Second Examination *solutions*

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Problem 1. *Write the following ordinals in Cantor normal form (that is, in base ω). All exponents should themselves be in normal form. (* Exception: *The exponent* 1 *and the coefficient* 1 *need not be written. In strict normal form, ω should be written as ω^{ω^0} or even $\omega^{\omega^0 \cdot 1} \cdot 1$; but this is not required.)*

Solution.

a) $\omega + 2$ [already in normal form]

b) $2 + \omega = \omega$

c) $\omega^2 + \omega$ [already in normal form]

d) $\omega + \omega^2 = \omega^2$

e) $(\omega + 2) \cdot 2 = \omega \cdot 2 + 2$

f) $2 \cdot (\omega + 2) = 2 \cdot \omega + 2 \cdot 2 = \omega + 4$

g) $(\omega + 2) \cdot \omega = \omega \cdot \omega = \omega^2$

h) $(\omega + 2) \cdot (\omega + 2) = (\omega + 2) \cdot \omega + (\omega + 2) \cdot 2 = \omega^2 + \omega \cdot 2 + 2$ [by (g) and (e)]

i) $(\omega + 2)^2 = \omega^2 + \omega \cdot 2 + 2$ [by (h)]

j) $2^{\omega+2} = 2^\omega \cdot 2^2 = \omega \cdot 4$

k) $(\omega + 2)^\omega = \omega^\omega$

l) $(\omega + 2)^{\omega+2} = (\omega + 2)^\omega \cdot (\omega + 2)^2 = \omega^\omega \cdot (\omega^2 + \omega \cdot 2 + 2) = \omega^{\omega+2} + \omega^{\omega+1} \cdot 2 + \omega^\omega \cdot 2$ [by (k) and (i)]

m) $(\omega^{\omega^\omega})^{\omega^{\omega^\omega}} = \omega^{\omega^\omega \cdot \omega^{\omega^\omega}} = \omega^{\omega^{\omega + \omega^\omega}} = \omega^{\omega^{\omega^\omega}}$

n) $(\omega^{\omega^{\omega^\omega}})^{\omega^{\omega^\omega}} = \omega^{\omega^{\omega^\omega} \cdot \omega^{\omega^\omega}} = \omega^{\omega^{\omega^\omega + \omega^\omega}} = \omega^{\omega^{\omega^\omega \cdot 2}}$

o) $2^{\omega^2} = (2^\omega)^\omega = \omega^\omega$

Remark. It is essential to distinguish between $2 + \omega$ (which is ω) and $\omega + 2$ (which is not). One cannot do anything with ordinals without having internalized this distinction (made it a part of oneself).

Some people misremembered various rules of arithmetic, or forgot the special conditions under which they apply. I don't try to memorize them, myself; I figure out what they must be, and I play with them, so that I develop a feeling for *why* they should be true.

Problem 2.

a) State the recursive definition of ordinal addition.

b) Prove from this definition that $0 + \alpha = \alpha$ for all α.

Solution.

a) $\alpha + 0 = \alpha$
$\alpha + \beta' = (\alpha + \beta)'$
$\alpha + \gamma = \sup\{\alpha + x \colon x < \gamma\}$ if γ is a limit

b) $0 + 0 = 0$.
If $0 + \alpha = \alpha$, then $0 + \alpha' = (0 + \alpha)' = \alpha'$.
If β is a limit, and $0 + \alpha = \alpha$ whenever $\alpha < \beta$, then

$$0 + \beta = \sup_{\alpha<\beta}(0 + \alpha) = \sup_{\alpha<\beta} \alpha = \beta.$$

Problem 3. *Prove that*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

for all ordinals. Use only the recursive *definitions of* $+$ *and* $\cdot$. *You may use the normality of the functions* $x \mapsto \alpha + x$ *and* $x \mapsto \beta \cdot x$ *where* $\beta > 0$, *and you may use the theorem that makes normality useful (as the instructions above suggest).*

Solution. We use induction on γ.

i) $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + 0 = \alpha \cdot \beta + \alpha \cdot 0$.

ii) If $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$, then

$$\begin{aligned}\alpha \cdot (\beta + \gamma') &= \alpha \cdot (\beta + \gamma)' \\ &= \alpha \cdot (\beta + \gamma) + \alpha \\ &= (\alpha \cdot \beta + \alpha \cdot \gamma) + \alpha \\ &= \alpha \cdot \beta + (\alpha \cdot \gamma + \alpha) \\ &= \alpha \cdot \beta + \alpha \cdot \gamma'.\end{aligned}$$

iii) If δ is a limit, and $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ whenever $\gamma < \delta$, then

$$\begin{aligned}\alpha \cdot (\beta + \delta) = \alpha \cdot \sup_{\gamma<\delta}(\beta + \gamma) &= \sup_{\gamma<\delta}(\alpha \cdot (\beta + \gamma)) \\ &= \sup_{\gamma<\delta}(\alpha \cdot \beta + \alpha \cdot \gamma) \\ &= \alpha \cdot \beta + \sup_{\gamma<\delta}(\alpha \cdot \gamma) = \alpha \cdot \beta + \alpha \cdot \delta\end{aligned}$$

Remark. The induction must be on γ; nothing else works. With ordinals, all of the action that we know about from definitions happens on the *right*.

Problem 4. *Assuming $\alpha > 1$, prove that the function $x \mapsto \alpha^x$ on* **ON** *is normal.*

Solution. By definition, if β is a limit, then $\alpha^\beta = \sup\{\alpha^x \colon x < \beta\}$. Therefore it remains to show

$$\beta < \gamma \Rightarrow \alpha^\beta < \alpha^\gamma.$$

We prove this for all β, by induction on γ.

i) The claim is vacuously true when $\gamma = 0$ [since it is never true that $\beta < 0$].

ii) Suppose the claim is true when $\gamma = \delta$. If $\beta < \delta'$, then $\beta \leqslant \delta$, so

$$\alpha^\beta \leqslant \alpha^\delta < \alpha^\delta \cdot \alpha = \alpha^{\delta'}.$$

iii) Suppose δ is a limit, and the claim holds when $\gamma < \delta$. If now $\beta < \delta$, then $\beta < \beta' < \delta$, so

$$\alpha^\beta < \alpha^{\beta'} \leqslant \sup_{\gamma<\delta} \alpha^\gamma = \alpha^\delta.$$

Scores:

	EA	PC	AF	Mİ	MM	OŞ	NT	ÖT
1	10	13	8	2	7	13	13	9
2	2	5	5	2	0	5	5	4
3	0	5	5	0	0	0	5	0
4	2	0	2	0	0	4	2	2
	14	23	20	4	7	22	25	15