Fourth and Final Examination *solutions*

Math 320, David Pierce

June 11 (Saturday), 2011, at 13:30 in M-04

Instructions. Each of the five numbered problems is worth 8 points. As usual, write solutions on separate sheets, keeping *this* sheet. Then enjoy the holiday!

Problem 1. *Write down formulas defining the following classes. (Use only the symbols $\in$, $\neg$, $($, $\Rightarrow$, $)$, and $\exists$; variables; and the constant a.)*

a) $\mathscr{P}(a)$

b) $\bigcup a$

Solution.

a) $x \subseteq a$, that is, $\forall y\ (y \in x \Rightarrow y \in a)$, that is,

$$\neg\exists y\ \neg(y \in x \Rightarrow y \in a).$$

b) $\exists y\ (y \in a \land x \in y)$.

Remark. The formulas *defining* the classes are as given. Then for example the class $\mathscr{P}(a)$ itself is

$$\{x\colon \neg\exists y\ \neg(y \in x \Rightarrow y \in a)\}.$$

Problem 2. *Prove or disprove:*

a) *Every set is a class.*

b) *Every class is a set.*

Solution.

a) Every set a is the class $\{x\colon x \in a\}$.

b) Not every class is a set. Indeed, the class $\{x\colon x \notin x\}$ is not a set, for if it were the set a, then

$$\forall x\ (x \in a \Leftrightarrow x \notin x),$$

and in particular

$$a \in a \Leftrightarrow a \notin a,$$

which is a contradiction.

Problem 3. *Perform the following ordinal computations, giving the answers in Cantor normal form.*

Solution.

a) $3 \cdot (\omega + 4) = 3 \cdot \omega + 3 \cdot 4 = \omega + 12$

b) $(\omega + 4) \cdot 3 = \omega \cdot 3 + 4$

c) $(\omega+5)^2 = (\omega+5)\cdot(\omega+5) = (\omega+5)\cdot\omega+(\omega+5)\cdot 5 = \omega^2 + \omega \cdot 5 + 5$

d) $9^{\omega+2} = 9^{\omega} \cdot 9^2 = \omega \cdot 81$

e) $(\omega+5)^{\omega+2} = (\omega+5)^{\omega} \cdot (\omega+5)^2 = \omega^{\omega} \cdot (\omega^2 + \omega \cdot 5 + 5) = \omega^{\omega+2} + \omega^{\omega+1} \cdot 5 + \omega^{\omega} \cdot 5$

f) $(\omega^{\omega})^{\omega^{\omega}} = \omega^{\omega \cdot \omega^{\omega}} = \omega^{\omega^{1+\omega}} = \omega^{\omega^{\omega}}$

g) $(\omega^{\omega^{\omega}})^{\omega^{\omega}} = \omega^{\omega^{\omega} \cdot \omega^{\omega}} = \omega^{\omega^{\omega \cdot 2}}$

h) $6^{\omega^{1330}} = (6^{\omega})^{\omega^{1329}} = \omega^{\omega^{1329}}$

Problem 4. *Prove, for all ordinals α, β, and γ such that $\alpha > 1$,*

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma} \tag{$*$}$$

Use the recursive definitions, and normality, of $x \mapsto \alpha^x$, $x \mapsto \beta + x$, and $x \mapsto \delta \cdot x$ (where $\delta > 0$). You may use other known ordinal identities, besides $(*)$ itself.

Solution. We use induction on γ. Since

$$\alpha^{\beta+0} = \alpha^{\beta} = \alpha^{\beta} \cdot 1 = \alpha^{\beta} \cdot \alpha^0,$$

the claim holds when $\gamma = 0$. Suppose the claim holds when $\gamma = \delta$. Then

$$\alpha^{\beta+\delta'} = \alpha^{(\beta+\delta)'} = \alpha^{\beta+\delta} \cdot \alpha = \alpha^{\beta} \cdot \alpha^{\delta} \cdot \alpha = \alpha^{\beta} \cdot \alpha^{\delta'},$$

so the claim holds when $\gamma = \delta'$.

Suppose finally δ is a limit, and the claim holds when $\gamma < \delta$.

Then

$$\begin{aligned}
\alpha^{\beta+\delta} &= \alpha^{\sup_{\gamma<\delta}(\beta+\gamma)} && \text{[by definition of } x \mapsto \beta + x\text{]} \\
&= \sup_{\gamma<\delta} \alpha^{\beta+\gamma} && \text{[by normality of } x \mapsto \alpha^x\text{]} \\
&= \sup_{\gamma<\delta} (\alpha^\beta \cdot \alpha^\gamma) && \text{[by inductive hypothesis]} \\
&= \alpha^\beta \cdot \sup_{\gamma<\delta} \alpha^\gamma && \text{[by normality of } x \mapsto \alpha^\beta \cdot x\text{]} \\
&= \alpha^\beta \cdot \alpha^\delta, && \text{[by definition of } x \mapsto \alpha^x\text{]}
\end{aligned}$$

so the claim holds when $\gamma = \delta$.

Problem 5. *Define the function* $\alpha \mapsto V_\alpha$ *on* **ON** *by*

$$V_0 = 0, \qquad V_{\alpha+1} = \mathscr{P}(V_\alpha), \qquad V_\beta = \bigcup_{\alpha<\beta} V_\alpha,$$

where β *is a limit. Find* $\operatorname{card}(V_\alpha)$ *in the following cases. Your answer should be a natural number, an aleph* $\aleph_\beta$*, or a beth* $\beth_\gamma$*.*

a) $\alpha \in \omega$

b) $\alpha = \omega$

c) $\alpha = \omega + 320$

d) $\alpha = \omega \cdot 6$

e) $\alpha = \omega \cdot 11 + 2011$

f) $\alpha = \omega^2$

g) $\alpha = \aleph_1$

h) $\alpha = \beth_1$

Solution.

a) $\operatorname{card}(V_0) = 0$, and $\operatorname{card}(V_{k+1}) = 2^{\operatorname{card}(V_k)}$ if $k \in \omega$.

b) $\operatorname{card}(V_\omega) = \sup_{k\in\omega} \operatorname{card}(V_k) = \aleph_0$.

c) $\operatorname{card}(V_{\omega+1}) = 2^{\operatorname{card}(V_\omega)} = 2^{\aleph_0} = 2^{\beth_0} = \beth_1$, and in general
$$\operatorname{card}(V_{\omega+k}) = \beth_k$$
if $k \in \omega$; in particular, $\operatorname{card}(V_{320}) = \beth_{320}$.

d) $\operatorname{card}(V_{\omega\cdot 2}) = \sup_{k\in\omega} \operatorname{card}(V_{\omega+k}) = \sup_{k\in\omega} \beth_k = \beth_\omega$, and in general
$$\operatorname{card}(V_{\omega\cdot(n+1)}) = \beth_{\omega\cdot n}$$
if $n \in \omega$; in particular, $\operatorname{card}(V_{\omega\cdot 6}) = \beth_{\omega\cdot 5}$.

e) In general,
$$\operatorname{card}(V_{\omega+\alpha}) = \beth_\alpha \qquad (\dagger)$$
for all ordinals α; in particular, $\operatorname{card}(V_{\omega\cdot 11+2011}) = \beth_{\omega\cdot 10+2011}$.

f) Since $\omega^2 = \omega + \omega^2$, we have $\operatorname{card}(V_{\omega^2}) = \beth_{\omega^2}$.

g) $\operatorname{card}(V_{\aleph_1}) = \beth_{\aleph_1}$

h) $\operatorname{card}(V_{\beth_1}) = \beth_{\beth_1}$

Remark. The rule ($\dagger$) can be proved by induction, but this was not required. Note the resemblance to the rule for powers of natural numbers, which can be written as
$$n^{\omega^{1+\alpha}} = n^{\omega\cdot\omega^\alpha} = (n^\omega)^{\omega^\alpha} = \omega^{\omega^\alpha}.$$
where $1 < n < \omega$.

Scores

	EA	PC	AF	Mİ	OŞ	NT	ÖT
1	—	3	7	—	1	7	0
2	—	3	7	0	4	7	0
3	8	7	7	—	6	7	6
4	5	8	7	—	6	8	6
5	0	2	4	—	2	7	0
	13	23	32	0	19	36	12