Third Examination *solutions*

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Instructions. Write solutions on separate sheets; you may keep *this* sheet. In Problem 3, do not assume the Axiom of Choice. Problem 1 is worth 15 points; the other three problems are worh 5 points each.

Problem 1. *For each of the following sets, write its cardinality as* $\aleph_\alpha$ *or* $\beth_\alpha$ *for some ordinal* α*. All operations involving numbers* $\aleph_\alpha$ *or* $\beth_\alpha$ *are cardinal operations; all operations involving* ω *are ordinal operations.*

a) ω

b) ω^ω

c) $\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$

d) the set of countable ordinals

e) $\mathbb{R}$

f) ${}^\omega\mathbb{R}$

g) the set of uncountable subsets of $\mathbb{R}$

h) $\aleph_{\omega^\omega} + \aleph_\omega$

i) $\aleph_\omega + \aleph_{\omega^\omega}$

j) $\aleph_{\omega^\omega} \cdot \aleph_\omega$

k) ${\aleph_0}^{\aleph_0}$

l) $\sup\{\aleph_0, {\aleph_0}^{\aleph_0}, {\aleph_0}^{{\aleph_0}^{\aleph_0}}, \dots\}$

m) $\aleph_{\omega^2\cdot 3+\omega}{}^{\aleph_{\omega^\omega}}$

n) ${\beth_{\omega+1}}^{\beth_\omega}$

o) $\mathscr{P}(\beth_\omega)$

Solution.

a) $\omega = \aleph_0$ [it is already a cardinal, so the cardinality of ω is itself, which is $\aleph_0$]

b) ω^ω has cardinality $\aleph_0$ [remember that ω^ω is the *ordinal* power; see §5.4 of the notes]

c) $\sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\} = \bigcup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$, the union of a nonempty countable set of countably infinite sets [by part (b)], so its cardinality is $\aleph_0$ [this is a special case of Theorem 123 of the notes]

d) the set of countable ordinals is exactly the first uncountable ordinal, which is therefore itself a cardinal, namely $\aleph_1$

e) $\mathbb{R}$ has cardinality $\beth_1$ [since $\mathbb{R} \approx \mathscr{P}(\omega) \approx {}^{\omega}2 \approx 2^{\aleph_0}$ (the cardinal power), which is $\beth_1$ by definition]

f) $\mathrm{card}({}^{\omega}\mathbb{R}) = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \beth_1$

g) The set of uncountable subsets of $\mathbb{R}$ has cardinality $\beth_2$. [Denote the set of uncountable subsets of $\mathbb{R}$ by a. Then $\mathscr{P}(\mathbb{R}) \smallsetminus a$ is the set b of countable subsets of $\mathbb{R}$. Since $b \preccurlyeq {}^{\omega}\mathbb{R}$, we have $\mathrm{card}(b) \leqslant \beth_1$ by part (f). Hence

$$\begin{aligned}\beth_2 = \mathrm{card}(\mathscr{P}(\mathbb{R})) = \mathrm{card}(a \cup b) &\leqslant \mathrm{card}(a) + \mathrm{card}(b)\\ &\leqslant \mathrm{card}(a) + \beth_1 = \max(\mathrm{card}(a), \beth_1).\end{aligned}$$

Since $\beth_1 < \beth_2$, we must have $\beth_2 \leqslant \mathrm{card}(a)$. But also

$$\mathrm{card}(a) \leqslant \mathrm{card}(\mathscr{P}(\mathbb{R})) = \beth_2.$$

Therefore $\mathrm{card}(a) = \beth_2$. Similarly, whenever $c \subset d$ and $\mathrm{card}(c) < \mathrm{card}(d)$, but d is infinite, then $\mathrm{card}(d \smallsetminus c) = \mathrm{card}(d)$.]

h) $\aleph_{\omega^\omega} + \aleph_\omega = \aleph_{\omega^\omega}$ [the greater of the two alephs]

i) $\aleph_\omega + \aleph_{\omega^\omega} = \aleph_{\omega^\omega}$ [as in part (i)]

j) $\aleph_{\omega^\omega} \cdot \aleph_\omega = \aleph_{\omega^\omega}$ [as in parts (i) and (j)]

k) ${\aleph_0}^{\aleph_0} = 2^{\aleph_0} = \beth_1$ [since $2^{\aleph_0} \leqslant {\aleph_0}^{\aleph_0} \leqslant (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$; see Exercise 30]

l) $\sup\{\aleph_0, {\aleph_0}^{\aleph_0}, {\aleph_0}^{{\aleph_0}^{\aleph_0}}, \dots\} = \sup\{\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots\}$ [as in (k)], and

$$\sup\{\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \dots\} = \sup\{\beth_0, \beth_1, \beth_2, \dots\} = \beth_\omega$$

[by definition of the beths; compare Exercise 35]

m) ${\aleph_{\omega^2 \cdot 3+\omega}}^{\aleph_{\omega^\omega}} = 2^{\aleph_{\omega^\omega}}$ [as in (k), since $2 \leqslant \omega^2 \cdot 3 + \omega \leqslant \omega^\omega$; the cardinal $2^{\aleph_{\omega^\omega}}$ is $\aleph_\alpha$ for some unknown α, but I do not know whether it is $\beth_\beta$ for any β; see Exercise 34; note that the given cardinal must be understood as κ^λ, where $\kappa = \aleph_{\omega^2 \cdot 3+\omega}$ and $\lambda = \aleph_{\omega^\omega}$]

n) ${\beth_{\omega+1}}^{\beth_\omega} = (2^{\beth_\omega})^{\beth_\omega} = 2^{\beth_\omega \cdot \beth_\omega} = 2^{\beth_\omega} = \beth_{\omega+1}$

o) $\mathrm{card}(\mathscr{P}(\beth_\omega)) = 2^{\beth_\omega} = \beth_{\omega+1}$

Remark. Five of the exercises involved the beths (the numbers $\beth_\alpha$).

Problem 2.

a) Write the definitions of the cardinal product $\kappa \cdot \lambda$ and the cardinal power κ^λ.

b) Show that $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

Solution.

a) $\kappa \cdot \lambda = \operatorname{card}(\kappa \times \lambda)$ and $\kappa^\lambda = \operatorname{card}({}^\lambda\kappa)$.

b) [*Short version:*] There is a bijection from ${}^\mu({}^\lambda\kappa)$ to ${}^{\lambda\times\mu}\kappa$, namely the function that converts a function f on μ (where $f(\alpha)$ is a function $x \mapsto f(\alpha)(x)$ from λ to κ for all α in μ) to the function
$$(x, y) \mapsto f(y)(x).$$
[*Long version:*] We show there is a bijection Φ from ${}^\mu({}^\lambda\kappa)$ to ${}^{\lambda\times\mu}\kappa$. An element of ${}^\mu({}^\lambda\kappa)$ is a function f from μ to ${}^\lambda\kappa$. In particular, if $\alpha \in \mu$, then $f(\alpha)$ is a function from λ to κ. We can denote this function by
$$x \mapsto f(\alpha)(x).$$
We can convert f into a function $\Phi(f)$ from $\lambda \times \mu$ into κ by defining
$$\Phi(f)(x, y) = f(y)(x).$$
Then Φ is the desired bijection. Indeed, we can define a function Ψ from ${}^{\lambda\times\mu}\kappa$ to ${}^\mu({}^\lambda\kappa)$ so that, if $g \in {}^{\lambda\times\mu}\kappa$, and $\alpha \in \mu$, then $\Psi(g)(\alpha)$ is the function
$$x \mapsto g(x, \alpha)$$
from λ to κ. Then Ψ is the inverse of Φ, since
$$\Phi(\Psi(g))(x, y) = \Psi(g)(y)(x) = g(x, y),$$
so $\Phi(\Psi(g)) = g$, and
$$\Psi(\Phi(f))(y)(x) = \Phi(f)(x, y) = f(y)(x),$$
so $\Psi(\Phi(f)) = f$.

Remark. Part (b) was part of Exercise 25.

Problem 3. *Let a be some nonempty set.*

a) What is a choice-function *for a?*

b) Define a set b such that every subset of b that is linearly ordered by $\subset$ has an upper bound in b, and every maximal element of b (with respect to $\subset$) is a choice-function for a.

Solution.

a) A choice-function for a is a function from $\mathscr{P}(a) \smallsetminus \{0\}$ (or $\mathscr{P}(a)$) to a such that
$$f(x) \in x$$
for all nonempty subsets x of a.

b) Let b be the set of functions f such that the domain of f is a subset of $\mathscr{P}(a) \smallsetminus \{0\}$ and $f(x) \in x$ whenever x is a nonempty subset of a. In particular, if the domain of f is all of $\mathscr{P}(a) \smallsetminus \{0\}$, then f is a choice-function for a. Suppose $g \in b$, but the domain of g is a proper subset of $\mathscr{P}(a) \smallsetminus \{0\}$. Then some element c of $\mathscr{P}(a) \smallsetminus \{0\}$ is not in the domain of g. Then c has an element d, and therefore $g \cup \{(c, d)\}$ is an element of b that is greater (with respect to $\subset$) than g; so g is not maximal. Thus a maximal element of b must have domain $\mathscr{P}(a) \smallsetminus \{0\}$ and therefore be a choice-function for a.

Remark. Part (b) was part of Exercise 24.

Problem 4. *Recall the definition*

$$\mathbf{R}(0) = 0, \qquad \mathbf{R}(\alpha + 1) = \mathbf{R}(\alpha), \qquad \mathbf{R}(\beta) = \bigcup\{\mathbf{R}(x) \colon x \in \beta\},$$

where β is a limit. Show that, for every subset a of $\bigcup \mathbf{R}[\mathbf{ON}]$, there is

a) α such that $a \subseteq \mathbf{R}(\alpha)$,

b) β such that $a \in \mathbf{R}(\beta)$.

Solution. The definition of $\mathbf{R}$ is given incorrectly in the statement of the problem. [This was my mistake. The correct definition had been given in the exercises, just before Exercise 28.] Under the given incorrect definition, $\mathbf{R}(\alpha) = 0$ for all α, so that $\bigcup \mathbf{R}[\mathbf{ON}] = 0$. The only subset of this is 0, and this is a subset of each $\mathbf{R}(\alpha)$, but it is not an element of any $\mathbf{R}(\alpha)$, since they are all empty. [This would have been an acceptable answer.]

Under the correct definition, $\mathbf{R}(\alpha + 1) = \mathscr{P}(\mathbf{R}(\alpha))$. Then the problem can be solved as follows.

a) Note that $\bigcup \mathbf{R}[\mathbf{ON}] = \bigcup\{\mathbf{R}(\alpha) \colon \alpha \in \mathbf{ON}\}$. If b is a member of this, let $f(b)$ be the least ordinal α such that $b \in \mathbf{R}(\alpha)$. Let $\gamma = \sup\{f(x) \colon x \in a\}$. Then

$$a \subseteq \bigcup\{\mathbf{R}(\delta) \colon \delta \leqslant \gamma\} \subseteq \bigcup\{\mathbf{R}(\delta) \colon \delta < \gamma + \omega\} = \mathbf{R}(\gamma + \omega)$$

(since $\gamma + \omega$ is a limit). So we can let $\alpha = \gamma + \omega$.

b) If α is as in (a), then $a \in \mathscr{P}(\mathbf{R}(\alpha))$, which is $\mathbf{R}(\alpha + 1)$; so we can let $\beta = \alpha + 1$.

Remark. In part (a), the ordinal $f(b)$ must be a successor, which is $\operatorname{rank}(b) + 1$ by definition of the rank function.

Scores.

	EA	PC	AF	Mİ	OŞ	NT	ÖT
1	4	5	5	1	5	11	5
2	1	2	3	1	3	5	4
3	—	2	1	—	1	4	2
4	—	1	2	—	0	3	0
	5	10	11	2	9	23	11