagger)$ numerically (in the manner suggested by Cardano); your steps should be justifiable. Your answer will involve square roots of negative numbers.

Problem 2. This problem shows that every line through the center of an ellipse is a diameter with certain properties. The method is based on Apollonius; but the algebraic geometry of Descartes makes some simplifications possible.



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Straight line AB is given, and angle BAK is given. The point C moves along AB, and as it moves, straight line CD remains parallel to AK. But D moves along DC as C moves, so that D traces out a curvilinear figure ADB, as shown above with two possible positions of DC.

Recall that the curvilinear figure ADB is an **ellipse** with **diameter** AB and **ordinates** parallel to AK if and only if

$$CD^2 \propto AC \times CB \tag{‡}$$

(that is, the square on CD varies as the rectangle formed by AC and CB).

Let E be chosen at random on ADB, and let straight line EF be drawn parallel to KA, meeting AB at F. Let straight line EG be drawn, meeting BA extended at G so that

$$\frac{AG}{GB} = \frac{AF}{FB}. \tag{§}$$

Let H be the midpoint of AB, and let straight line HE be drawn and extended to meet AK at K. Let L be taken on AB (extended if necessary) so that straight line DL is parallel to GE. Finally, let M be the point of intersection of DC and HK (both extended if necessary).

For computations, let

$$AH = b, \qquad EF = c, \qquad HF = d, \qquad CD = x, \qquad CH = y.$$

Also, let a be such that

$$\frac{a^2}{b^2} = \frac{EF^2}{AF \times FB} = \frac{c^2}{b^2 - d^2}. \tag{¶}$$

A. Show that (‡) holds if and only if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{‖}$$

B. Find HG in terms of b and d.

C. Show that (‡) holds if and only if

$$\triangle CDL = \triangle AHK - \triangle CHM. \tag{$**$}$$

(Angle BAK is not assumed to be a right angle; but the computations can be performed as if it were.)

D. Assuming (‡) holds (and hence ($**$) holds, for *all* possibilities for C), show

$$\triangle AHK = \triangle GHE.$$

E. Assume (‡) holds. Let EH be extended to meet the ellipse again at N, and let EN meet DL (extended as necessary) at P. Show that the curvilinear figure ADB is an ellipse with diameter EN whose ordinates are parallel to EG. (You will probably want to use part **C**, translated appropriately.)

Bonus. What are your suggestions for improving the course?

Geldiğiniz için teşekkürler. İyi tatiller!

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