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IV. Algebra

The earliest extant textbook on algebra is the *Compendium on Calculation by Completion and Reduction*, by Muḥammad ibn-Mūsā al-Khwārizmī. This work was dedicated to the Caliph al-Ma'mūn, who ruled from 813 to 833 and established in Baghdad a research center called the House of Wisdom. We include several excerpts from this text, including parts of the preface, the introduction, the methods of solution of two types of mixed quadratic equations, the geometric justification for one of the solution methods, a discussion of multiplication of algebraic expressions of degree one, and some problems on legacies from the last section of the work.

Al-Khwārizmī's Compendium on Calculation by Completion and Reduction

From the preface

That fondness for science, by which God has distinguished the Imam al-Mamun, the Commander of the Faithful (besides the caliphate which He has vouchsafed unto him by lawful succession, in the robe of which He has invested him, and with the honors of which He has adorned him), that affability and condescension which he shows to the learned, that promptitude with which he protects and supports them in the elucidation of obscurities and in the removal of difficulties—has encouraged me to compose a short work on Calculating by (the rules of) Completion and Reduction, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, lawsuits, and trade, and in all their dealing with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned—relying on the goodness of my intention therein, and hoping that the learned will reward it, by obtaining (for me) through their prayers the excellence of the Divine mercy; in requital of which, may the choicest blessings and the abundant bounty of God be theirs! My confidence rests with God, in this as in every thing, and in Him I put my trust. He is the Lord of the Sublime Throne. May His blessing descend upon all the prophets and heavenly messengers!

[26]The Arabic is "add it to it to it."

Methods for solving equations

When I considered what people generally want in calculating, I found that it always is a number. I also observed that every number is composed of units, and that any number may be divided into units. Moreover, I found that every number, which may be expressed from one to ten, surpasses the preceding by one unit; afterwards the ten is doubled or tripled, just as before the units were; thus arise twenty, thirty, etc., until a hundred; then the hundred is doubled and tripled in the same manner as the units and the tens, up to a thousand; then the thousand can be thus repeated at any complex number; and so forth to the utmost limit of numeration.

I observed that the numbers which are required in calculating by Completion and Reduction are of three kinds, namely, roots, squares, and simple numbers relative to neither root nor square.

A root is any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending. A square is the whole amount of the root multiplied by itself. A simple number is any number which may be pronounced without reference to root or square.

A number belonging to one of these three classes may be equal to a number of another class; you may say, for instance, "squares are equal to roots," or "squares are equal to numbers," or "roots are equal to numbers."

Of the case in which *squares are equal to roots*, this is an example. "Square is equal to five roots of the same"; the root of the square is five, and the square is twenty-five, which is equal to five times its root.

So you say, "one third of the square is equal to four roots"; then the whole square is equal to twelve roots; that is a hundred and forty-four; and its root is twelve.

Or you say, "five squares are equal to ten roots"; then one square is equal to two roots; the root of the square is two, and its square is four.

In this manner, whether the squares be many or few (i.e., multiplied or divided by any number), they are reduced to a single square; and the same is done with the roots, which are their equivalents; that is to say, they are reduced in the same proportion as the squares.

[...]

I found that these three kinds, namely, roots, squares, and numbers, may be combined together, and thus three compound species arise; that is, "squares and roots equal to numbers"; "squares and numbers equal to roots"; "roots and numbers equal to squares."

Roots and Squares are equal to Numbers; for instance, "one square, and ten roots of the same, amount to thirty-nine dirhams"; that is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: you halve the number of the roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine.

The solution is the same when two squares or three, or more or less are specified; you reduce them to one single square, and in the same proportion you reduce also the roots and simple numbers which are connected therewith.

For instance, "two squares and ten roots are equal to forty-eight dirhams"; that is to say, what must be the amount of two squares which, when summed up and added to ten times the root of one of them, make up a sum of forty-eight dirhams? You must at first reduce the two squares to one; and you know that one square of the two is the half of both. Then reduce everything mentioned in the statement to its half, and it will be the same as if the question had been, a square and five roots of the same are equal to twenty-four dirhams; or, what must be the amount of a square which, when added to five times its root, is equal to twenty-four dirhams? Now halve the number of the roots; the half is two and a half. Multiply that by itself; the product is six and a quarter. Add this to twenty-four; the sum is thirty dirhams and a quarter. Take the root of this; it is five and a half. Subtract from this the half of the number of the roots, that is two and a half; the remainder is three. This is the root of the square, and the square itself is nine.

The proceeding will be the same if the instance be, "half of a square and five roots are equal to twenty-eight dirhams"; that is to say, what must be the amount of a square, the half of which, when added to the equivalent of five of its roots, is equal to twenty-eight dirhams? Your first business must be to complete your square, so that it amounts to one whole square. This you effect by doubling it. Therefore, double it, and double also that which is added to it, as well as what is equal to it. Then you have a square and ten roots, equal to fifty-six dirhams. Now halve the roots; the half is five. Multiply this by itself; the product is twenty-five. Add this to fifty-six; the sum is eighty-one. Extract the root of this; it is nine. Subtract from this the half of the number of roots, which is five; the remainder is four. This is the root of the square which you sought for; the square is sixteen, and half the square eight.

Squares and Numbers are equal to Roots; for instance, "a square and twenty-one in numbers are equal to ten roots of the same square." That is to say, what must be the amount of a square, which, when twenty-one dirhams are added to it, becomes equal to the equivalent of ten roots of that square? Solution: Halve the number of the roots; the half is five. Multiply this by itself; the product is twenty-five. Subtract from this the twenty-one which are connected with the square; the remainder is four. Extract its root; it is two. Subtract this from the half of the roots, which is five; the remainder is three. This is the root of the square which you required, and the square is nine. Or you may add the root to the half of the roots; the sum is seven; this is the root of the square which you sought for, and the square itself is forty-nine.

When you meet with an instance which refers you to this case, try its solution by addition, and if that does not serve, then subtraction certainly will. For in this case both addition and subtraction may be employed, which will not answer in any other of the three cases in which the number of the roots must be halved. And know, that, when in a question belonging to this case you have halved the number of the roots and multiplied the half by itself, if the product be less than the number of dirhams connected with the square, then the instance is impossible; but if the product be equal to the dirhams by themselves, then the root of the square is equal to the half of the roots alone, without either addition or subtraction.

In every instance where you have two squares, or more or less, reduce them to one entire square, as I have explained under the first case.

[...]

Geometrical justification

When a square plus twenty-one dirhams are equal to ten roots, we depict the square as a square surface *AD* of unknown sides. Then we join it to a parallelogram, *HB*, whose width, *HN*, is equal to one of the sides of *AD* [see fig. 5.2]. The length of the two surfaces together is equal to the side *HC*. We know its length to be ten numbers since every square has equal sides and angles, and if one of its sides is multiplied by one, this gives the root of the surface, and if by two, two of its roots. When it is declared that the square plus twenty-one equals ten of its roots, we know that the length of the side *HC* equals ten numbers because the side *CD* is a root of the square figure. We divide the line *CH* into two halves by the point *G*. Then you know that line *HG* equals line *GC*, and that line *GT* equals line *CD*. Then we extend line *GT* a distance equal to the difference between line *CG* and line *GT* to make the quadrilateral. The line *TK* equals line *KM* making a quadrilateral *MT* of equal sides and angles. We know that the line *TK* and the other sides equal five. Its surface is twenty-five obtained by the multiplication of one-half the roots by itself, or five by five equals twenty-five. We know that the surface *HB* is the twenty-one that is added to the square. From the surface *HB*, we cut off a piece by line *TK*, one of the sides of the surface *MT*, leaving the surface *TA*. We take from the line *KM* line *KL* which is equal to line *GK*. We know that line *TG* equals line *ML* and that line *LK* cut from line *MK* equals line *KG*. Then the surface *MR* equals surface *TA*. We know that surface *HT* plus surface *MR* equals surface *HB*, or twenty-one. But surface *MT* is twenty-five. And so, we subtract from surface *MT*, surface *HT* and surface *MR*, both [together] equal to twenty-one. We have remaining a small surface *RK*, or twenty-five less twenty-one or 4. Its root, line *RG*, is equal to line *GA*, or two. If we subtract it from line *CG*, which is one-half the roots, there remains line *AC* or three. This is the root of the first square. If it is added to line *GC* which is one-half the roots, it comes to seven, or line *RC*, the root of a larger square. If twenty-one is added to it, the result is ten of its roots. This is the figure:

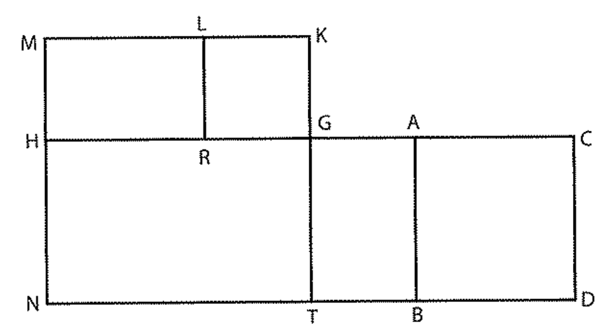

FIGURE 5.2

On multiplication

I shall now teach you how to multiply the unknown numbers, that is to say, the roots, one by the other, if they stand alone, or if numbers are added to them, or if numbers are subtracted from them, or if they are subtracted from numbers; also how to add them one to the other, or how to subtract one from the other.

Whenever one number is to be multiplied by another, the one must be repeated as many times as the other contains units.

If there are greater numbers combined with units to be added to or subtracted from them, then four multiplications are necessary; namely, the greater numbers by the greater numbers, the greater numbers by the units, the units by the greater numbers, and the units by the units.

If the units, combined with the greater numbers, are positive, then the last multiplication is positive; if they are both negative, then the fourth multiplication is likewise positive. But if one of them is positive, and one negative, then the fourth multiplication is negative.

For instance, "ten and one to be multiplied by ten and two." Ten times ten is a hundred; once ten is ten positive; twice ten is twenty positive, and once two is two positive; this altogether makes a hundred and thirty-two.

But if the instance is "ten less one, to be multiplied by ten less one," then ten times ten is a hundred; the negative one by ten is ten negative; the other negative one by ten is likewise ten negative, so that it becomes eighty; but the negative one by the negative one is one positive, and this makes the result eighty-one.

Or if the instance be "ten and two, to be multiplied by ten less one," then ten times ten is a hundred, and the negative one by ten is ten negative; the positive two by ten is twenty positive; this together is a hundred and ten; the positive two by the negative one gives two negative. This makes the product a hundred and eight.

I have explained this, that it might serve as an introduction to the multiplication of unknown sums, when numbers are added to them, or when numbers are subtracted from them, or when they are subtracted from numbers.

If the instance be "ten minus thing to be multiplied by ten minus thing," then ten times ten is a hundred; and minus thing by ten is minus ten things; and again, minus thing by ten is minus ten things. But minus thing multiplied by minus thing is a positive square. The product is therefore a hundred and a square, minus twenty things.

In like manner if the following question be proposed to you: "one dirham minus one-sixth to be multiplied by one dirham minus one-sixth"; that is to say, five sixths by themselves, the product is five and twenty parts of a dirham, which is divided into six and thirty parts, or two-thirds and one-sixth of a sixth. Computation: You multiply one dirham by one dirham, the product is one dirham; then one dirham by minus one-sixth, that is one-sixth negative; then, again, one dirham by minus one-sixth is one-sixth negative; so far, then, the result is two-thirds of a dirham; but there is still minus one-sixth to be multiplied by minus one-sixth, which is one-sixth of a sixth positive; the product is, therefore, two-thirds and one sixth of a sixth.

On legacies

On Capital, and Money lent: A man dies, leaving two sons behind him, and bequeathing one-third of his capital to a stranger. He leaves ten dirhams of property and a claim of ten dirhams upon one of the sons.

Computation: You call the sum which is taken out of the debt thing. Add this to the capital, which is ten dirhams. The sum is ten and thing. Subtract one-third of

this, since he has bequeathed one-third of his property, that is, three dirhams and one-third plus one-third of thing. The remainder is six dirhams and two-thirds plus two-thirds of thing. Divide this between the two sons. The portion of each of them is three dirhams and one-third plus one-third of thing. This is equal to the thing which was sought for. Reduce it, by removing one-third from thing, on account of the other third of thing. There remain two-thirds of thing, equal to three dirhams and one-third. It is then only required that you complete the thing, by adding to it as much as one half of the same; accordingly, you add to three and one-third as much as one-half of them. This gives five dirhams, which is the thing that is taken out of the debts.

On another Species of Legacy: A man dies, leaving his mother, his wife, and two brothers and two sisters by the same father and mother with himself; and he bequeaths to a stranger one-ninth of his capital.

Computation: You constitute their shares by taking them out of forty-eight parts. You know that if you take one-ninth from any capital, eight-ninths of it will remain. Add now to the eight-ninths one-eighth of the same, and to the forty-eight also one-eighth of them, namely, six, in order to complete your capital. This gives fifty-four. The person to whom one-ninth is bequeathed receives six out of this, being one-ninth of the whole capital. The remaining forty-eight will be distributed among the heirs, proportionally to their legal shares.

Thābit ibn Qurra (836–901) was from Ḥarrān, in northern Mesopotamia. According to one account, the Banū Mūsā met him as a money changer in his hometown, but were so impressed with his linguistic capabilities that they brought him back to Baghdad to work with them in the House of Wisdom. Thābit's goal in this passage is to show how the procedures of the algebraists (such as al-Khwārizmī) for solving quadratic equations can be represented geometrically and justified by an appeal to Propositions II, 5 and II, 6 of Euclid's *Elements*. Book II of the *Elements* was a fundamental part of the toolkit of Greek and Islamic mathematics. It begins with ten theorems, each asserting that if a straight line is divided into two or more segments in certain ways then certain relationships hold between rectangles and squares formed from the line and these segments. And it ends with applications of these theorems to four advanced geometrical problems.

Particularly important are Propositions 5 and 6 of Book II, which deal with a straight line bisected at one point and (in the case of 5) divided arbitrarily at another point or (in the case of 6) extended by an arbitrary straight line. How these are used in Euclidean geometry is well illustrated by Euclid's use of Proposition 6 to prove (in Prop. 11) a crucial lemma for the construction of a regular pentagon and his use of Prop. 5 to construct (in Prop. 14) a square equal to a given polygon. Thābit's treatise extends the usefulness of these theorems by showing how they may be used to justify the known procedures for solving quadratic equations, a concept not found in the *Elements*.

The order in which Thābit lists the three basic forms of quadratic equations that have both square and linear terms is the same as the order found in al-Khwārizmī's *Algebra*. It is interesting, also, that in the case of $x^2 + c = bx$ he notes, as did al-Khwārizmī, that there are two solutions. Because of the interest in the exact terminology used in the early history of algebra we have tried to make the translation fairly literal. Since the first two proofs show how the two propositions of Euclid referred to above are used, we translate only them and the statement of the third type of equation. We have translated the following excerpt from the German translation by P. Luckey of the Arabic text in the manuscript Aya Sofya 2457,3[27] which we have also consulted in a number of places.

Thābit ibn Qurra on the geometric proof of the correctness of procedures for solving quadratic equations

In the name of God, the Merciful, the Compassionate! A letter by Abū al-Ḥasan Thābit ibn Qurra on the verification of problems in algebra by geometric proofs.

The basic forms, to which most problems of algebra are reducible, are three.

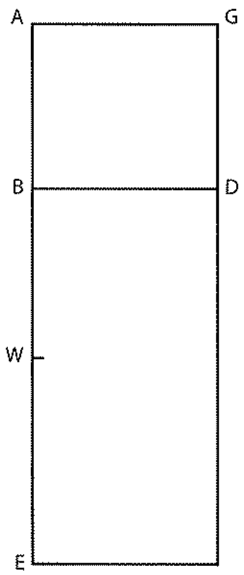

FIGURE 5.3

(1) The first basic form is *māl* and roots[28] equal a number. The way and means of the solution of same, by the sixth [proposition] of Book II of Euclid's book [*The Elements*], is as I now describe: We make [fig. 5.3] the *māl* into the square *ABGD*, we make in *BE* (an amount) of the multiples of the unit by which the lines are measured like the given amount of roots contained,[29] and we complete the area *DE*.

Then the root is obviously *AB* since the *māl* is the square *ABGD*, and this (squaring) is in the domain of the calculation. And the number is like the product of *AB* in the unit by which the lines are measured. Therefore the product of *AB* in the unit by which the lines are measured, is the root on the side [in the domain] of calculation and number. Now however in *BE* is [contained an amount] of these units like the given amount of roots; therefore, the product *AB* in *BE* equals the roots of the problem in the domain of calculation and number. But the product *AB* in *BE* is the area *DE*, since *AB* is like *BD*. Thus, in this way, the area *DE* is equal to the roots of the problem, so the whole area, *GE*, is like the *māl* together with the roots. Now, however, the *māl* together with the roots is like a known number. Thus the area *GE* is known and it is like the product *EA* in *AB*, since *AB* is like *AG*.

Thus the product *EA* in *AB* is known, and the line *BE* is known since the number of its units is known. Thus the matter is reduced to a given geometric problem, to wit: The line *BE* is known. To it is added *AB*, and thereby the product *EA* in *AB* is known. Now it was proved in proposition six of Book II of the *Elements* that, if the line *BE* is halved at the point *W*, the product *EA* in *AB*, together with the square of *BW*, is like the square of *AW*. But the product *EA* in *AB* is known and the square of *BW* is known. Thus, the

[27] We have compared Luckey's translation with the Arabic and have made a number of changes, not because Luckey's translation is wrong but simply because, due to the interest in the vocabulary of early algebra texts, we have decided to translate more literally than Luckey does in some places. Specifically we have translated one standard Arabic expression for "the product of *A* by *B*" as "the product of *A* in *B*," and we have translated *mithl* as "the like of" rather than "equal," or as "its like" rather than "itself."

[28] *Māl* in this context refers to the square of the root. (One ordinary meaning of this Arabic word is "asset," and Luckey translated the Arabic word *māl* by the German word for "asset.") "Roots" refers to what we write as *bx*. When Thābit wants to speak of just the coefficient *b* he refers to "the number of roots."

[29] I.e., make *BE* equal to as many units of length, as there are units in the coefficient of the linear term.

square of *AW* is known, and so *AW* is known.[30] And if *BW*, which is known, is subtracted from it, then *AB* is left as known, and that is the root. And if we multiply it in itself, then the square *ABGD* is known, i.e. the *māl*, and that is what we wanted to show.

This procedure agrees with the procedure of the algebraists in solving this problem. Namely, their taking one half of the number of the roots, is as if we take half of the line *BE*. That they multiply it in itself, is just as if we take the square of half of the line *BE*. That they add the number to the result obtained is like our adding the product *EA* in *AB*. So that, out of all this, the square of the sum of *AB* and half of the line [*BW* = *BE*/2] are put together. That they take the root of the result is as if we say: The sum of *AB* and half the line [*BE*] is known when its square is known.[31] That they subtract from this <half the number of the roots, so that they obtain the remainder, namely the root, is as if we take away half of *BE*>[32] so that the remainder results, as *AB* resulted for us. They multiply it in its like, and thus they determine the *māl*, [just] as we determined from *AB* its square, and that is the *māl*.[33]

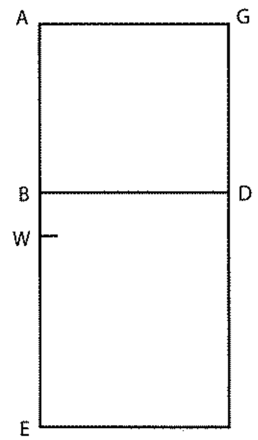

FIGURE 5.4

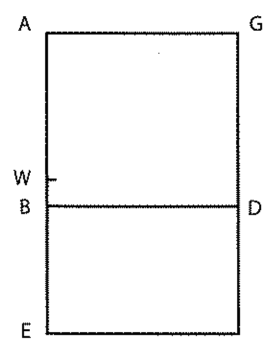

FIGURE 5.5

(2) The second fundamental form is that *māl* and number are equal to roots. The guiding precept in solving that is from the fifth [theorem] of Book II of the writings of Euclid, as I describe it: We make [see figs. 5.4 and 5.5] the *māl* the square *ABGD* and make (it so that there is) in *AE* [a number of] the multiples of the unit, by which the lines will be measured, equal to the given number of roots.

Then obviously *AE* is longer than *AB*, since the roots, which in the domain of calculation are the product *GA* in *AE*, are bigger than the *māl*.[34] We complete the area *GE* and prove, as was said, that it equals the roots according to the ways of the calculation. And when *BG* [the *māl*] is subtracted from it, *DE*, equal to the number, remains. Thus *DE* is known, and it is like the product *AB* in *BE*. And the line *AE* is [also] known. So the matter now amounts to the fact that the line *AE* is known and is divided at *B* so that the product *AB* in *BE* is known. Now it is proven in the fifth [proposition] of Book II of the writing of Euclid, that, if *AE* is halved at *W*, then the product *AB* in *BE* together with the square of *BW* is like the square on *AW*. But *AW* is known, and its square is known, and the product *AB* in *BE* is known. So the square of *BW* is known, and hence *BW* is known. And if it is (fig. 5.4) subtracted from *AW* or (fig. 5.5) added to it, *AB* results as known. And that is the root. And if we multiply it in its like, then *ABGD*, i.e., the *māl*, is known. And that is what we wanted to prove.

[30] This is a consequence of Prop. 55 of Euclid's *Data*.

[31] The Arabic says the sum is known "if it is a known square." The reference, again, is to Euclid's *Data*, Prop. 55.

[32] The words within the pointed brackets are based on Luckey's restoration of a portion of the text that has been destroyed.

[33] Computing the square of the unknown was part of the solution of the problem in Thābit's time.

[34] Since the roots are equal to *māl* plus a number.

This procedure too agrees with the procedure of the algebraists in the calculation of this problem. Therefore this allows both types of procedure, the [use of] augmentation and deduction in regards to the line *BD*.

(3) The third fundamental form (is): Number and roots are equal to *māl*.

Abū Kāmil ibn Aslam (c. 850–930), "the Egyptian calculator," used justifications similar to those of Thābit ibn Qurra in his own algebra text, a text similar to that of al-Khwārizmī but with many more complicated examples and geometrical applications. Of the two excerpts we present here, the first asks for three positive numbers x, y, and z so that $x+y+z=10$, $z^2=x^2+y^2$, and $xz=y^2$. To solve these he uses an ancient method known as "false position," in which he provisionally sets $x=1$ and, with this assumption, solves the last two equations for provisional values y and z. He then compares the value of the sum of the provisional values with the desired value, 10. Because the three equations are homogeneous he can now find the desired values by multiplying each of the provisional values by 10 divided by their sum.

The first half of the translated text sets up the problem and finds an expression for the value of the least of the three unknowns. The second half, beginning with the words "But, we know," works out the value of this unknown in simpler terms. A sometimes faulty English translation of the whole solution can be found in [Levey 1966, 186–192]. (Levey translated from a Hebrew version of the work by the fifteenth-century Jewish scholar Mordecai Finzi of Mantua.)

Our second problem from Abū Kāmil asks for the computation of the side of an equilateral pentagon inscribed in a square (in such a way that the one of the vertices of the pentagon coincides with a vertex of the square). Considerably later in the tenth century Abū Sahl al-Kūhī, in a selection we give in our section on geometry, constructed such a pentagon, but one with none of its vertices on a vertex of the square.

We have translated both problems from the Arabic text of Abū Kāmil's *Algebra* as found in a facsimile edition of the Arabic text from the Beyazid Library in Istanbul (Qara Mustafa Pasha, No. 379). In the first excerpt, Abū Kāmil's methods are sufficiently clear and his achievement sufficiently impressive from the first part of the solution, where he finds the value of the smallest of the three unknowns, that we have translated only that part. An exposition of Abū Kāmil's solution in modern mathematical symbols may be found in [Berggren 1986, 110–111].[35]

Abū Kāmil on the solution of three equations in three unknowns

And if it is said to you that you will divide ten *dirhams* into three parts so that you will multiply the smallest by its like and the middle by its like and it [the result of putting the two together] is like the largest [multiplied] by its like. And you multiply the smallest by the largest and it will be the like of the middle multiplied by its like. So, you omit for now [the condition about] the ten *dirhams*. Then we say that when you are faced with three different quantities [such that] if you multiply the smallest by its like and the middle by its like, it [the sum] is the like of [the product of] the biggest by its like, and when you multiply the smallest by the biggest it is the like of the middle [multiplied] by its like, then:

35Where it is wrongly stated that Abū Kāmil's work is a commentary on the *Algebra* of al-Khwārizmī.

Its rule [for solution] is that we make the smallest [quantity] a *dirham*, the middle a thing, and the largest a square (since, when you multiply the smallest by the largest, it is like the product of the middle by its like). Then we multiply the smallest by its like and the middle by its like and we put them together, so that it (the problem) will be: A square and one *dirham* is equal to a square square (which is like the product of the largest by itself).

And so we do as I described to you and the square will be half a *dirham* and the root of one and a fourth, and it is the largest [of the three quantities]. And the middle quantity is the root of this, and it is the root of the sum of half a *dirham* and the root of one and a fourth.[36] And the small quantity is a *dirham*. And you add the three quantities together and it is a *dirham* and a half and the root of one and a fourth and the root of the sum of half a *dirham* and the root of one and a fourth.

Then you return to [the condition about] the 10 *dirhams*.[37] And we say: We divide 10 *dirhams* by 1½, and the root of 1¼, and the root of the sum of ½ *dirham* and the root of 1¼. And so there results thing.[38] But, we know that whenever we multiply the quotient by the divisor the dividend results. And so we multiply thing by 1½ *dirhams* and the root of 1¼ and the root of the sum of ½ *dirham* and the root of 1¼. And so 1½ things and the root of 1¼ *māl*[39] and the root of the sum of ½ *māl* and the root of 1¼ *māl māl* will be equal to ten *dirhams*.

So subtract 1½ things and the root of 1¼ *māl* from 10 *dirhams* and there remain 10 *dirhams* less 1½ things[40] and less the root of 1¼ *māl* equal to the root of the sum of ½ *māl* and the root of 1¼ *māl māl*. And so multiply 10 *dirhams* less 1½ things[41] and less the root of 1¼ *māl* by its like, so 100 *dirhams* and 3½ *māl* and the root of 11¼ *māl māl* less 30 things and less the root of 500 *māl* will be equal to ½ *māl* and the root of 1¼ *māl māl*. And so complete the 100 *dirhams* by 30 things and the root of 500 *māl* and add the like to ½ *māl* and the root of *māl māl* and ¼ of *māl māl*. And throw away the half *māl* from the 3½ *māl*, and throw away the root of 1¼ of *māl māl* from the root of 11¼ *māl māl*. And so there remain 100 *dirhams* and 3 *māl* and the root of 5 *māl māl*, equal to 30 things and the root of 500 *māl*.

And so refer each of your terms to a [single] *māl*,[42] i.e., we multiply it by ¾ less the root of ¼ and ½ of 1/8. And so you multiply everything you have by ¾ less the root of ¼ and ½ of 1/8. And so *māl* and 75 *dirhams* less the root of 3125 *dirhams* will be equal to 10 things. Thus, halve the things, so it will be 5, and multiply it by its like, and it will be twenty-five. And throw away from it the 75 less the root of 3125. So there remains the root of 3125 less 50. And so the root of that is subtracted from 5, and what remains is the smallest of the three parts whose sum is 10 *dirhams*.

And if you want to know the largest part then make the largest of the three quantities a *dirham*, the smallest a thing, and the middle the root of the thing. Etc.

36We have modernized the text here and at other appearances of the same phrase. The Arabic could be more literally translated "it is half a *dirham* and the root of one and a fourth, its root taken."

37From now on we replace the text's verbal form of the numbers by our usual ciphers.

38This is now the value of the "thing" for the original problem.

39In this context *māl* refers to the square of the thing.

40The text says "one thing and a half."

41Again, text says "a thing and a half."

42 We would say, "Multiply your equation by a number that makes the coefficient of x^2 equal to 1."