Exam 1 solutions

Math 320, David Pierce

April 7, 2011

[Instructions given with exam:]

- *This examination assumes the axioms of Equality, Null Set, Adjunction, Separation, Replacement, Union, and Infinity.*
- *Proofs are not required, unless they are explicitly asked for.*
- *In proofs, you may use any theorem that we know, unless you are being asked to prove that theorem.*
- *All problems have equal weight.*

Problem 1. *Let a and b be sets.*

a) Write down a formula that defines the class denoted by $a \times b$. If you use any symbols other than a, b, $\in$, $=$, and logical symbols, you should define them.

b) Prove that $a \times b$ is a set.

Solution.

a) Such a formula is

$$\exists x\ \exists y\ (z = (x, y) \wedge x \in a \wedge y \in b),$$

where:

- $z = (x, y)$ stands for $z = \{\{x\}, \{x, y\}\}$,
- $z = \{u, v\}$ stands for $\forall x\ (x \in z \Leftrightarrow x = u \vee x = v)$,
- $x = \{u\}$ stands for $\forall y\ (y \in x \Leftrightarrow y = u)$.

b) By the Null Set and Adjunction axioms, ordered pairs are sets. Therefore, for each c in a, there is a well-defined function

$$y \mapsto (c, y)$$

on b. The image of b under this function is the class $\{c\} \times b$; this class is a set, by the Replacement Axiom. Therefore there is a well-defined function

$$x \mapsto \{x\} \times b$$

on a. The image of a under this function is the class

$$\{\{x\} \times b \colon x \in a\};$$

this is a set, again by Replacement. By the Union Axiom, the class

$$\bigcup\{\{x\} \times b \colon x \in a\}$$

is a set; but this class is just $a \times b$.

Remark. This problem was Exercise 18; it is also Theorem 74 of the notes. For example, if $a = 3 = \{0, 1, 2\}$, then

$$a \times b = (\{0\} \times b) \cup (\{1\} \times b) \cup (\{2\} \times b) = \bigcup\{\{k\} \times b \colon k \in 3\}.$$

Problem 2. *Write down:*

a) A transitive set that is not an ordinal.

b) A set that is well-ordered by membership, but is not an ordinal.

Solution.

a) $\{0, \{0\}, \{\{0\}\}\}$.

b) $\{\{0\}\}$.

Remark. There are many possible answers; those given are probably the simplest. One can approach this problem as follows:

a) Start with a set a that is not an ordinal, then find the smallest set b that contains a and is transitive. The simplest set that is not an ordinal is $\{1\}$, that is, $\{\{0\}\}$; let this be a. Then $a \in b$, so we must also have $a \subseteq b$, which means $1 \in b$. So $\{a, 1\} \subseteq b$. But since $1 \in b$, we must have $1 \subseteq b$, that is, $0 \in b$. So $\{a, 1, 0\} \subseteq b$. We are done: the set $\{a, 1, 0\}$, is now transitive, but it is not an ordinal, since a is not an ordinal.

b) Every set of ordinals is well-ordered by membership. So take a set of ordinals that is not an ordinal. A set of *one* ordinal is enough, as long as that ordinal is not 0.

Problem 3. *Either prove or give a counterexample:*

a) Every set of ordinals has a supremum.

b) Every class of ordinals has a supremum.

Solution.

a) Let a be a set of ordinals. Then its supremum is $\bigcup a$: we prove this as follows.

First, $\bigcup a$ is an ordinal. For, each ordinal is a set of ordinals, so $\bigcup a$ is a set of ordinals, and therefore it is well-ordered by membership. Moreover, if $\alpha \in \bigcup a$, then $\alpha \in \beta$ for some β in a, so $\alpha \subset \beta$, but also $\beta \subseteq \bigcup a$, so $\alpha \subset \bigcup a$. Thus $\bigcup a$ is also transitive. Therefore it is an ordinal.

Now, if $\alpha \in a$, then $\alpha \subseteq \bigcup a$. Thus $\bigcup a$ is an upper bound of a. If β is an upper bound, then for all α in a, we have $\alpha \subseteq \beta$; but this shows $\bigcup a \subseteq \beta$. Thus $\bigcup a$ is the least upper bound of a.

b) The class **ON** itself has no supremum, since it is closed under $x \mapsto x'$, and $x \in x'$.

Remark. The offered solution uses implicitly the theorem that, on **ON**, the relations $\in$ and $\subset$ are the same (and are the relation by which **ON** is well-ordered). Part (a) is really Theorem 69 of the notes.

Problem 4.

a) Find a set of successor ordinals whose supremum is a limit ordinal.

b) Prove that there is no set of limit ordinals whose union is a successor ordinal.

Solution.

a) $\omega = \sup\{n + 1 \colon n \in \omega\}$.

b) Say a is a set of limit ordinals, and let $\beta = \sup(a)$. If $\beta \in a$, it is a limit. Say $\beta \notin a$. Then for all α, if $\alpha < \beta$, then $\alpha < \gamma < \beta$ for some γ in a, and then $\alpha' \leqslant \gamma < \beta$. Thus β is still a limit, or 0.

Problem 5. *Prove or disprove:*

a) $k + n = n + k$ for all natural numbers k and n.

b) $\alpha + \beta = \beta + \alpha$ for all ordinals α and β.

Solution.

a) The statement is true. To prove it, we shall use the definition of addition on ω:

$$k + 0 = k, \qquad\qquad k + n' = (k + n)'.$$

We first show $0 + k$ by induction:
i) $0 + 0 = 0$ by definition of $+$.
ii) If $0 + k = k$, then

$$\begin{aligned} 0 + k' &= (0 + k)' && \text{[by definition of +]} \\ &= k' && \text{[by inductive hypothesis].} \end{aligned}$$

Next, we show $n' + k = (n + k)'$ by induction:
i) $n' + 0 = n' = (n + 0)'$.
ii) If $n' + k = (n + k)'$, then

$$\begin{aligned} n' + k' &= (n' + k)' && \text{[by definition of +]} \\ &= (n + k)'' && \text{[by inductive hypothesis]} \\ &= (n + k')' && \text{[by definition of +].} \end{aligned}$$

Now we can prove the original claim by induction:
i) $n + 0 = n = 0 + n$.
ii) If $n + k = k + n$, then

$$\begin{aligned} n + k' &= (n + k)' \\ &= (k + n)' && \text{[by inductive hypothesis]} \\ &= k' + n. \end{aligned}$$

b) The statement is false:

$$\begin{aligned} 1 + \omega &= \sup\{1 + n \colon n \in \omega\} \\ &= \sup\{n + 1 \colon n \in \omega]\} \\ &= \omega \\ &\neq \omega + 1. \end{aligned}$$

Remark. In part (a), it was not strictly required to prove the preliminary lemmas, since it is permitted to assume Lemma 7 of the notes. What is to be proved in part (a) is Theorem 31 of the notes; and doing this was Exercise 8.