Minimalist set theory

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Preface

This book is for use in Math 320 (Set Theory) at METU in the spring semester of 2010/11. The book is based on notes I have used in teaching the course in the past; but I have rewritten many sections.

The catalogue description of Math 320 is:

> Language and axioms of set theory. Ordered pairs, relations and functions. Order relation and well ordered sets. Ordinal numbers, transfinite induction, arithmetic of ordinal numbers. Cardinality and arithmetic of cardinal numbers. Axiom of choice, generalized continuum hypothesis.

The set theory presented in this book is a version of what is called ZFC: **Z**ermelo–**F**raenkel set theory with the Axiom of **C**hoice. I call the presentation minimalist, as in the title, for several reasons:

1. The only basic relation between sets is membership; equality of sets is a *defined* notion.
2. Classes as such have no *formal* existence: they are not individuals in the theory, though we can treat them in some respects as if they were.
3. Axioms are introduced only when further progress is otherwise hindered.
4. The form of many axioms, namely that such-and-such a class is a set, is used even for the Axiom of Infinity: the *class* ω of natural numbers is obtained without first assuming that it exists as a set.

See Appendix F for further discussion.

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1. Introduction

In this book, we—the writer and the reader—shall develop an axiomatic theory of sets:

1. We shall study *sets,* which are certain kinds of *collections.*
2. We shall do so by the *axiomatic method.*

There are various reasons why one might want to do this. I see them as follows.

1. All concepts of mathematics can be defined by means of sets. Among such concepts are the numbers *one, two, three,* and so on—numbers that we learn to count with at an early age.

2. Theorems about sets can be as elegant, as beautiful, as any theorems in mathematics.

3. The axiomatic method is of general use in mathematics, and set theory is an example of its application.

4. Set theory provides a *fundamental* or *foundational* example of the axiomatic method, in the following sense. The field of mathematics called *model theory* can be considered as a formal investigation of the axiomatic method, as it is used in ordinary mathematics. In model theory, one defines *structures,* which are to be considered as *models* of certain *theories.* All of these notions—structure, model, theory—are defined in terms of sets.[1]

By the *axiomatic method,* I mean:

1) the identification of certain fundamental properties of some mathematical structure (or kind of structure);
2) from these alone, the derivation of other properties of the structure.

[1] In a strict sense, a *theory* is a kind of collection of formal sentences. Sentences are not normally considered as sets, though they can be. A *structure* is a set considered with certain basic *relations,* that is, subsets of the Cartesian powers of the set. The theory of a structure is the collection of *true* sentences about the structure. The determination of these sentences is made by considering the interactions of the basic relations. This is all done, assuming we already have a set theory. It is true that not everybody likes the set-theoretic conception of model theory: see Angus Macintyre's speculative paper, 'Model theory: geometrical and set-theoretic aspects and prospects' [26].

The fundamental properties are called *axioms* (or *postulates*).

For example, *groups* are studied by the axiomatic method. The structure of interest in group theory is based on the set of *permutations* or *symmetries* of a given set. If the given set is A, then the symmetries of A are just the invertible functions from A to itself. These symmetries compose a set, which we may call $\mathrm{Sym}(A)$. This set always has at least one element, the identity on A, which can be denoted by id_A or simply id. Every element σ of $\mathrm{Sym}(A)$ has an inverse, σ^{-1}. Any two elements σ and τ of $\mathrm{Sym}(A)$ have the composites $\sigma \circ \tau$ and $\tau \circ \sigma$. In short, we have a *structure* on $\mathrm{Sym}(A)$, which we may denote by

$$(\mathrm{Sym}(A), \mathrm{id}, {}^{-1}, \circ). \tag{1.1}$$

Some fundamental properties of this structure are that, for all x, y, and z in $\mathrm{Sym}(A)$,

$$x \circ \mathrm{id} = x, \qquad x \circ x^{-1} = \mathrm{id}, \qquad x \circ (y \circ z) = (x \circ y) \circ z. \tag{1.2}$$

We have been referring to sets, such as A and $\mathrm{Sym}(A)$; this illustrates reasons 1 and 4 above to study sets. Everything here can be reduced to sets as follows.

1. Every element of $\mathrm{Sym}(A)$ is a certain kind of subset of $A \times A$.
2. $A \times A$ is the set $\{(x, y) \colon x \in A \mathbin{\&} y \in A\}$.
3. (a, b) is the set $\{\{a\}, \{a, b\}\}$.
4. id or id_A is the set $\{(x, x) \colon x \in A\}$.
5. ${}^{-1}$ is the set $\{(x, x^{-1}) \colon x \in \mathrm{Sym}(A)\}$.
6. σ^{-1} is the set $\{(y, x) \colon (x, y) \in \sigma\}$.
7. $\circ$ is the set $\{((x, y), x \circ y) \colon (x, y) \in \mathrm{Sym}(A) \times \mathrm{Sym}(A)\}$.
8. $\sigma \circ \tau$ is the set $\{(x, z) \colon \exists y\, ((x, y) \in \tau \mathbin{\&} (y, z) \in \sigma)\}$.

The properties in (1.2) are the *group axioms.*[2] Suppose $(G, \mathrm{e}, {}^*, \cdot)$ is a structure satisfying these axioms: that is, for all x, y, and z in G,

$$x \cdot \mathrm{e} = x, \qquad x \cdot x^* = \mathrm{e}, \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

Then $(G, \mathrm{e}, {}^*, \cdot)$ is called a *group;* but more importantly, there is a set A such that $(G, \mathrm{e}, {}^*, \circ)$ *embeds* in $(\mathrm{Sym}(A), \mathrm{id}, {}^{-1}, \circ)$; that is, the former structure can be considered as a *substructure* of the latter. Indeed, A can

[2] Usually two more axioms are given, namely $\mathrm{id} \circ x = x$ and $x^{-1} \circ x = \mathrm{id}$; but these can be derived from the others.

be G itself, and the embedding of G in $\mathrm{Sym}(G)$ is the function that takes an element g of G to the symmetry $\{(x, g \cdot x) : x \in G\}$ of G. This result is known as *Cayley's Theorem.* It is an example of complete success with the axiomatic method.

Axiomatic set theory does not have the same success of axiomatic group theory. It *cannot* have the same success. This result is known as *Gödel's Incompleteness Theorem,* and it is proved in Appendix C. This result can be taken to illustrate reason 2 to study set theory. Sets are logically *prior* to the rest of mathematics; we cannot expect to identify all of their properties, even implicitly. We shall identify enough for some wonderful consequences though, such as the existence of *transfinite* ordinal and cardinal numbers (covered in Chapters 3, 4, and 5).

Euclid's *Elements* [13] is the world's most popular textbook, having been in use for over two thousand years. It has been taken as the prototypical example of the axiomatic method. Before Euclid, many theorems of geometry were known, such as:

1. the so-called Pythagorean Theorem (Fig. 1.1): the square on the

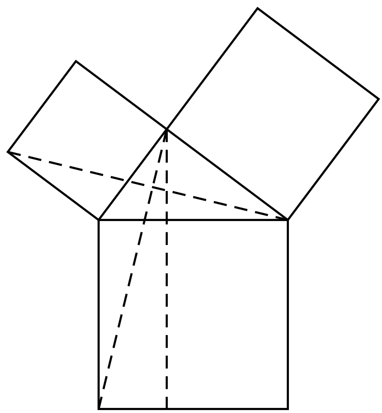

Figure 1.1. The figure of the Pythagorean Theorem

   hypotenuse of a right triangle is equal to the sum of the squares on the other two sides;
2. the existence of five so-called Platonic solids, such as the dodecahedron (Fig. 1.2).

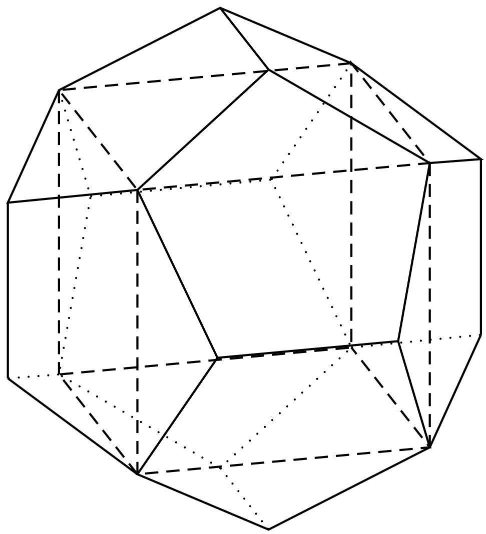

Figure 1.2. The dodecahedron, about a cube

Euclid's innovation is to arrange all of these theorems in a *system.* Euclid starts with some basic facts, which we call axioms or postulates:[3] for example,

- It is possible to draw a straight line from one point to another.
- All right angles are equal.

Euclid uses these axioms to prove some theorems. (His first theorem is that it is possible to construct an equilateral triangle with a given side.) He uses these theorems to prove other theorems, and so on.

Today it is often believed that Euclid's axioms are insufficient to the task of establishing all of his theorems of geometry. I would say rather

[3] Euclid calls them αἰτήματα, which is the plural of αἰτήμα. (The Greek text of Euclid as established by Heiberg [12] can be found in various places on the Web. See Appendix A for the Greek alphabet.) The ordinary meaning of αἰτήμα is *request, demand.* A Latin translation of the word is *postulatum,* and the English translation *postulate* is derived from this. The Greek noun comes from the verb αἰτέω *ask.* The Greek ἀξίωμα means *that which is thought fit,* from the adjective ἄξιος *worthy,* from the verb ἄγω meaning *lead* etc. Several Greek compounds using this verb have found their way into English; one example is *pedagogy.*

that Euclid's methods of proof are not the same as the methods we use today.[4] In any case, the development of axiomatic set theory has been inspired in part by an analysis of Euclid.

The text of Euclid's *Elements* that we have today begins, not with axioms, but with *definitions* of objects like points and lines. These definitions are never explicitly used to prove anything, and it is possible that they have been added to Euclid's original text by later editors. Let us call them *informal* definitions. We shall start our own work with some informal definitions.

One of my typographical conventions in this work is to put important technical terms in **boldface** when they are being defined, or when an an important example of their use is being given. A technical term may be in *italics* if it is of less importance, or if it is not yet being defined.

Some writers use the expression *if and only if* when making definitions. For example, they may write,

> An animal that walks on two legs is a human if and only if it has no feathers.[5]

However, if one knows that this is a definition, then the *and only if* is not needed; it is enough to say,

> An animal that walks on two legs is a **human** if it has no feathers.

Some (but not all) important definitions in this book are explicitly labelled as such:

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[4]See Avigad *et al.,* 'A formal system for Euclid's *Elements*' [1], for a modern analysis of Euclid's methods. The abstract of this paper of 2009 reads:

> We present a formal system, E, which provides a faithful model of the proofs in Euclid's *Elements,* including the use of diagrammatic reasoning.

There is a sense in which the model is not faithful: its reliance on special symbolism—its very *formalism*—is foreign to the spirit of Euclid.

[5]In the *Lives of the Eminent Philosophers* [11, 6.2.40], Diogenes Laërtius wrote the following about Diogenes of Sinope (today's Sinop):

> Plato had defined Man as an animal, biped and featherless, and was applauded. Diogenes plucked a fowl and brought it into the lecture-room with the words, 'Here is Plato's man.' In consequence of which there was added to the definition, 'having broad nails.'

I do not take this anecdote to have much value beyond amusement value; Diogenes Laërtius lived some centuries after Diogenes of Sinope, and he does not document his claims. (The quotation was taken from the Perseus Digital Library `http://www.perseus.tufts.edu/`, February 17, 2011.)

Definition 0. An animal that walks on two legs is a **human** if it has no feathers.

Even without the labelling, the boldface typography of the key word should be enough to distinguish a definition.

If a theorem is given without a proof or with a sketchy proof, it is usually assumed that the reader can supply a proof or the missing details of the proof.

2. The logic of sets

2.1. Sets and classes

One of our earliest mathematical activities is *counting.* Counting is an activity involving a *thing* that is also many *things.* We may count the days in a week: the days are many things, but the week that they make up is one thing. We cannot count things unless we can also consider them together as one thing. I propose to refer to such a thing as a **collection.** A collection is made up of **individuals.** In counting a collection, we take its individuals one by one, while uttering words like *one, two, three,* and so on.

The word *collection* is a *collective noun,* and we shall use it as the most general collective noun.[1] Other collective nouns are words like *pair, flock, deck* (of cards), *number* (of things), *group, family,* and so on.[2] In English, such nouns can be used as subjects of singular or plural verbs:

It's where my family lives.
It's where my family live.[3]

[1] Most general for our purposes, that is; there is no most general collective noun in an absolute sense, because of the Russell Paradox, formalized as Theorem 12 below. There is no collection of all collections, since if there were, then there would be a collection of all collections that do not contain themselves; and this collection can neither contain nor fail to contain itself.

[2] Despite the earlier example of days in a week, I do not think that *week* is a collective noun. We didn't count the week; we counted the collection of days in a week. Likewise, *meter* is not a collective noun, even though a meter is made up of 100 centimeters.

[3] From Evelyn Waugh's 1945 novel *Brideshead Revisited:*

> 'Well?' said Sebastian, stopping the car. Beyond the dome lay receding steps of water and round it, guarding and hiding it, stood the soft hills. 'Well?'
>
> 'What a place to live in!' I said.
>
> 'You must see the garden front and the fountain.' He leaned forward and put the car into gear. 'It's where my family live'; and even then, rapt in the vision, I felt, momentarily, an ominous chill at the words he used—not, 'that is my house', but 'it's where my family live'.
>
> 'Don't worry,' he continued, 'they're all away. You won't have to meet

The individuals that make up a collection will be called **elements** or **members** of the collection. They are **in** the collection, and the collection **contains** them. Collections will be allowed to have just one element or no element. A collection is said to **consist of,** or **comprise,** its members, and the members are said to **compose** the collection.[4] The members of a collection share some *property* with one another, but with nothing else.

There may be two apparently different properties that are shared by exactly the same individuals. An example of two such properties is

1) being in Washington,
2) being in the District of Columbia.

The city of Washington, which is the capital of the United States of America, lies within a region called the District of Columbia, and this is why the city is referred to as Washington, D.C. Originally, the District also contained two other cities (namely Alexandria and Georgetown), along with unincorporated land. Today, the city of Washington has been enlarged, and the District shrunken, so that they have the same boundaries (which include Georgetown, but not Alexandria). In a word, the city and the District are today the same in **extension.** However, they differ in **intension**—they differ in what is intended or *meant* by their names. By *Washington,* we refer to the capital of the USA; by *District of Columbia,* we refer merely to the area in which that capital lies.[5] Hence the collection of people living in Washington differs in intension, but not in extension, from the collection of people living in the District of Columbia.

We shall develop a theory of a certain kind of collection—a kind of collection which will be called a **set.** Sets will be certain collections considered in extension, not intension; that is, two sets with the same members will be considered as the same set. Sets will have the peculiarity

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them.'

(Text taken from `http://www.en8848.com.cn/fiction/Fiction/Classic/2008-03-20/59319_4.html`, February 18, 2011.)

[4] Unfortunately the words *comprise* and *compose* are confused, even by native English speakers. The former, cognate with *comprehend,* has the root meaning of *take together;* the latter, *put together.*

[5] I do not know of a current *legal* distinction between the terms *Washington* and *District of Columbia.* In my experience, *Washington* may refer to the metropolitan region of which the city is the center; then *District* may be used to refer to the city itself.

that all of their members are themselves sets.[6] We shall start with a set with no members: there will be only one such set, called the *empty set,* denoted by $\varnothing$ or 0. From this we shall obtain the set $\{0\}$, whose sole element is 0; the new set will be called 1. Then we shall be able to obtain the set $\{0, 1\}$, called 2, whose elements are 0 and 1. Continuing this way, we shall obtain *formal* definitions of each of the so-called *natural numbers.* The next question will be, What property do these numbers share with one another, but with no other sets, so that we can define the set of natural numbers?

We do not yet have these formal natural numbers, officially, because we have no axioms yet to justify their definition. (See the Introduction, p. 34, and Appendix E.) But meanwhile, we may note another peculiarity of sets. The number 2 will be the set $\{0, \{0\}\}$. In this expression, 0 occurs twice. But 0 is *one* set. The set 0 is both a member of, and a member of a member of, the set 2. However, 0 is *not* a member of the set $\{\{0\}\}$; the only member of *this* set is $\{0\}$, that is, 1. Such a situation does not often arise in ordinary life. If I put a spoon in a teacup, and the teacup in a cupboard, then the spoon is automatically counted as being in the cupboard.

If b is a set, and a is a member of b, we write

$$a \in b;$$

if a is not a member of b, we write

$$a \notin b.$$

if $a \in b$, then, by our convention, a must be a set itself; even if $a \notin b$, we shall understand a to be a set in our system.

The symbol $\in$ is derived from the Greek minuscule letter epsilon (ε): this is the first letter of the Greek verb ἐστί, which just means *is.* The original idea[7] was that $a \in b$ means a is b, in the sense that a cat *is* a mammal: a is one of the b. This way of thinking is potentially ambiguous; for us, $a \in b$ means simply a is *in* b.

[6]Some writers allow sets to have members that are not sets. In that case, the sets that *we* shall use are called something like *hereditary sets* [23, p. 9] or *pure sets* [30, §9.1, p. 238].

[7]It was Peano's idea [28]. Writing in Latin, he said he would use ε to mean *est*—the Latin for *is.*

We cannot expect the collection of all sets to be a set itself. It will be called a *class*. More precisely, the collection of all sets is the **universal class,** and we shall denote it by

$$\mathbf{V}.$$

This class will be our object of study. Our axioms will be about this class and its members. More precisely, $\mathbf{V}$ will be a world, a *universe,* in which our axioms are true. A *class* will be a certain kind of collection of elements of $\mathbf{V}$. In particular, every set will be a class, although not every class will be a set.

The symbol $\in$, and also the *formula* $x \in y$, can be understood to denote a certain *binary relation* on $\mathbf{V}$, namely the relation of **membership.** There are other relations on $\mathbf{V}$, and each of them is denoted by one or more formulas. It will turn out that all relations in this sense can be understood as classes. We shall have various *formulas,* and each will be the name of a class. To say what this means, we need to develop the *logic* of sets.

2.2. Formulas

In school algebra, one encounters *equations* like

$$ax^2 + bx + c = 0; \tag{2.1}$$

in analytic geometry,

$$\ell x = y^2, \tag{2.2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{2.3}$$

Each of these equations is a *formula* in a logic for the set $\mathbb{R}$ of real numbers. I refer to $\mathbb{R}$ as a set, because our axioms will ultimately ensure that $\mathbb{R}$ belongs to $\mathbf{V}$. (See §5.6.) In formulas like (2.1), (2.2), and (2.3), letters like a, b, c, and ℓ are used as *constants,* while letters like x, y, and z are used as *variables.*[8] Both the constants and the variables here

[8]This distinction of letters at the *beginning* of the alphabet from letters at the *end* is made by René Descartes in his *Geometry* [10] of 1637.

denote real numbers. Somewhat imprecisely, we can describe the distinction between constants and variables by saying that a constant denotes a *particular* real number, while a variable denotes all real numbers—not considered as one thing, which is $\mathbb{R}$ itself, but considered as individuals. However, in (2.1) for example, we cannot say *which* particular real numbers are denoted by a, b, and c; what is of interest is that, whatever real numbers are denoted by a, b, and c, there are at most two real numbers that can be denoted by x so that the equation is true.

In the logic of sets, we shall follow the same convention of using letters like a, b, and c as **constants,** and letters like x, y, and z as **variables.** These letters, in their different ways, will denote *sets.* A **term** is a letter that is either a constant or a variable.[9] If t and u are terms, then the string

$$t \in u$$

is called an **atomic formula.** Now, the letters t and u here are not actually symbols of our logic; they just *denote* symbols of our logic, namely symbols such as a or x. These letters t and u then, as well as letters like φ and ψ as used below, can be called **syntactic variables.**[10] (Again, see Appendix A for all of the Greek letters.)

Examples of atomic formulas include $a \in b$ as above, but also

$$a \in a, \qquad x \in a, \qquad b \in y, \qquad x \in y, \qquad z \in z.$$

We may sometimes understand a constant like a as a syntactic variable denoting an arbitrary constant, and a variable like x as a syntactic vari-

[9] I shall occasionally use the word *term* also as it is used in ordinary speech, as a word for a word or phrase that has a precise definition. I used *term* in this way in the Introduction.

[10] Or *syntactical variables.* Older logic books like Shoenfield [30, p. 7] and Church [7, p. 60] use this terminology; a newer book like Chiswell and Hodges [6] uses *metavariables.* Syntactical variables are part of the *syntax language;* Church [7, p. 60] traces the latter term to Carnap's *Logische Syntax der Sprache* (1934). In the 1937 translation of Carnap, one finds [5, §1, p. 4]:

> ... we are concerned with two languages: in the first place with the language which is the object of our investigation—we shall call this the **object-language**—and, secondly, with the language in which we speak *about* the syntactical forms of the object-language—we shall call this the **syntax-language.** As we have said, we shall take as our object-languages certain symbolic languages; as our syntax language we shall at first use the English language with the help of some additional Gothic symbols.

Our own object language consists of the formulas that we are in the process of defining.

able denoting an arbitrary variable. For example, if we should refer to an atomic formula of the form $x \in a$, we mean this formula, but also $y \in a$, and $x \in b$, and $y \in b$, and so on.

Our atomic formulas can be called more precisely **atomic $\in$-formulas** (atomic epsilon-formulas), to distinguish them from atomic formulas in other logics. The equations (2.1), (2.2), and (2.3) are in fact atomic formulas of the usual logic of $\mathbb{R}$; in that logic, polynomials are terms.

In our logic of sets, we do *not* have equations among the atomic formulas. Rather, we shall use equations as abbreviations of certain non-atomic formulas; see §2.8. Our formulas in general are defined as follows.

Definition 1. A **formula,** or more precisely an **$\in$-formula** (epsilon-formula), is a string of symbols that can be built up by application of any of the following rules, as many times as desired:

1. An atomic formula is a formula.
2. If a string φ is a formula, then the string $\neg\varphi$ is a formula.
3. If strings φ and ψ are formulas, then the string $(\varphi \Rightarrow \psi)$ is a formula.
4. If a string φ is a formula, and x is a variable, then the string $\exists x\,\varphi$ is a formula.

In rule 2 of the definition, the formula $\neg\varphi$ is the **negation** of φ. The negation of an atomic formula $t \in u$ is normally written, not as $\neg t \in u$, but as

$$t \notin u.$$

In rule 3, the formula $(\varphi \Rightarrow \psi)$ can be called an **implication,** whose **antecedent** is φ and whose **consequent** is ψ. In rule 4, the formula $\exists x\,\varphi$ is an **instantiation**[11] of φ. Note here that x serves as a syntactic variable; the formulas $\exists y\,\varphi$ and $\exists z\,\varphi$ and so on are also instantiations of φ.

Note also for example that the string $\neg\varphi$ in rule 2 is not a string of two symbols, $\neg$ and φ; it is the string that begins with $\neg$ and continues with all of the symbols that are in the string called φ.

[11] Unlike *negation* and *implication,* the term *instantiation* does not appear to be in common use, although it is found in Shoenfield [30, p. 18]. The formula $\exists x\,\varphi$ will be understood to say that φ is true in some *instance*—that is, for some value of x; so it makes sense to call the formula an instantiation.

If a formula is defined using only the first three rules of the definition, let us say that the formula is **quantifier-free.**[12] In other words, a quantifier-free formula is a formula in which the symbol $\exists$ does not occur.

I take formulas to be fundamental objects, more fundamental than numbers. In English at least, there is no system for assigning *words* to *all* of the numbers, because number names are not customarily repeated. Two twos are *four;* ten tens are a *hundred;* a thousand thousands are a *million.* A million millions were, in France in the 16th century, given the name *billion,* although this later came to understood as the name for a thousand millions. If we must speak of a billion billions, or a billion billion billions, then we do so; but if we must refer repeatedly to these numbers, we shall probably come up with new words for these numbers.

It is clearer in writing that all numbers can be named; but still it is not easy to write down the *algorithm* whereby all numbers can be written in order, as *numerals.* We must first understand that there are ten *digits.* These may best be understood as forming a circle as in Figure 2.1. For

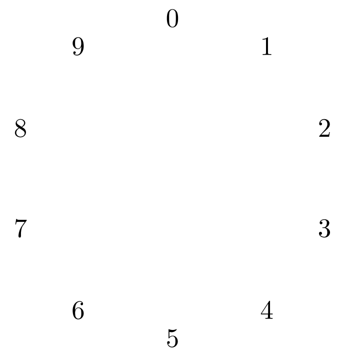

Figure 2.1. The digits

each digit, there is a *next* digit, namely the digit that comes next, in the clockwise direction, around the circle. A numeral is a string of digits. Each digit of the numeral occupies a *place* in the numeral. The *first* place is the leftmost place; the last place, the rightmost. A numeral may not have 0 in the first place.

[12] Shoenfield uses the term *open* [30, p. 36]; it is faster to say, but the meaning is not so obvious.

The numerals themselves have an ordering, in which the first numeral is 1. Given a numeral, we obtain the next numeral as follows. We replace the digit in the last place with its next digit in the circle. If this next digit is 0, then we also replace the digit in the next-to-last place of the numeral with *its* next digit in the circle. If *this* next digit is 0, then we replace the digit in the next place to the left with *its* next digit, and so on. If the digit in the first place becomes 0, then a new first place is added to the left, and the digit 1 is placed there.

Perhaps the first numerals to arise historically, and the easiest to describe, are just strings of marks: $\mathbf{I}$, $\mathbf{II}$, $\mathbf{III}$, $\mathbf{IIII}$, and so on. We can define these by:

1. $\mathbf{I}$ is a numeral.
2. If a string s is a numeral, then so is $s\mathbf{I}$.

Our definition of formulas, Definition 1, is only slightly more sophisticated than this. The definition of formula *does* assume we have indefinite lists of constants and variables. We *could* assume that our constants are a, a', a'', and so on; and our variables, x, x', x'', and so on. That is, we could use the definition:

1. a is a constant, and x is a variable.
2. If s is a constant or variable, then s' is respectively a constant or variable.

But I do not see a need to be this precise about defining constants and variables.

2.3. Logic

Formulas in general are part of the subject-matter of the field of logic. Propositional logic is about quantifier-free formulas; predicate logic takes up the rest.[13]

Our formulas, the $\in$-formulas, are just a tool to be used for understanding sets. Formulas and their symbols are not considered as sets themselves. Constants will only be *names* of sets, and formulas will be *names* of classes. Sets will be classes, though not every class will be a set. The situation can be depicted as in Figure 2.2. The arrows point as

[13]The sign $\in$ is an example of a predicate. Predicate logic may also be called *first-order logic,* but then there is something more: second-order logic. The distinction between first- and second-order logic is not meaningful in the context of set theory.

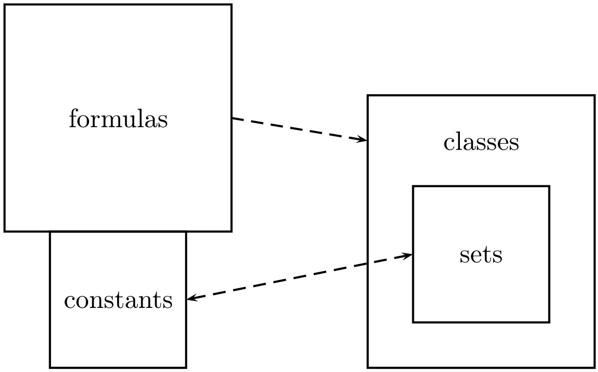

Figure 2.2. The logic of sets

they do, because again:

1. Every constant will denote a particular set.
2. Every set can be denoted by some constant.
3. Every formula will denote a class.

Conversely, every class is denoted by a formula; but the formula is not unique. In the terminology introduced in §2.1, a class can be understood as the *extension* of a formula; but different formulas can have the same extension.[14]

More precisely, the correspondences in Figure 2.2 occur *in a particular context:* they are not fixed once for all. This is why the arrows in the diagram are dashed. The letter a does not denote a set right now; but whenever we happen to have a particular set, we may denote it by some letter, and this letter can be a, unless a has already been used to denote another set. (We can however use different letters for the same set.)

The logic of sets can be constrasted with other logics, such as the logic of $\mathbb{R}$ mentioned above. The equations (2.1), (2.2), and (2.3) can be considered to denote their *solution-sets.*[15] These solution-sets are, respectively, a certain set of real numbers, a certain set of *ordered pairs* of real numbers, and a certain set of *ordered triples* of real numbers. A

[14]It will be possible to consider classes as *equivalence-classes* of formulas.

[15]The solution-sets of (2.2) and (2.3) are usually called *graphs:* a parabola and an hyperboloid of one sheet, respectively.

real number itself cannot be any of these sets. The situation is as in Figure 2.3. A remarkable point about the logic of sets is seen in the

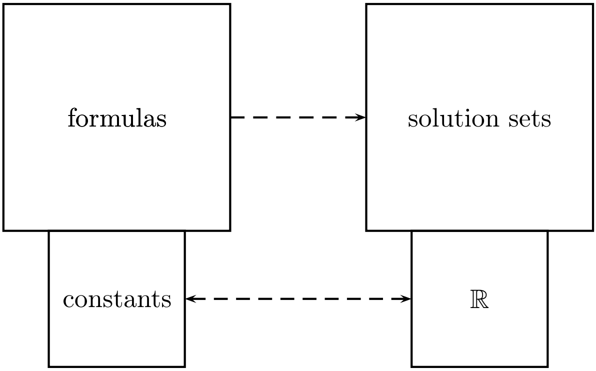

Figure 2.3. The logic of $\mathbb{R}$

difference between figures 2.2 and 2.3: in the logic of sets, constants denote some of the *same things* that formulas denote.

Definition 1, of *formula* (more precisely, $\in$-*formula*), makes it possible to prove theorems about the collection of formulas that will be useful for us. Indeed, the definition of *formula* is the kind of definition that can be described by either of two adjectives: it is

- **recursive,** because it involves *recurrent* (repeated) application of certain rules;
- **inductive,** because theorems about the collection of things with the given definition can by proved by *induction.*

I prefer to call such a definition *recursive,* leaving the word *inductive* to describe the kind of proof that the definition allows.

For example, the natural numbers can be given a recursive definition:

1. 0 is a natural number.
2. If n is a natural number, then so is $n + 1$.

Such a definition will be made officially as Definition 12 in §3.1, and then the collection of natural numbers will be called $\mathbb{N}$. Our theoretical work in this chapter and the next will give clear meanings to the expressions 0 and $n + 1$. In Definition 15, we shall have a non-recursive definition of a class denoted by ω, and then the classs denoted by $\bigcup \omega$ will be called the class of *formal natural numbers.* We shall show that we *may* assume $\mathbb{N}$ is just $\bigcup \omega$; but we cannot *prove* that they are the same. After Theorem 43, we shall have that $\bigcup \omega$ and ω are the same.

Meanwhile, we do not officially have the natural numbers. Unofficially, we may note that their recursive definition makes possible the usual method of inductive proof, whereby every natural number has some property if

1) 0 has the property, and

2) $n+1$ has the property on the assumption that n has the property.

In general, an inductive proof has as many parts as the corresponding recursive definition. Our first example of an inductive proof concerning formulas will be Lemma 2 below; the proof will have four parts, like Definition 1.

When a string is a formula, then the history of its construction *as* a formula can be shown in a *tree,* called the **parsing tree** of the formula.[16] For example, the parsing tree for the formula

$$\neg(\exists x\, x \in x \Rightarrow x \notin x)$$

is as in Figure 2.4. In the definition of formula, if implications did not

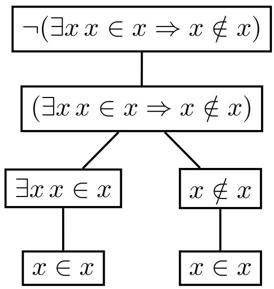

Figure 2.4. A parsing tree

have parentheses, then the parsing tree for a given formula might not be

[16] I take the terminology from Chiswell and Hodges [6], who make much use of parsing trees. Strictly, such trees are examples of *labelled* trees. The Wikipedia article on parsing trees in the present sense has the heading *Parse tree* (on March 6, 2011), although I added the alternative form *parsing tree.*

unique; there might be more than one way to construct the same formula. With the definition of formula as it is, the parsing tree *is* unique. This is a consequence of Theorem 1 below, which takes a bit of work to prove. *Part* of the proof is easy; it is the following immediate consequence of the definitions:

Lemma 1.

1. *Atomic formulas begin with terms.*
2. *Negations begin with* $\neg$.
3. *Implications begin with* (.
4. *Instantiations begin with* $\exists$.

Note well that none of the symbols $\neg$, (, and $\exists$ is a term, and none is the same as any other. So an atomic formula cannot also be a negation, a negation cannot also be an implication, and so forth. There remains the question of whether an implication can be an implication in more than one way. Does an implication have a unique antecedent and consequent? We shall be able to settle this by means of the next lemma.

We shall use the following terminology. A **proper initial segment** of a string is the string that results from deleting one or more (but not all) symbols from the end. An **initial segment** of a string is either a proper initial segment, or the string itself.

Lemma 2. *No proper initial segment of a formula is a formula.*

Proof. We prove the lemma in the following version: for all formulas φ,

- no proper initial segment of φ is a formula, and
- φ is not a proper initial segment of a formula.

By Lemma 1, if an initial segment of an atomic formula is itself a formula, it must be an atomic formula; if of a negation, a negation; and so on. Now we can complete the proof by means of induction. Since the recursive definition of formula has four parts, so will our inductive proof.

1. Our claim holds for atomic formulas, since each of these is exactly three symbols long.

2. If the claim holds for φ, then it holds for $\neg\varphi$. Indeed, every proper initial segment of $\neg\varphi$ is $\neg S$ for some proper initial segment S of φ. If S must not be a formula, then $\neg S$ is not a formula either. Similarly, every string of which $\neg\varphi$ is a proper initial segment is $\neg T$ for some string T of which φ is a proper initial segment; if T must not be a formula, then neither is $\neg T$ a formula.

3. Suppose the claim holds for φ and ψ. Then it holds for $(\varphi \Rightarrow \psi)$. Indeed, if an initial segment of the last formula is a formula itself, then it must be $(\theta \Rightarrow \rho)$ for some formulas θ and ρ. If θ and φ are the same formula, then ρ is an initial segment of ψ. The other possibility is that θ is an initial segment of φ, or φ is an initial segment of θ. By our inductive hypothesis, θ must be φ, and then ρ must be ψ. Similarly, if $(\varphi \Rightarrow \psi)$ is an initial segment of a formula, then that formula must be $(\varphi \Rightarrow \psi)$ itself.

4. Finally, if the claim holds for φ, then it holds for $\exists x\, \varphi$, just as it holds for $\neg\varphi$. This completes the induction. □

The following will justify various recursive definitions of *functions* on collections of formulas.

Theorem 1 (Unique readability)**.** *Each formula is of only one of the four kinds: atomic formulas, negations, implications, and instantiations. Moreover, a formula is of one of these kinds in only one way. In particular, if φ, ψ, θ, and ρ are formulas, and the two implications $(\varphi \Rightarrow \psi)$ and $(\theta \Rightarrow \rho)$ are the same formula, then φ and θ are the same formula, and so are ψ and ρ.*

Proof. The first claim follows from Lemma 1. The second claim follows from Lemma 2, since (in the notation of the claim) one of φ and θ is an initial segment of the other, so they are the same, and hence ψ and ρ are the same. □

So far we have worked out the **syntax** of formulas: the rules for their construction, and some consequences of these rules. The next job is to work out the **semantics** of formulas: what they *mean,* and which of them can be called *true.* The distinction between syntax and semantics is not always clear. In §2.5, we shall develop two notions: *logical entailment,* and *syntactic derivation.* The former notion can be called semantic. They differ in intension; but they will turn out to be the same in extension.

2.4. Sentences and truth

Quantifier-free formulas that have no variables are called *quantifier-free sentences.* Among such formulas are the atomic formulas $a \in b$, which we may obviously call **atomic sentences.** We have just given a non-recursive definition of the quantifier-free sentences. There is also a recursive definition:

Definition 2. The **quantifier-free sentences** are given by the following rules.

1. Every atomic sentence is a quantifier-free sentence.
2. If σ is a quantifier-free sentence, then so is $\neg\sigma$.
3. If σ and τ are quantifier-free sentences, then so is $(\sigma \Rightarrow \tau)$.

We know what it means for atomic sentences to be true or false. We extend the definition as follows.

Definition 3. A quantifier-free sentence is **false** if it is not *true;* and quantifier-free sentences are **true** under the conditions given by the following rules.

1. An atomic sentence $a \in b$ is true if $a \in b$. (That is, $a \in b$ is true if and only if a is a member of b.)
2. A quantifier-free sentence $\neg\sigma$ is true if σ is false.
3. If σ and τ are quantifier-free sentences, then $(\sigma \Rightarrow \tau)$ is true if either σ is false or τ is true.

This definition is **recursive.** It is not a recursive definition of a *collection.* Rather, it is a recursive definition of a *function,* namely a function on the collection of quantifier-free sentences. This collection has the recursive definition above, in three parts, and the definition of the function on this collection has three corresponding parts. But we must check that the definition of the function is valid: we must check that there really is such a function. In the definition, rule 3 assumes that σ and τ are uniquely determined by the whole formula $(\sigma \Rightarrow \tau)$. This assumption is justified by Theorem 1. Some books overlook the need for such justification; but if implications did not have parentheses, then truth could not be unambiguously defined.

If σ is true, we may write simply σ (as we did in rule 1). Then $(\sigma \Rightarrow \tau)$ is true if and only if the English sentence

$$\text{If } \sigma, \text{ then } \tau$$

is true. We can compute whether an arbitrary quantifier-free sentence is true or false by means of a *truth table.* The reader may well be familiar with truth-tables; but different writers treat them differently. I understand them as follows.

In the parsing tree for any formula, the various formulas that occur are just the **subformulas** of the original formula. Each subformula that is not atomic is obtained from one or two other subformulas by application of one of the symbols $\neg$, $\Rightarrow$, and $\exists$. (We also add parentheses when the symbol is $\Rightarrow$, and we add a variable when the symbol is $\exists$.) If the original formula is a quantifier-free sentence σ, then all of the subformulas are quantifier-free sentences. We consider one of these quantifier-free sentences to have the value 1 if it is true, 0 if it is false.[17] This value, 1 or 0, is the **truth value** of the sentence; a sentence denotes its truth value, as a constant denotes a set. We can compute the truth values of the subformulas of σ in turn, from the atomic subformulas all the way up to σ itself. Suppose, in the construction of σ, we have used letters like P, Q, and R in place of the atomic sentences: we can think of these letters as syntactic variables, either for atomic sentences, or for their possible truth values. Then each subformula corresponds to a single symbol in σ: either one of the letters just mentioned, or $\neg$, or $\Rightarrow$. We can write out the whole of σ, and write the values of its subformulas under the corresponding symbols. We may include all possible values of the atomic sentences that occur; then we get a **truth table.**

The rules of computation are shown in Table 2.1. The parsing tree of a particular quantifier-free sentence is shown in Figure 2.5; the truth table of this quantifier-free sentence is worked out in stages in Table 2.2; the truth table itself is in Table 2.3. This particular quantifier-free sentence, $(P \Rightarrow (\neg Q \Rightarrow \neg(P \Rightarrow Q)))$, happens to take the value 1, no matter what values are assigned to the atomic sentences P and Q; therefore it can be called a **tautology.** A quantifier-free sentence that always takes the value 0 is a **contradiction.**

Definition 4. An arbitrary $\in$-formulas is a **sentence,** or more precisely an **$\in$-sentence,** if it has no *free* variables. The collection of **free variables** of a formula is defined recursively:

1. The free variables of an atomic formula are just the variables that occur in the atomic formula.
2. The free variables of $\neg\varphi$ are the free variables of φ.
3. The free variables of $(\varphi \Rightarrow \psi)$ are the free variables of φ or ψ.
4. The free variables of $\exists x\,\varphi$ are those of φ, except x.

[17] Some writers, as Stoll [32, Ch. 4, Exercise 3.7], use 0 and 1 in the opposite sense.

$\neg$	σ
1	0
0	1

$(\sigma$	$\Rightarrow$	$\tau)$
0	1	0
1	0	0
0	1	1
1	1	1

Table 2.1. The two basic truth tables

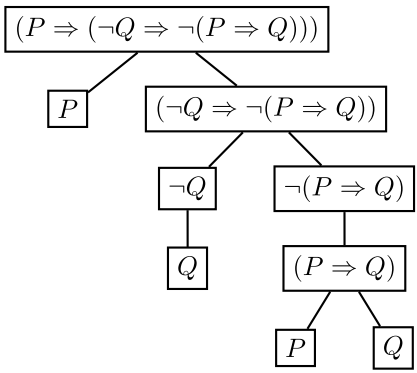

Figure 2.5. The parsing tree of a quantifier-free sentence

Here again, part 3 relies on Theorem 1. Of course we want to define truth and falsity of arbitrary $\in$-sentences. To do this, we must deal with a complication. The same variable may occur several times in a formula. We distinguish between:

1. A variable that occurs in a formula.
2. The particular occurrences of that variable in the formula.

So for example only one variable occurs in the formula $x \in x$, but this variable has two occurrences in the formula. Now, every occurrence of a variable x in a formula φ is also an occurrence in one or more subformulas of φ. If one of these subformulas is an instantiation $\exists x\, \psi$, then the occurrence of x in φ is said to be a **bound occurrence.**[18] Occurrences

[18] A bound variable is so called because the symbol $\exists$ **binds** it—ties it down. The

$(P$	$\Rightarrow$	$(\neg$	Q	$\Rightarrow$	$\neg$	$(P$	$\Rightarrow$	$Q)))$
0			0			0		0
1			0			1		0
0			1			0		1
1			1			1		1
0		1	0			0	1	0
1		1	0			1	0	0
0		0	1			0	1	1
1		0	1			1	1	1
0		1	0		0	0	1	0
1		1	0		1	1	0	0
0		0	1		0	0	1	1
1		0	1		1	1	1	1
0		1	0	0	0	0	1	0
1		1	0	1	1	1	0	0
0		0	1	1	0	0	1	1
1		0	1	1	1	1	1	1
0	1	1	0	0	0	0	1	0
1	1	1	0	1	1	1	0	0
0	1	0	1	1	0	0	1	1
1	1	0	1	1	1	1	1	1

Table 2.2. The filling-out of a truth table

that are not bound are **free occurrences.** For example, the formula $\neg(\exists x\, x \in x \Rightarrow x \notin x)$ has the free variable x; but only the last two occurrences of x are free; the first three are bound. Thus it is possible that, in a formula, some *occurrences* of a free variable are bound and not free. In this case, the variable is free only because some *other* occurrences are free.

In practice, we never need such formulas. All we need are **good formulas,** which can be defined recursively as follows.

relevant verb is *bind, bound, bound,* whose ancestor is found in Old English (English as spoken before the Norman Conquest of 1066). There is an unrelated verb *bound, bounded, bounded,* which is also used in mathematics; this verb is derived from the noun *bound,* which came to English from French (more precisely, Anglo-Norman) in the 13th century [20].

$(P$	$\Rightarrow$	$(\neg$	Q	$\Rightarrow$	$\neg$	$(P$	$\Rightarrow$	$Q)))$
0	1	1	0	0	0	0	1	0
1	1	1	0	1	1	1	0	0
0	1	0	1	1	0	0	1	1
1	1	0	1	1	1	1	1	1

Table 2.3. A truth table

1. Every atomic formula is good.
2. The negation of a good formula is good.
3. If φ and ψ are good, and every variable that occurs in *both* of them is a *free* variable of *each* of them, then $(\varphi \Rightarrow \psi)$ is good.
4. If φ is good, and x is a free variable of φ, then $\exists x\,\varphi$ is good.

A good formula should be a formula in which, for every variable x, the string $\exists x$ does not occur twice, and if it occurs once, then x is not a free variable of the formula. The latter condition is more important; we prove that it is satisfied as follows.

Theorem 2. *In a good formula, every occurrence of a free variable is a free occurrence.*

Proof. We use induction.

1. Since the symbol $\exists$ does not occur in atomic formulas, every occurrence of a variable in an atomic formula is a free occurrence.

2. Suppose x is a free variable of $\neg\varphi$. Then x is a free variable of φ. If every occurrence of x in φ is a free occurrence, then the same is true of $\neg\varphi$, since the subformulas of $\neg\varphi$ are the same as those of φ, except for $\neg\varphi$ itself, and this is not an instantiation.

3. Suppose x is a free variable of $(\varphi \Rightarrow \psi)$, and this is a good formula. Then x is a free variable of φ or of ψ, and if x does occur in one of these formulas, then it is a free variable of that formula. Suppose further that every occurrence in φ of a free variable of φ is a free occurrence, and the same is true for ψ. Then, in particular, every occurrence of x in φ or in ψ is a free occurrence. Therefore every occurrence of x in $(\varphi \Rightarrow \psi)$ is a free occurrence, since the subformulas of this formula are the same as those of φ and ψ, except for $(\varphi \Rightarrow \psi)$ itself, which is not an instantiation.

4. Finally, suppose x is a free variable of $\exists y\,\varphi$, and in φ, every occurrence of a free variable is a free occurrence. Then y is not x, so every

occurrence of x in $\exists y\,\varphi$ is an occurrence in φ and therefore a free occurrence in φ. Also every free occurrence of x in φ is a free occurrence in $\exists y\,\varphi$. $\square$

Suppose φ is a **singulary formula,** namely a formula with just one free variable; and let that variable be x. (For example, the formula could be $x \in x$ or $\exists y\, y \in x$.) Then we may denote φ by

$$\varphi(x).$$

If t is a term, and if we replace every *free* occurrence of x in φ with t, we obtain the formula denoted by

$$\varphi(t).$$

We may say that we obtain $\varphi(t)$ from $\varphi(x)$ by **substitution** of t for x. If φ is a good formula, then *every* occurrence of x in φ is a free occurrence, by Theorem 2. Note however that, if y is a variable other than x that occurs in $\varphi(x)$, then $\varphi(y)$ need not be a good formula. Such is the case when φ is $\exists y\, y \in a \mathbin{\&} x \in z$. In practice, we always avoid this situation.

We use the letters σ and τ as syntactic variables for $\in$-sentences.

Definition 5. An $\in$-sentence is **false** if it is not *true.* An $\in$-sentence is **true** according to the following rules, which include the rules given above for truth of quantifier-free sentences:

1. An atomic sentence $a \in b$ is true if $a \in b$.
2. A sentence $\neg\sigma$ is true if σ is false.
3. A sentence $(\sigma \Rightarrow \tau)$ is true if either σ is false or τ is true.
4. A sentence $\exists x\,\varphi(x)$ is true if there is a set a such that $\varphi(a)$ is true.

Because of rule 4, the sentence $\exists x\,\varphi(x)$ is said to result from $\varphi(x)$ by **existential quantification** of the variable x.

We have not quite given a recursive definition of the $\in$-sentences, and the definition of their truth is not quite recursive in the foregoing sense. But it is close enough. We can understand the definition of the truth of a sentence as the definition of the truth of the sentences that *result* from a formula when its free variables are replaced with constants.

2.5. The theory of sets

Our goal is to identify all true $\in$-sentences. The collection of these true sentences is called the **theory of sets** or **set theory.** In one sense, a weak sense, we have already achieved our goal, simply by *defining* what it means for a sentence to be true. However, if possible, we should like to have an *algorithm* for determining whether a given $\in$-sentence is true. Truth tables give us an algorithm for determining whether *quantifier-free* sentences are true; but many $\in$-sentences are not quantifier-free sentences.

The truth value of an arbitrary $\in$-sentence is still determined by the truth values of atomic sentences. So we are able to consider whether a given sentence would be true under a possibly incorrect assignment of truth values to atomic sentences. In other words, we are able to consider whether a given sentence would be true under an arbitrary **interpretation** of the symbol $\in$.

Definition 6. If a sentence would be true under every interpretation of $\in$, then the sentence is called **logically true.** If σ is logically true, we may express this by writing

$$\models \sigma.$$

Here the symbol $\models$ is the **semantic turnstile.**

The tautologies as defined so far are examples of logically true sentences; but there are other examples. Indeed, suppose σ is a one of these tautologies. For each atomic sentence P that occurs as a subformula of σ, we choose some sentence τ, and we replace each occurrence of P in σ with τ. Call the resulting sentence σ'. We enlarge the meaning of **tautology** to include such sentences σ'.

Theorem 3. *Tautologies are logically true.*

Proof. I take the claim to be an obvious consequence of the definitions. In the notation just used, the truth value of σ' can be read off from the truth-table of σ, when one knows the truth value of the sentences τ; but then the former truth value will always by 1. □

Examples of logically true sentences that are not tautologies are given by the following.

Theorem 4. *If σ is a sentence in which x occurs, but y does not, and the sentence σ' is obtained from σ by replacing* each *occurrence of x with y, then $(\sigma \Rightarrow \sigma')$ is logically true:*

$$\models (\sigma \Rightarrow \sigma').$$

If also σ is a good *sentence, then so is σ'.*

Proof. The sentence σ' is what results from σ if the variable x is considered to be y, and y to be x. Such a change does not affect the truth or goodness of a sentence. □

In the theorem, note that $(\sigma \Rightarrow \sigma')$ is not a good sentence unless no bound variable other than x occurs in σ.

Some true sentences are not logically true. We shall establish some such sentences by means of our intuition for what sets are, and we shall declare these sentences as *axioms.* An **axiom** then is just a true sentence, considered as being 'obviously' true.[19] We continue from the axioms by means of the following notion.

Definition 7. Given a collection Σ of sentences, suppose τ is a sentence that *would be* true in any interpretation of $\in$ in which each sentence in Σ was true. Then τ is a **logical consequence** of Σ, and Σ **logically entails** τ. We may express this with the semantic turnstile by writing

$$\Sigma \models \tau.$$

We may say now that the symbol $\models$ denotes the relation of **logical entailment.**[20]

The logically true sentences are those that are logical consequences of every collection of formulas, in particular the empty collection. We shall want to find the logical consequences of our axioms. How one discovers

[19]It may be that a particular axiom is not obviously *true,* but obviously *useful* for proving interesting theorems.

[20]Strictly, there is no need to use the adjective *logical;* we can just refer to the relation as *entailment.* It is the relation defined in the Wikipedia article of that name, `http://en.wikipedia.org/wiki/Entailment` (accessed Febuary 20, 2011). But the word *logical* should clarify the relation better than simply *entailment.* We are going to define a second way to get sentences from collections of sentences, and it will be important to distinguish this from logical entailment.

interesting logical consequences is a difficult question; but if they *have* been discovered, then it will be possible to derive them mechanically from our axioms, by means of *rules of inference.* The following theorem gives an example of such a rule.

Theorem 5 (Detachment)**.** *From the truth of σ and $(\sigma \Rightarrow \tau)$, the truth of τ can be inferred.*

Proof. The claim follows immediately from the definition of the truth of an implication. □

The Rule of Detachment can be stated as an imperative:

From σ and $(\sigma \Rightarrow \tau)$, infer τ.

We can also write this rule as

$$\sigma, (\sigma \Rightarrow \tau) \models \tau.$$

We can now define a **rule of inference** as a theorem that a sentence of a certain kind is a logical consequence of certain other sentences. The Detachment Rule is the rule of inference that τ is a logical consequence of σ and $(\sigma \Rightarrow \tau)$. Because of this rule, every logically true implication yields a rule of inference. For example, the following implications are tautologies:

$$\begin{gathered} ((P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)), \\ ((\neg P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P)), \\ ((P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow \neg P)), \\ ((\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)). \end{gathered}$$

These give us the rules of inference that we can refer to collectively as **Contraposition:**

From $(\sigma \Rightarrow \tau)$, infer $(\neg\tau \Rightarrow \neg\sigma)$.
From $(\neg\sigma \Rightarrow \tau)$, infer $(\neg\tau \Rightarrow \sigma)$.
From $(\sigma \Rightarrow \neg\tau)$, infer $(\tau \Rightarrow \neg\sigma)$.
From $(\neg\sigma \Rightarrow \neg\tau)$, infer $(\tau \Rightarrow \sigma)$.

From the tautologies

$$(P \Rightarrow \neg\neg P), \qquad\qquad (\neg\neg P \Rightarrow P),$$

we have the rules of **Double Negation:**

$$\text{From } \sigma, \text{ infer } \neg\neg\sigma.$$
$$\text{From } \neg\neg\sigma, \text{ infer } \sigma.$$

To prove a sentence σ by **Contradiction,** we find some false sentence τ such that $(\neg\sigma \Rightarrow \tau)$ is true. By Contraposition, we infer $(\neg\tau \Rightarrow \sigma)$, and then, since $\neg\tau$ is true, we can infer σ by Detachment. In short, we have the first of the following rules of inference; the others follow similarly.

$$\text{From } (\neg\sigma \Rightarrow \tau) \text{ and } \neg\tau, \text{ infer } \sigma.$$
$$\text{From } (\sigma \Rightarrow \tau) \text{ and } \neg\tau, \text{ infer } \neg\sigma.$$
$$\text{From } (\neg\sigma \Rightarrow \neg\tau) \text{ and } \tau, \text{ infer } \sigma.$$
$$\text{From } (\sigma \Rightarrow \neg\tau) \text{ and } \tau, \text{ infer } \neg\sigma.$$

To continue our investigations of rules of inference, it is convenient to have some abbreviations of formulas:

Definition 8.

1. For $(\neg\varphi \Rightarrow \psi)$, we write
$$(\varphi \vee \psi).$$
2. For $\neg(\neg\varphi \vee \neg\psi)$, we write
$$(\varphi \mathbin{\&} \psi).$$
3. For $((\varphi \Rightarrow \psi) \mathbin{\&} (\psi \Rightarrow \varphi))$, we write
$$(\varphi \Leftrightarrow \psi).$$
4. For $\neg\exists x\,\neg\varphi$, we write
$$\forall x\,\varphi.$$

The abbreviations so defined are, respectively, **disjunctions, conjunctions, equivalences,** and **generalizations.**

Let us acknowledge that these abbreviations mean what they are supposed to mean:

Theorem 6. *Suppose σ and τ are sentences, and $\varphi(x)$ is a singulary formula.*

1. *The sentence $(\sigma \vee \tau)$ is true if and only if at least one of the two sentences σ and τ is true.*

2. *The sentence* $(\sigma \mathbin{\&} \tau)$ *is true if and only if both sentences* σ *and* τ *are true.*
3. *The sentence* $(\sigma \Leftrightarrow \tau)$ *is true if and only if either both sentences* σ *and* τ *are true or both are false.*
4. *The sentence* $\forall x\, \varphi(x)$ *is true if and only if, for each set* a*, the sentence* $\varphi(a)$ *is true.*

Since, for example, $((P \mathbin{\&} Q) \Rightarrow P)$ is a tautology, we have the rule,

$$\text{From } (\sigma \mathbin{\&} \tau), \text{ infer } \sigma.$$

Similarly, there is a rule,

$$\text{From } \sigma, \text{ infer } (\sigma \vee \tau).$$

Such rules of inference can be multiplied as needed. More logical truths are as follows:

Theorem 7 (Specialization). *For all singulary formulas* $\varphi(x)$ *and sets* a*, the sentences*

$$(\varphi(a) \Rightarrow \exists x\, \varphi(x)), \qquad (\forall x\, \varphi(x) \Rightarrow \varphi(a))$$

are logically true.

Proof. The first sentence follows immediately from the definitions of truth of implication and instantiation. Then the second is obtained by Contraposition (and the definition of a generalization). □

We now have two more rules of inference:

$$\text{From } \forall x\, \varphi(x), \text{ infer } \varphi(a).$$
$$\text{From } \varphi(a), \text{ infer } \exists x\, \varphi(x).$$

Another rule of inference is given by the following.

Theorem 8 (Generalization). *Suppose* $\varphi(x)$ *is a formula, and* a *is a constant not occurring in* $\varphi(x)$ *or the sentence* σ*.*

1. *If* $(\varphi(a) \Rightarrow \sigma)$ *is* logically *true, then so is the sentence*

$$(\exists x\, \varphi(x) \Rightarrow \sigma).$$

2. *If* $(\sigma \Rightarrow \varphi(a))$ *is logically true, then so is the sentence*

$$(\sigma \Rightarrow \forall x\, \varphi(x)).$$

Proof. In the first part, because $(\varphi(a) \Rightarrow \sigma)$ is logically true, it is true for all sets a. If $\exists x\, \varphi(x)$ is false, then $(\exists x\, \varphi(x) \Rightarrow \sigma)$ is true. Suppose $\exists x\, \varphi(x)$ is true. Then $\varphi(b)$ for some b. We can give b the name a, since the constant a does not occur in $\varphi(x)$ or σ. Then $\varphi(a)$ is true, and therefore σ, by Detachment. In each case then, $(\exists x\, \varphi(x) \Rightarrow \sigma)$ is true. The second part then follows by Contraposition. □

Note the importance of the several conditions in the theorem:

1. If σ is false, then we have $((a \in b \Rightarrow a \notin b) \Rightarrow \sigma)$, but not $\exists x\, (x \in b \Rightarrow a \notin b) \Rightarrow \sigma)$; here a still occurs in $(x \in b \Rightarrow a \notin b)$.
2. If $a \notin b$, but $\exists x\, x \in b$, we have $(a \in b \Rightarrow a \in b)$, but not $\exists x\, x \in b \Rightarrow a \in b)$; here a occurs in $a \in b$.
3. If $a \notin b$, but $c \in b$, We have $(a \in b \Rightarrow c \notin b)$, but not $(\exists x\, x \in b \Rightarrow c \notin b)$; here $(a \in b \Rightarrow c \notin b)$ is not *logically* true.

We now know enough to be able to establish *every* logical entailment mechanically. The first step is to establish the logical truths, by the process implicit in the next definition.

Definition 9. As **logical axioms,** we take:

1) the tautologies,
2) the sentences $(\varphi(a) \Rightarrow \exists x\, \varphi(x))$.

Then the **logical theorems**[21] are defined as follows.

1. Logical axioms are logical theorems.
2. If σ and $(\sigma \Rightarrow \tau)$ are logical theorems, then so is τ.
3. If $(\varphi(a) \Rightarrow \sigma)$ is a logical theorem, and a does not occur in $\varphi(x)$ or σ, then $(\exists x\, \varphi(x) \Rightarrow \sigma)$ is a logical theorem.

If σ is a logical theorem, we may express this by writing

$$\vdash \sigma.$$

Here the symbol $\vdash$ is the **syntactic turnstile.**

[21] Some sources, such as Shoenfield [30], will refer to logical theorems simply as *theorems;* but they should be distinguished from the sentences in ordinary language (with some symbolism) that are labelled as theorems in books of mathematics like the present one.

A sentence σ that is a logical theorem is so called because it has a **formal proof:** a list τ, τ', τ'', …, σ of sentences, ending with σ, in which each entry is either:

1. a logical axiom, or
2. a sentence ρ, where sentences π and $(\pi \Rightarrow \rho)$ come earlier in the list, or
3. a sentence $(\exists x\, \varphi(x) \Rightarrow \rho)$, where the sentence $(\varphi(a) \Rightarrow \rho)$ comes earlier in the list, and a does not occur in $\varphi(x)$ or ρ.

We generally do not want to write down such formal proofs; it is enough to convince ourselves that they exist. Meanwhile, we should note:

Theorem 9. *Every logical theorem is logically true: if $\vdash \sigma$, then $\models \sigma$.*

Proof. We use induction.

1. The logical axioms are logically true by Theorems 3 and 7.

2. If σ and $(\sigma \Rightarrow \tau)$ are logical theorems that are logically true, then τ is logically true, by Theorem 5.

3. If $(\varphi(a) \Rightarrow \sigma)$ is a logical theorem that is logically true, and a does not occur in $\varphi(x)$ or σ, then $(\exists x\, \varphi(x) \Rightarrow \sigma)$ is logically true, by Theorem 8. □

The converse of this theorem is **Gödel's Completeness Theorem,** which is Theorem 167 in Appendix B. The theorem is in an appendix, because we shall never need to *appeal* to the theorem in justification of anything we do. We may use informal methods to establish some particular logical truth σ. If one knows the Completeness Theorem, then one knows that a formal proof of σ can always be found. If one doubts this though, one can just go ahead and find the formal proof.

The following establishes a useful abbreviating convention for writing formulas.

Definition 10.

1. We need not write the outer parentheses of a formula (if it has them).
2. We can remove internal parentheses by understanding & and $\vee$ to have priority over $\Rightarrow$ and $\Leftrightarrow$, so that for example $\varphi \mathbin{\&} \psi \Rightarrow \chi$ means $(\varphi \mathbin{\&} \psi) \Rightarrow \chi$, which in turn means $((\varphi \mathbin{\&} \psi) \Rightarrow \chi)$.
3. When the symbol $\Rightarrow$ is repeated, the occurrence on the right has priority, so $\varphi \Rightarrow \psi \Rightarrow \chi$ means $\varphi \Rightarrow (\psi \Rightarrow \chi)$.

For an example of a logical theorem, let σ be the sentence

$$(\exists x\, \varphi(x) \Rightarrow \tau) \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau).$$

It is not hard to see that σ is logically *true.* Indeed, suppose $\exists x\, \varphi(x) \Rightarrow \tau$ is true. We can consider two cases:

1. If $\exists x\, \varphi(x)$ is also true, then τ is true. Consequently, $\varphi(a) \Rightarrow \tau$ is true (no matter what a is), so $\exists x\, (\varphi(x) \Rightarrow \tau)$ is true.
2. If $\exists x\, \varphi(x)$ is false, then, no matter what a is, we have $\neg\varphi(a)$, so $\varphi(a) \Rightarrow \tau$, and again $\exists x\, (\varphi(x) \Rightarrow \tau)$.

In either case, we get $\exists x\, (\varphi(x) \Rightarrow \tau)$ on the assumption that $\exists x\, \varphi(x) \Rightarrow \tau$. This means σ is logically true. By the Completeness Theorem, σ must be a logical theorem; but to establish this directly is more laborious. We can do it though. Keeping in mind the conventions of Definition 10, we have the following logical axioms:

$$\tau \Rightarrow \varphi(a) \Rightarrow \tau,$$
$$(\varphi(a) \Rightarrow \tau) \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau).$$

We also have, as a logical axiom, the tautology

$$(P \Rightarrow Q) \Rightarrow (Q \Rightarrow R) \Rightarrow (P \Rightarrow R),$$

which we use in the form

$$\begin{aligned}&(\tau \Rightarrow \varphi(a) \Rightarrow \tau)\\ &\qquad \Rightarrow ((\varphi(a) \Rightarrow \tau) \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau)) \Rightarrow (\tau \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau)).\end{aligned}$$

Using this with the first axiom above, by Detachment we derive

$$((\varphi(a) \Rightarrow \tau) \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau)) \Rightarrow (\tau \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau)).$$

Using this with the second axiom above, by Detachment we derive

$$\tau \Rightarrow \exists x\, (\varphi(x) \Rightarrow \tau).$$

We also have the tautology

$$(P \Rightarrow Q) \Rightarrow R \Rightarrow (R \Rightarrow P) \Rightarrow Q,$$

which we use in the form

$$(\tau \Rightarrow \exists x\,(\varphi(x) \Rightarrow \tau)) \Rightarrow \exists x\,\varphi(x) \Rightarrow (\exists x\,\varphi(x) \Rightarrow \tau) \Rightarrow \exists x\,(\varphi(x) \Rightarrow \tau).$$

By Detachment again, we derive

$$\exists x\,\varphi(x) \Rightarrow (\exists x\,\varphi(x) \Rightarrow \tau) \Rightarrow \exists x\,(\varphi(x) \Rightarrow \tau).$$

We have now written down a formal proof (with explanations) of this last sentence. This basically takes care of case 1 above. For case 2, we have the tautologies

$$\neg\varphi(a) \Rightarrow (\varphi(a) \Rightarrow \tau),$$
$$\varphi(a) \Rightarrow \exists x\,\varphi(x).$$

By Detachment by means of the appropriate tautologies, we derive

$$\neg\exists x\,\varphi(x) \Rightarrow \neg\varphi(a),$$
$$\neg\varphi(a) \Rightarrow \exists x\,(\varphi(x) \Rightarrow \tau),$$
$$\neg\exists x\,\varphi(x) \Rightarrow \exists x\,(\varphi(x) \Rightarrow \tau).$$

We can combine the two cases by Detachment and tautologies to derive σ. So σ is a logical theorem—which we already knew, if we accepted the Completeness Theorem.

Now that we know how to derive the logical truths, the next step is to be able derive true sentences from axioms:

Definition 11. Suppose Γ is a collection of sentences. The sentences that are **derivable** from Γ, or that can be **derived** from Γ, are defined recursively:

1. Every logical theorem is derivable from Γ.
2. Every sentence in Γ is derivable from Γ.
3. If σ and $(\sigma \Rightarrow \tau)$ are derivable from Γ, then so is τ.

If a sentence σ is derivable from Γ, we can express this with the syntactic turnstile, writing

$$\Gamma \vdash \sigma.$$

We may want to have a name for the relation thus symbolized by $\vdash$. I propose to call it **derivation,** or more precisely **syntactic derivation.**[22]

[22]The adjective *syntactic* serves as a reminder of what is involved. See footnote 20 above.

If σ is derivable from Γ, then, as with logical theorems, this is shown with a **formal proof:** this is now a list τ, τ', τ'', …, σ of sentences, ending with σ, in which each entry is either:

1. a logical theorem, or
2. a sentence in Γ, or
3. a sentence ρ, where sentences π and $(\pi \Rightarrow \rho)$ come earlier in the list.

Theorem 10. *Sentences that are derivable from a collection are logical consequences of that collection: if $\Gamma \vdash \sigma$, then $\Gamma \models \sigma$.*

Proof. Let Γ be a collection of sentences. We prove by induction that the collection of sentences derivable from Γ is a collection of logical consequences of Γ.

1. Logical theorems are logically true by Theorem 9 and are therefore logical consequences of Γ.

2. Every sentence in Γ is trivially a logical consequence of Γ.

3. If σ and $(\sigma \Rightarrow \tau)$ are derivable from Γ and are logical consequences of Γ, then τ is a logical consequence of Γ by Theorem 5. □

The converse of this theorem is the Completeness Theorem, given as a porism to Theorem 167 in Appendix B. This theorem ensures that our method of obtaining logical consequences through syntactic derivation is *complete* in the sense that every logical consequence can be so derived. The syntactic notion of derivation completely captures the semantic notion of logical entailment.

Nonetheless, suppose Γ is a collection of axioms that we can write down or at least describe. (It may be an infinite collection.) The collection of logical consequences of Γ must itself be *incomplete,* in that there will be some sentence such that neither itself nor its negation is a logical consequence of Γ. This is **Gödel's Incompleteness Theorem,** which is proved in Appendix C as Theorem 168. The precise statement of this result requires a precise formulation of what it means to *write down* Γ. There should be an *algorithm* for writing down the sentences of Γ. Consequently, there is no algorithm for writing down *all* true $\in$-sentences. The best we can do is try to identify those axioms from which all theorems of mathematics known so far can be derived.

Our axioms are supposed to be *true.* We cannot *prove* their truth in any meaningful way, unless they are *logically true.* Each axiom does constitute a one-line proof of itself from the collection of all axioms, but this tells us nothing. We might hope to prove that our axioms are **consistent,** that is, no contradiction is derivable from them. This hope is dashed by Gödel's **Second Incompleteness Theorem,** also in Appendix C. In Chapter 6, however, we shall be able to show that, if certain collections of axioms are indeed consistent, then certain larger collections are also consistent.

Meanwhile, another standard tool in deriving logical consequences is the following.

Theorem 11. *These sentences are logically true:*

$$(\forall x\, \neg\varphi(x) \Leftrightarrow \neg\exists\varphi\,(x)), \qquad (\exists x\, \neg\varphi(x) \Leftrightarrow \neg\forall x\, \varphi(x)).$$

Proof. We show that they are logical theorems. We have

$$\begin{gathered}
\vdash (\varphi(a) \Rightarrow \exists x\, \varphi(x)),\\
\vdash (\neg\neg\varphi(a) \Rightarrow \varphi(a)),\\
\vdash ((\neg\neg\varphi(a) \Rightarrow \varphi(a)) \Rightarrow ((\varphi(a) \Rightarrow \exists x\, \varphi(x)) \Rightarrow (\neg\neg\varphi(a) \Rightarrow \exists x\, \varphi(x)))),\\
\vdash ((\varphi(a) \Rightarrow \exists x\, \varphi(x)) \Rightarrow (\neg\neg\varphi(a) \Rightarrow \exists x\, \varphi(x))),\\
\vdash (\neg\neg\varphi(a) \Rightarrow \exists x\, \varphi(x)).
\end{gathered}$$

Assuming a does not occur in $\varphi(x)$, we have then

$$\vdash (\exists x\, \neg\neg\varphi(x) \Rightarrow \exists x\, \varphi(x)).$$

Similarly,

$$\vdash (\exists x\, \varphi(x) \Rightarrow \exists x\, \neg\neg\varphi(x)).$$

By using the tautology $((P \Rightarrow Q) \Rightarrow ((Q \Rightarrow P) \Rightarrow (P \Leftrightarrow Q)))$, we obtain

$$\vdash (\exists x\, \neg\neg\varphi(x) \Leftrightarrow \exists x\, \varphi(x)).$$

Then, by means of the tautology $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$, we obtain the first claim. The second claim is established similarly. □

We shall not generally write out proofs in such detail.

In our theorems so far, we have made no reference to sets or the symbol $\in$. Now we do, in our first theorem that is specifically about sets. Russell expressed this observation in a letter [29] to Frege.

Theorem 12 (Russell Paradox)**.** *There is no set consisting precisely of the sets that do not contain themselves:*

$$\neg\exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x).$$

Proof. Suppose on the contrary that

$$\exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x).$$

By definition, there is a set a such that

$$\forall x\,(x \in a \Leftrightarrow x \notin x).$$

Then in particular, by Specialization,

$$a \in a \Leftrightarrow a \notin a.$$

This is a contradiction; in particular, it is false. Therefore, by Contradiction, the claim holds. □

I said this theorem was about sets. And yet the theorem is logically true, since we have

$$\begin{gathered}
\vdash (\forall x\,(x \in a \Leftrightarrow x \notin x) \Rightarrow (a \in a \Leftrightarrow a \notin a)),\\
\vdash (\forall x\,(x \in a \Leftrightarrow x \notin x) \Rightarrow (b \in b \Leftrightarrow b \notin b)),\\
\vdash (\exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x) \Rightarrow (b \in b \Leftrightarrow b \notin b)),\\
\vdash \neg\exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x).
\end{gathered}$$

So really, the theorem is about what happens when we have a binary predicate (which is what $\in$ is).

For example, we might suppose x and y stand for men in a village, and instead of $x \in y$, we can consider the formula x *is shaved by* y (or y *shaves* x). Then we get the **Barber Paradox,** reported by Russell: There can be no man in a village who shaves precisely those men in the village that do not shave themselves; for if there were such a man, he would shave himself if and only if he didn't. (There could however be a *woman* in the village who shaves exactly those men that do not shave themselves.)

2.6. Relations and classes

We can now start to complete the picture given above in Figure 2.2. Suppose $\varphi(x)$ is a singulary formula. If $\varphi(a)$ is true for some set a, then a is said to **satisfy** φ. The collection of those sets that satisfy φ can be denoted by

$$\{x\colon \varphi(x)\}.$$

Such a collection is called a **class.** It can also be called a **singulary relation** on $\mathbf{V}$. The relation $\{x\colon \varphi(x)\}$ is said to be **defined** by the formula φ.

The Russell Paradox, Theorem 12 above, is that the class $\{x\colon x \notin x\}$ is not a set.[23] Indeed, we have defined classes in general only after defining the class $\mathbf{V}$ of all sets. Therefore we cannot just assume that an arbitrary class will be a set, since in that case the class must already have been a member of $\mathbf{V}$.

We may choose to denote the class $\{x\colon \varphi(x)\}$ by a boldface[24] capital letter, such as $\boldsymbol{C}$. Then, instead of $\varphi(a)$, we may write

$$a \in \boldsymbol{C}.$$

The letter $\boldsymbol{C}$ here, like φ, is a syntactic variable. The reason for introducing it is twofold.

1. It is easier to write $\boldsymbol{C}$ than $\{x\colon \varphi(x)\}$, especially if φ is a long formula.
2. Different formulas may define the *same* class.

Indeed, we *define* two classes to be **equal,** or the **same,** if they have the same members. In other words, we consider classes only in extension. Equality is denoted by the sign

$$=,$$

the **equals-sign.** So we have

$$\{x\colon \varphi(x)\} = \{x\colon \psi(x)\}$$

if and only if we have

$$\forall x\,(\varphi(x) \Leftrightarrow \psi(x)).$$

[23] Frege had in effect assumed that all classes *were* sets. There is some scholarship aimed at recovering what is sound in Frege's work: see Burgess, *Fixing Frege* [4].

[24] In writing, boldface is indicated by a wavy underline.

If the latter sentence is indeed true, then the formulas φ and ψ can be called **equivalent.** So two formulas are equivalent if and only if they define the same class. The following is obvious, and indeed we assume it when we say that two classes are equal, rather than saying more precisely that one class is equal *to* another: the extra precision is unneeded.

Theorem 13. *For all classes* $\boldsymbol{C}$, $\boldsymbol{D}$, *and* $\boldsymbol{E}$,

$$\boldsymbol{C} = \boldsymbol{C},$$
$$\boldsymbol{C} = \boldsymbol{D} \Rightarrow \boldsymbol{D} = \boldsymbol{C},$$
$$\boldsymbol{C} = \boldsymbol{D} \;\&\; \boldsymbol{D} = \boldsymbol{E} \Rightarrow \boldsymbol{C} = \boldsymbol{E}.$$

As mentioned above (p. 17), there are also *binary* relations on $\mathbf{V}$; these are defined by **binary formulas,** which are formulas that have just two free variables. Suppose ψ is such a formula, and its free variables are x and y. Then we can write ψ as

$$\psi(x, y).$$

If t and u are terms, then by substituting t for each free occurrence of x, and u for each free occurrence of y, we obtain the formula denoted by

$$\psi(t, u).$$

We might obtain for example $\psi(x, x)$ or $\psi(y, x)$.[25]

If ψ is not a good formula, it might happen that, when we form $\psi(y, x)$ from $\psi(x, y)$, a new occurrence of y is bound, although (of course) the old occurrence of x at the same place was free. For example, suppose $\psi(x, y)$ is

$$\exists y\,(y \in x \;\&\; y \notin y) \;\&\; y \in x.$$

Then $\psi(a, b)$ is $\exists y\,(y \in a \;\&\; y \notin y) \;\&\; b \in a$, which will turn out be true for some a and b. However, $\psi(y, x)$ is

$$\exists y\,(y \in y \;\&\; y \notin y) \;\&\; x \in y,$$

which can be written as $\varphi(x, y)$; then $\varphi(a, b)$ is $\exists y\,(y \in y \;\&\; y \notin y) \;\&\; a \in b$, which is always false. In particular, although $\varphi(x, y)$ is $\psi(y, x)$,

[25] Note that $\psi(y, x)$ will never be the formula ψ. We wrote ψ as $\psi(x, y)$ because x comes before y in the alphabet.

the formula $\varphi(a, b)$ is not $\psi(b, a)$. We shall always avoid this problem by using good formulas.

Given the binary formula $\psi(x, y)$, we may introduce a symbol such as $\boldsymbol{R}$, and then, as another way of saying that $\psi(a, b)$ is true, we may write

$$a \mathrel{\boldsymbol{R}} b.$$

Then $\boldsymbol{R}$ can be understood to denote a **binary relation** on $\mathbf{V}$, namely the relation **defined** by $\psi(x, y)$. For the moment, $\boldsymbol{R}$ is a new kind of thing. It can be understood as the collection of *ordered pairs* (a, b) such that $a \mathrel{\boldsymbol{R}} b$; but we do not yet *officially* know what ordered pairs are (though we mentioned them on page 22). Later we shall define ordered pairs as certain sets, and then $\boldsymbol{R}$ will indeed by a class.

If we wish, we can define *ternary* relations, *quaternary* relations, and so forth, as far as we need to go.

2.7. Relations between classes and collections

We have defined the notion of a *class* and of a binary relation on $\mathbf{V}$. More informally, we may consider the collection of all classes, along with some binary relations on this collection. Indeed, we have already defined one such relation: equality. Then of course we have *inequality:* if classes $\boldsymbol{C}$ and $\boldsymbol{D}$ are not equal, they are **unequal,** and we may write

$$\boldsymbol{C} \neq \boldsymbol{D}.$$

Now suppose $\boldsymbol{C}$ is the class $\{x \colon \varphi(x)\}$, and $\boldsymbol{D}$ is $\{x \colon \psi(x)\}$. If $\forall x\,(\varphi(x) \Rightarrow \psi(x))$, then we write

$$\boldsymbol{C} \subseteq \boldsymbol{D},$$

saying that $\boldsymbol{C}$ is a **subclass** of $\boldsymbol{D}$, and $\boldsymbol{D}$ **includes** $\boldsymbol{C}$. If $\boldsymbol{C} \subseteq \boldsymbol{D}$, but $\boldsymbol{C} \neq \boldsymbol{D}$, then $\boldsymbol{C}$ is a **proper** subclass of $\boldsymbol{D}$, and $\boldsymbol{D}$ **properly** includes $\boldsymbol{C}$, and we may write

$$\boldsymbol{C} \subset \boldsymbol{D}.$$

Theorem 14. *For all classes $\boldsymbol{C}$ and $\boldsymbol{D}$,*

$$\boldsymbol{C} = \boldsymbol{D} \Leftrightarrow \boldsymbol{C} \subseteq \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \subseteq \boldsymbol{C}.$$

Proof. The claim is

$$\forall x\,(\varphi(x) \Leftrightarrow \psi(x)) \Leftrightarrow \forall x\,(\varphi(x) \Rightarrow \psi(x))\ \&\ \forall x\,(\psi(x) \Rightarrow \varphi(x)).$$

But this means $\varphi(a) \Leftrightarrow \psi(a)$ for every set a if and only if both $\varphi(a) \Rightarrow \psi(a)$ and $\psi(a) \Rightarrow \varphi(a)$ for every set a; and this is true. □

We shall have some occasion to use similar terminology and notation for collections in general. For example, the collection of $\in$-formulas includes the collection of quantifier-free $\in$-formulas.

2.8. Sets as classes

If a is a set, then the formula $x \in a$ defines a class. We shall consider this class to be the set a itself. Then a set is equal to a class if they have the same members, and two sets are equal if they have the same members. In particular, if $\boldsymbol{C}$ is the class $\{x\colon \varphi(x)\}$, then we can write

$$x = \boldsymbol{C}$$

as an abbreviation of the formula

$$\forall y\,(y \in x \Leftrightarrow \varphi(y)),$$

where y is a variable not occurring in $\varphi(x)$. As an abbreviation of the formula

$$\forall z\,(z \in x \Leftrightarrow z \in y),$$

we can write

$$x = y.$$

Since sets are now classes, Theorem 13 applies to them. A class $\boldsymbol{C}$ is a set if and only if

$$\exists y\,\forall x\,(x \in y \Leftrightarrow x \in \boldsymbol{C}).$$

For some kinds of classes, there will be easier ways to say that they are sets. Meanwhile, there is now another way to prove the Russell Paradox: Let $\boldsymbol{C}$ be the class defined by the singulary formula $x \notin x$. If a is a set, then $a \in \boldsymbol{C} \Leftrightarrow a \notin a$, so $\boldsymbol{C}$ and a have different members, and therefore $\boldsymbol{C} \neq a$. In short, $\boldsymbol{C}$ is not a set.

Things that are equal ought to have the same behavior. We can derive this from our first axiom: it is our first true sentence that is not *logically* true.

Axiom 1 (Equality). *Equal sets are members of the same sets:*

$$\forall x\,\forall y\,\forall z\,(x = y \Rightarrow (x \in z \Leftrightarrow y \in z)). \tag{2.4}$$

The expression in (2.4) is really an abbreviation for

$$\forall x\,\forall y\,(\forall z\,(z \in x \Leftrightarrow z \in y) \Rightarrow \forall z\,(x \in z \Leftrightarrow y \in z)).$$

For all sets a, we now have

$$\forall x\,\forall y\,(x = y \Rightarrow (x \in a \Leftrightarrow y \in a)). \tag{2.5}$$

By the definition of equality of sets, we have

$$\forall x\,\forall y\,(x = y \Rightarrow (a \in x \Leftrightarrow a \in y)). \tag{2.6}$$

Each of the last two sentences is a part of Theorem 16 below. This is about singulary formulas, and we shall prove it by induction. Now, we did not exactly define singulary formulas recursively. We defined *formulas* recursively, and we defined the *free variables* of formulas recursively; but then we took the non-recursive step of defining singulary formulas as formulas with just one free variable. Nonetheless, our inductive proof will be justified by the following.

Theorem 15. *Suppose x is a variable, and Γ is a collection of formulas meeting the following conditions.*

1. *Every singulary atomic formula $\varphi(x)$ is in Γ.*
2. *If $\varphi(x)$ is in Γ, then so is $\neg\varphi(x)$.*
3. *If $\varphi(x)$ and $\psi(x)$ are in Γ, and σ is a sentence, then $(\varphi(x) \Rightarrow \psi(x))$ and $(\varphi(x) \Rightarrow \sigma)$ and $(\sigma \Rightarrow \psi(x))$ are in Γ.*
4. *Suppose $\varphi(x, y)$ is a binary formula such that, for each constant a, the formula $\varphi(x, a)$ is in Γ. Then $\exists y\,\varphi(x, y)$ is in Γ.*

Then Γ consists of the singulary formulas with the free variable x.

Theorem 16. *For all singulary formulas $\varphi(x)$ in which y does not occur,*

$$\forall x\,\forall y\,(x = y \Rightarrow (\varphi(x) \Leftrightarrow \varphi(y))).$$

Proof. We prove the claim by induction, as follows.

1. The cases where $\varphi(x)$ is $x \in a$ or $a \in x$ are taken care of by (2.5) and (2.6). Now suppose $\varphi(x)$ is $x \in x$. Given arbitrary sets a and b such that $a = b$, we want to show

$$a \in a \Leftrightarrow b \in b.$$

By (2.5) and (2.6), we have $a \in a \Leftrightarrow b \in a$ and $b \in a \Leftrightarrow b \in b$.

2. If the claim is true when φ is ψ, then it is true when φ is $\neg\psi$, because of the tautology

$$(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q).$$

3. If the claim is true when φ is ψ or χ, and σ is a sentence, then the claim is true when φ is $\psi \Rightarrow \chi$ or $\psi \Rightarrow \sigma$ or $\sigma \Rightarrow \psi$, because of the tautologies

$$(P \Leftrightarrow Q) \mathbin{\&} (R \Leftrightarrow S) \Rightarrow ((P \Rightarrow R) \Leftrightarrow (Q \Rightarrow S)),$$
$$(P \Leftrightarrow Q) \Rightarrow ((P \Rightarrow R) \Leftrightarrow (Q \Rightarrow R)),$$
$$(P \Leftrightarrow Q) \Rightarrow ((R \Rightarrow P) \Leftrightarrow (R \Rightarrow Q)).$$

4. Finally, suppose that, for some binary formula $\psi(x, z)$, for all sets a, the claim is true when $\varphi(x)$ is $\psi(x, a)$. We want to show

$$x = y \Rightarrow (\exists z\, \psi(x, z) \Leftrightarrow \exists z\, \psi(y, z))$$

(where y does not occur in $\psi(x, z)$). But if $b = c$, and $\exists y\, \psi(b, y)$, then $\psi(b, a)$ for some set a, and then $\psi(c, a)$ by inductive hypothesis, so $\exists z\, \psi(c, z)$. This establishes what is desired. □

In another version of the logic of set theory, equality is accepted, along with membership, as a fundamental notion. This means making the following adjustments:

1. Equations $t = u$ (where t and u are terms) are counted as atomic formulas.
2. The equation $a = b$ is defined to be true if $a = b$.
3. Theorem 16 is counted as being logically true: it is a logical axiom.
4. In particular, Axiom 1 is counted as a logical axiom.
5. A nonlogical axiom is then needed, namely

$$\forall x\, \forall y\, (\forall z\, (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$$

(which for us is true by definition); this axiom is called something like the **Axiom of Extension.**

Either way, we get to where we are now.

2.9. Operations on classes

There are many ways to combine two singulary formulas into a new singulary formula. These correspond to ways of combining classes. Some of these ways are given special names and symbols:

$$\begin{aligned}\boldsymbol{C} \smallsetminus \boldsymbol{D} &= \{x\colon x \in \boldsymbol{C} \mathbin{\&} x \notin \boldsymbol{D}\},\\ \boldsymbol{C} \cap \boldsymbol{D} &= \{x\colon x \in \boldsymbol{C} \mathbin{\&} x \in \boldsymbol{D}\},\\ \boldsymbol{C} \cup \boldsymbol{D} &= \{x\colon x \in \boldsymbol{C} \lor x \in \boldsymbol{D}\};\end{aligned}$$

these are the **complement** of $\boldsymbol{D}$ in $\boldsymbol{C}$, the **intersection** of $\boldsymbol{C}$ and $\boldsymbol{D}$, and the **union** of $\boldsymbol{C}$ and $\boldsymbol{D}$. The complement of $\boldsymbol{D}$ in $\mathbf{V}$ is simply the **complement** of $\boldsymbol{D}$ and can be denoted by

$$\boldsymbol{D}^{\mathrm{c}}.$$

None of these combinations of $\boldsymbol{C}$ and $\boldsymbol{D}$ makes special use of the relation of membership of *sets* symbolized by $\in$. We used the symbol $\in$, but we could have done without this. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are defined by $\varphi(x)$ and $\psi(x)$ respectively, then, for example, $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ is defined by $\varphi(x) \mathbin{\&} \neg\psi(x)$.

By making use of membership of sets, we can obtain new classes from a single class as follows:

$$\begin{aligned}\bigcup \boldsymbol{C} &= \{x\colon \exists y\,(y \in \boldsymbol{C} \mathbin{\&} x \in y)\},\\ \bigcap \boldsymbol{C} &= \{x\colon \forall y\,(y \in \boldsymbol{C} \Rightarrow x \in y)\},\\ \mathscr{P}(\boldsymbol{C}) &= \{x\colon \forall y\,(y \in x \Rightarrow y \in \boldsymbol{C}\};\end{aligned}$$

these are the **union, intersection,** and **power class** of $\boldsymbol{C}$. We have

$$\mathscr{P}(\boldsymbol{C}) = \{x\colon x \subseteq \boldsymbol{C}\}.$$

Finally, classes can be formed from no, one, and two sets:

$$\begin{aligned}0 &= \{x\colon x \neq x\},\\ \{a\} &= \{x\colon x = a\},\\ \{a, b\} &= \{x\colon x = a \lor x = b\};\end{aligned}$$

these are the **null class,** the **singleton** of a, and the **pair** of a and b. If $a = b$, then the pair of a and b is the singleton of a. If $\boldsymbol{C} \cap \boldsymbol{D} = 0$, then $\boldsymbol{C}$ is **disjoint** from $\boldsymbol{D}$. In this case, since $\boldsymbol{C} \cap \boldsymbol{D} = \boldsymbol{D} \cap \boldsymbol{C}$, the two classes themselves are simply **disjoint.**

3. The natural numbers

3.1. The collection of natural numbers

Having constants in our language commits us to the existence of sets. Let us now say something about *which* sets exist. Since all sets are classes, we shall generally try to say which classes are sets.

We have to be careful. If $\boldsymbol{C}$ is the class $\{x\colon x \notin x\}$, then we know by the Russell Paradox that $\boldsymbol{C}$ is not a set. However, if $\boldsymbol{C}$ were a set, it would be a member of itself. In particular, we cannot know which sets belong to $\boldsymbol{C}$ unless we know whether $\boldsymbol{C}$ is a set.[1] Our next axioms do not appear to have this ambiguity.

Axiom 2 (Null set). *0 is a set:*

$$\exists x\, \forall y\, (y \notin x).$$

Axiom 3 (Adjunction). $a \cup \{b\}$ *is always a set:*

$$\forall x\, \forall y\, \exists z\, \forall w\, (w \in z \Leftrightarrow w \in x \lor w = y).$$

We can immediately derive:

Theorem 17 (Singling and Pairing). $\{a\}$ *and* $\{a, b\}$ *are always sets.*

Proof. $\{a\} = 0 \cup \{a\}$ and $\{a, b\} = \{a\} \cup \{b\}$. □

As a special case, we have the sets 0, $\{0\}$, $\{\{0\}\}$, $\{\{\{0\}\}\}$, $\{\{\{\{0\}\}\}\}$, and so on. These sets *could* serve as definitions of the natural numbers 0, 1, 2, 3, 4, and so on.[2] An inconvenience is that the sets all have one element each. However, given a set a, we also have that $a \cup \{a\}$ is a set. Let us write

$$a' = a \cup \{a\}.$$

Then we have the sets 0, $0'$, $0''$, $0'''$, and so on. We shall take *these* as the official natural numbers:

[1] The term is that the definition of $\boldsymbol{C}$ is *impredicative*.
[2] Zermelo [35] defines the natural numbers this way.

Definition 12. The **natural numbers** are given recursively by two rules:

1. 0 is a natural number.
2. If n is a natural number, then so is n'.

Let us denote the collection of natural numbers by

$$\mathbb{N}.$$

Then we may write

$$\mathbb{N} = \{0, 0', 0'', \dots\}.$$

There are standard names for some elements of $\mathbb{N}$:

$$\begin{aligned} 1 &= 0' = \{0\}, \\ 2 &= 1' = \{0, 1\}, \\ 3 &= 2' = \{0, 1, 2\}, \\ 4 &= 3' = \{0, 1, 2, 3\}, \end{aligned}$$

and so on. Note that 1 is now a set with just one element, 2 has just two elements, 3 has just three elements, and so forth. We may write

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

It is not clear whether $\mathbb{N}$ is a *class,* much less a set. The definition gives us a way to confirm that a particular set a is in $\mathbb{N}$: we just compare a with 0, 1, 2, and so on until we find a number that is equal to a. However, if $a \notin \mathbb{N}$, the definition does not show us a way to prove this. We shall investigate $\mathbb{N}$ further after looking at another consequence of our axioms; the existence of the *ordered pair* as a set.

3.2. Relations and functions

By Theorem 17, given sets a and b we can define

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

This set is the **ordered pair** of a and b. In case $a = b$, we have $(a, b) = (a, a) = \{\{a\}\}$. The sole purpose of the definition of an ordered pair is to make the following true.

Theorem 18. $(a,b)=(c,d) \Leftrightarrow a=c \,\&\, b=d.$

A binary formula $\varphi(x,y)$ can now be understood to define the class

$$\{z\colon \exists x\,\exists y\,(z=(x,y) \,\&\, \varphi(x,y))\}.$$

We may write this class also as

$$\{(x,y)\colon \varphi(x,y)\}. \tag{3.1}$$

A binary relation is now such a class. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are classes, then the class $\{(x,y)\colon x\in\boldsymbol{C} \,\&\, y\in\boldsymbol{D}\}$ is denoted by

$$\boldsymbol{C}\times\boldsymbol{D};$$

this is the **Cartesian product** of $\boldsymbol{C}$ and $\boldsymbol{D}$.

The notation in (3.1) is similar to the notation for the *image* of a class under a *function.* A binary relation $\boldsymbol{F}$ is a **function** if

$$\forall x\,\forall y\,\forall z\,(x\;\boldsymbol{F}\;y \,\&\, x\;\boldsymbol{F}\;z \Rightarrow y=z).$$

When $\boldsymbol{F}$ is such, and $a\;\boldsymbol{F}\;b$, we can use $\boldsymbol{F}(a)$ as a name for b. Then we can use for $\boldsymbol{F}$ itself the notation

$$x\mapsto\boldsymbol{F}(x).$$

For example, if $\boldsymbol{F}$ is the function $\{(x,y)\colon y=x\cup\{x\}\}$, that is, $\{(x,y)\colon y=x'\}$, then we can write this function as $x\mapsto x'$. In general, the **domain** of a function $\boldsymbol{F}$ is the class $\{x\colon \exists y\, x\;\boldsymbol{F}\;y\}$; this can be denoted by

$$\mathrm{dom}(\boldsymbol{F}).$$

If $\boldsymbol{C}\subseteq\mathrm{dom}(\boldsymbol{F})$, then the class

$$\{y\colon \exists x\,(x\in\boldsymbol{C} \,\&\, x\;\boldsymbol{F}\;y)\}$$

can be denoted by either of

$$\{\boldsymbol{F}(x)\colon x\in\boldsymbol{C}\}, \qquad\qquad \boldsymbol{F}[\boldsymbol{C}];$$

this is the **image** of $\boldsymbol{C}$ under $\boldsymbol{F}$. If $\boldsymbol{C}=\mathrm{dom}(\boldsymbol{F})$, then $\boldsymbol{F}[\boldsymbol{C}]$ is the **range** of $\boldsymbol{F}$ and can be denoted by

$$\mathrm{rng}(\boldsymbol{F}).$$

If also $\operatorname{rng}(\boldsymbol{F}) \subseteq \boldsymbol{D}$, then we may say that $\boldsymbol{F}$ is a function **from $\boldsymbol{C}$ to $\boldsymbol{D}$**. More generally, if $\boldsymbol{C} \subseteq \operatorname{dom}(\boldsymbol{F})$, we may want to consider the **restriction** of $\boldsymbol{F}$ to $\boldsymbol{C}$, namely the function $\{(x, y) \colon x \in \boldsymbol{C} \mathbin{\&} \boldsymbol{F}(x) = y\}$, which can be denoted by

$$\boldsymbol{F} \restriction \boldsymbol{C}.$$

For example, we may have two functions $\boldsymbol{F}$ and $\boldsymbol{G}$ whose domains include $\boldsymbol{C}$; if $\boldsymbol{F} \restriction \boldsymbol{C} = \boldsymbol{G} \restriction \boldsymbol{C}$, we may say that $\boldsymbol{F}$ and $\boldsymbol{G}$ **agree on $\boldsymbol{C}$**.

We may consider restrictions in a more general sense. If $\boldsymbol{R}$ is an arbitrary relation, and the relation $\{(x, y) \colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} y\}$ is a function whose domain is $\boldsymbol{C}$, then $\boldsymbol{R}$ may be described as being a function **on $\boldsymbol{C}$** (even though $\boldsymbol{R}$ itself is not a function, simply).

If we have two classes, $\boldsymbol{F}$ and $\boldsymbol{C}$, such that

1) $\boldsymbol{F}$ is a function on $\boldsymbol{C}$, and
2) $\boldsymbol{F}[C] \subseteq \boldsymbol{C}$,

then $\boldsymbol{C}$ is **closed** under $\boldsymbol{F}$, and $\boldsymbol{F}$ is a **singulary operation** on $\boldsymbol{C}$. If one of the two conditions is not met, then we may say that $\boldsymbol{F}$ is not a **well-defined** operation on $\boldsymbol{C}$.

If $\boldsymbol{R}$ is a binary relation, then the **converse** of $\boldsymbol{R}$ is the binary relation

$$\{(y, x) \colon x \mathrel{\boldsymbol{R}} y\};$$

this can be denoted by

$$\breve{\boldsymbol{R}}.$$

A function $\boldsymbol{F}$ is **injective** if

$$\forall x \, \forall y \, (x \in \operatorname{dom}(\boldsymbol{F}) \mathbin{\&} y \in \operatorname{dom}(\boldsymbol{F}) \mathbin{\&} \boldsymbol{F}(x) = \boldsymbol{F}(y) \Rightarrow x = y).$$

If $\boldsymbol{F}$ is a function with domain $\boldsymbol{C}$, and $\breve{\boldsymbol{F}}$ is a function with domain $\boldsymbol{D}$, then $\boldsymbol{F}$ is a **bijection** from $\boldsymbol{C}$ to $\boldsymbol{D}$, and $\boldsymbol{C}$ is **equipollent** to $\boldsymbol{D}$, and we may write

$$\boldsymbol{C} \approx \boldsymbol{D}.$$

Theorem 19. *If $\boldsymbol{F}$ is a bijection from $\boldsymbol{C}$ to $\boldsymbol{D}$, then both $\boldsymbol{F}$ and $\breve{\boldsymbol{F}}$ are injective.*

Given two binary relations $\boldsymbol{R}$ and $\boldsymbol{S}$, we can **compose** them to get the relation

$$\{(x, z) \colon \exists y \, (x \mathrel{\boldsymbol{R}} y \mathbin{\&} y \mathrel{\boldsymbol{S}} z\}.$$

This relation can be denoted by

$$\boldsymbol{R}/\boldsymbol{S},$$

although some people will write

$$\boldsymbol{S}\circ\boldsymbol{R}.$$

The latter notation is standard when $\boldsymbol{R}$ and $\boldsymbol{S}$ are functions such that the range of $\boldsymbol{R}$ is included in the domain of $\boldsymbol{S}$. In this case, $\boldsymbol{R}/\boldsymbol{S}$ or $\boldsymbol{S}\circ\boldsymbol{R}$ is a function with the same domain as $\boldsymbol{R}$. For example, if $\boldsymbol{F}$ is a bijection from $\boldsymbol{C}$ to $\boldsymbol{D}$, then $\boldsymbol{F}\circ\breve{\boldsymbol{F}}=\{(y,y)\colon y\in\boldsymbol{D}\}$ and $\breve{\boldsymbol{F}}\circ\boldsymbol{F}=\{(x,x)\colon x\in\boldsymbol{C}\}$.

Theorem 20. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\begin{gathered}\boldsymbol{C}\approx\boldsymbol{C},\\ \boldsymbol{C}\approx\boldsymbol{D}\Rightarrow\boldsymbol{D}\approx\boldsymbol{C},\\ \boldsymbol{C}\approx\boldsymbol{D}\mathbin{\&}\boldsymbol{D}\approx\boldsymbol{E}\Rightarrow\boldsymbol{C}\approx\boldsymbol{E}.\end{gathered}$$

Instead of saying that $\boldsymbol{C}$ is equipollent to $\boldsymbol{D}$, we are allowed by the theorem to say simply that $\boldsymbol{C}$ and $\boldsymbol{D}$ are equipollent (or $\boldsymbol{D}$ and $\boldsymbol{C}$ are equipollent).

If all we know is that $\boldsymbol{F}$ is an injective function with domain $\boldsymbol{C}$, and $\boldsymbol{F}[\boldsymbol{C}]\subseteq\boldsymbol{D}$, then $\boldsymbol{C}$ **embeds** in $\boldsymbol{D}$, and we may write

$$\boldsymbol{C}\preccurlyeq\boldsymbol{D}.$$

Immediately, we have

Theorem 21. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\begin{gathered}\boldsymbol{C}\approx\boldsymbol{D}\Rightarrow\boldsymbol{C}\preccurlyeq\boldsymbol{D},\\ \boldsymbol{C}\preccurlyeq\boldsymbol{C},\\ \boldsymbol{C}\preccurlyeq\boldsymbol{D}\mathbin{\&}\boldsymbol{D}\preccurlyeq\boldsymbol{E}\Rightarrow\boldsymbol{C}\preccurlyeq\boldsymbol{E}.\end{gathered}$$

The question of what happens when $\boldsymbol{C}\preccurlyeq\boldsymbol{D}$ and $\boldsymbol{D}\preccurlyeq\boldsymbol{C}$ will be dealt with in Chapter 5. Meanwhile, if $\boldsymbol{C}$ and $\boldsymbol{D}$ are not equipollent, we may write

$$\boldsymbol{C}\not\approx\boldsymbol{D}.$$

If $\boldsymbol{C}\preccurlyeq\boldsymbol{D}$, but $\boldsymbol{C}\not\approx\boldsymbol{D}$, we write

$$\boldsymbol{C}\prec\boldsymbol{D}.$$

Theorem 22. *For all classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$,*

$$\boldsymbol{C} \prec \boldsymbol{D} \Rightarrow \boldsymbol{C} \not\approx \boldsymbol{D},$$
$$\boldsymbol{C} \prec \boldsymbol{D} \,\&\, \boldsymbol{D} \prec \boldsymbol{E} \Rightarrow \boldsymbol{C} \prec \boldsymbol{E}.$$

3.3. The class of formal natural numbers

We have that $x \mapsto x'$ is a function on $\mathbf{V}$; let us refer to this function as **succession,** or **set-theoretic succession** if we need to be more precise. The recursive definition of $\mathbb{N}$, Definition 12, means simply that every collection of sets that contains 0 and is closed under succession includes $\mathbb{N}$. In short, the definition means that a certain kind of proof by induction is possible. Let us call this **finite induction** (because later there will be *transfinite induction*). Perhaps the most basic application of finite induction is the following:

Theorem 23. *Let $\boldsymbol{D}$ be the class of all sets a such that*
1) $0 \in a$ and
2) a is closed under $x \mapsto x'$.
Then

$$\mathbb{N} \subseteq \bigcap \boldsymbol{D}.$$

Proof. If $a \in \boldsymbol{D}$, then immediately by finite induction, $\mathbb{N} \subseteq a$. Therefore $\mathbb{N} \subseteq \bigcap \boldsymbol{D}$. □

In the notation of the theorem, if $a \in \boldsymbol{D}$, then $\bigcap \boldsymbol{D} \subseteq a$. This means the class $\bigcap \boldsymbol{D}$ allows proof by finite induction in a restricted sense: if a meets the conditions of being in $\boldsymbol{D}$, then all elements of $\bigcap \boldsymbol{D}$ are in a. This is a restricted sense of finite induction, because a must be a set, not an arbitrary collection. If $\mathbb{N}$ should be a set, then it would meet the conditions, so $\bigcap \boldsymbol{D} \subseteq \mathbb{N}$; by the theorem itself then, $\mathbb{N} = \bigcap \boldsymbol{D}$. But perhaps $\mathbb{N}$ is not a set. Indeed, for all we know so far, $\boldsymbol{D}$ may be empty, so that $\bigcap \boldsymbol{D} = \mathbf{V}$. In this case, there may still a proper class $\boldsymbol{C}$ that contains 0 and is closed under succession, although $\boldsymbol{C} \neq \mathbf{V}$; then $\boldsymbol{C} \subset \bigcap \boldsymbol{D}$, and therefore $\mathbb{N} \subset \bigcap \boldsymbol{D}$.

In many expositions of set theory, there is an *Axiom of Infinity,* which is that the class $\boldsymbol{D}$ is nonempty.[3] This axiom is a radical assumption,

[3] This is one of Zermelo's axioms [35].

and it would be premature to make it now; so we do not assume this axiom yet.

Even if we do not have $\mathbb{N}$ as a set, we know that some collections of its elements are sets. Indeed, the subclasses 0, $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, and so on are sets. In fact they are *elements* of $\mathbb{N}$ too, but let us ignore this for the moment. They are all members of the class $\boldsymbol{C}$ described in the following:

Theorem 24. *Let $\boldsymbol{C}$ be the class of all sets a such that, for all sets b in a, either*

1) $b = 0$, or

2) there is a set d in a such that $b = d'$.

That is, $\boldsymbol{C}$ is defined by $\forall y\,(y \in x \Rightarrow y = 0 \lor \exists z\,(z \in x \;\&\; y = z'))$. Then

$$\mathbb{N} \subseteq \bigcup \boldsymbol{C}.$$

Proof. We can prove the claim by finite induction.

1. Since $\{0\} \in \boldsymbol{C}$, we have $0 \in \bigcup \boldsymbol{C}$.
2. Suppose $n \in \bigcup \boldsymbol{C}$. Then $n \in a$ for some a in $\boldsymbol{C}$. Then $a \cup \{n'\} \in \boldsymbol{C}$, so $n' \in \bigcup \boldsymbol{C}$. Thus $\bigcup \boldsymbol{C}$ is closed under succession. □

We did not prove $\mathbb{N} = \bigcup \boldsymbol{C}$. Indeed, in the notation of the theorem, possibly $\boldsymbol{C}$ has an element a such that *every* element of a is b' for some element b of a. Such a set a is *ill-founded.* If such sets are allowed in $\boldsymbol{C}$, then $\bigcup \boldsymbol{C}$ may have elements that are definitely not in $\mathbb{N}$.

Definition 13. A class is **well-founded** if every nonempty subset has an element that is disjoint from that subset. That is, $\boldsymbol{C}$ is well-founded if and only if

$$\forall y\,(y \subseteq \boldsymbol{C} \;\&\; y \neq 0 \Rightarrow \exists z\,(z \in y \;\&\; z \cap y = 0)).$$

A class is **ill-founded** if it is not well-founded, that is, if it has a nonempty subset whose every element is not disjoint from it.

Some examples of ill-foundedness are as follows.

1. If $a \in a$, then $a \in a \cap \{a\}$, so $\{a\}$ is ill-founded. Since $\{a\} \subseteq a$, also a is ill-founded. Note here $a' = a$.
2. If $a \in b$ and $b \in a$, then $a \in b \cap \{a,b\}$ and $b \in a \cap \{a,b\}$, so $\{a,b\}$ is ill-founded. If also $a' = b$ and $b' = a$, then $a = b$.

3. If $a \in b$ and $b \in c$ and $c \in a$, then $\{a, b, c\}$ is ill-founded.
4. If there is an infinite *set* $\{a_0, a_1, a_2, \dots\}$, where $a_1 \in a_0$, and $a_2 \in a_1$, and so on, then the set is ill-founded.[4] Possibly $a_0 = a_1{}'$, and $a_1 = a_2{}'$, and so on.[5]

In the last theorem, we could require the elements of $\boldsymbol{C}$ to be well-founded. However, in the last example, it may be that $\{a_0, a_1, a_2, \dots\}$ is a proper class with no 'infinite' subsets.[6] Then the class is well-founded. This situation can arise when $\{a_0, a_1, a_2, \dots\} \cup \{0\}$ is itself a set, but every 'infinite' subset contains 0. This is actually not a problem in trying to obtain $\mathbb{N}$ as $\bigcup \boldsymbol{C}$ as in the theorem. At least, it is not a problem we can do anything about. I shall say more about this later in the section.

Meanwhile, another problem may arise. If $a_0 = a_1{}'$, and $a_1 = a_2{}'$, and so on, then

$$a_0 = a_1 \cup \{a_1\} = a_2 \cup \{a_2, a_1\} = a_3 \cup \{a_3, a_2, a_1\} = \cdots$$

For all we know, there may be some set that belongs to each of the sets a_0, a_1, a_2, and so on, but is not equal to any of them. This common element could be $a_0{}'$. Then $\{a_0, a_1, a_2, \dots\} \cup \{0\}$ could be a well-founded set as before, although $\{a_0{}'\} \cup \{a_0, a_1, a_2, \dots\} \cup \{0\}$ would be an ill-founded set, since neither element of the pair $\{a_0{}', a_0\}$ would be disjoint from the pair. Thus, in the last theorem, even if the elements of $\boldsymbol{C}$ are well-founded, maybe $\bigcup \boldsymbol{C}$ contains a_0, but not $a_0{}'$. To avoid this problem, we shall need another notion:

Definition 14. A class is called **transitive** if it *includes* each of its elements. That is, $\boldsymbol{C}$ is transitive if and only if

$$\forall y\, (y \in \boldsymbol{C} \Rightarrow y \subseteq \boldsymbol{C}),$$

or more suggestively (see Definition 20 below),

$$\forall x\, \forall y\, (x \in y \;\&\; y \in \boldsymbol{C} \Rightarrow x \in \boldsymbol{C}).$$

Now we define a subclass of the class defined in Theorem 24.

[4] One could write $\cdots \in a_2 \in a_1 \in a_0$, or $a_0 \ni a_1 \ni a_2 \ni \cdots$; but it must not be assumed that this implies, for example, $a_2 \in a_0$.

[5] In this case, $a_2 \in a_0$.

[6] We must speak informally here. We have no definition of infinite set.

Definition 15. We denote by

$$\omega$$

the class of all transitive, well-founded sets a that meet each of the following two conditions:

1. For all sets b in a, either
   a) $b = 0$, or
   b) there is a set c in a such that $b = c'$.
2. There is an element b of a such that $b' \notin a$.

Then the **formal natural numbers** compose the class

$$\bigcup \omega.$$

Evidently ω contains 0, and $\{0\}$, and $\{0, 0'\}$, and $\{0, 0', 0''\}$, and so on; but these are just the natural numbers themselves. Indeed, we shall be able to show

$$\bigcup \omega = \omega, \tag{3.2}$$

but this will take some work. Without the second condition on elements of ω, (3.2) might be false. Indeed, if this second condition were not imposed, and $\bigcup \omega$ were a set, then $\bigcup \omega$ would be an element of ω, but not of ω.

Lemma 3. *Every element of $\bigcup \omega$ is well-founded.*

Proof. If $n \in \bigcup \omega$, then $n \in a$ for some a in ω. But then a is transitive and well-founded, so $n \subseteq a$, and hence also n is well-founded. □

Lemma 4. *If $a \in \omega$ and $n \in a$, then $a \cup \{n'\} \in \omega$.*

Proof. The conclusion is trivially true if $n' \in a$; so we may assume $n' \notin a$. By transitivity of a, we have $n \subseteq a$. Since also $\{n\} \subseteq a$, we have $n' \subseteq a$. Thus $a \cup \{n'\}$ is transitive. Next, we show it is well-founded. We have $n' \notin n'$ (since $n' \notin a$). Then $\{n'\} \cap n' = 0$. Suppose $b \subseteq a \cup \{n'\}$ and $b \cap a \neq 0$. Then $b \cap a$ has an element c such that $c \cap b \cap a = 0$. But $c \subseteq a$, so c does not contain n', and therefore $c \cap b = 0$. Thus $a \cup \{n'\}$ is well-founded. Finally, since $n' \notin n'$, we have $n'' \neq n'$; therefore, if $n'' \in a \cup \{n'\}$, then $n'' \in a$, so $n'' \subseteq a$ and therefore $n' \in a$, which is assumed to be false. So $n'' \notin a \cup \{n'\}$. Therefore $a \cup \{n'\} \in \omega$. □

To prove any more, we shall need:

Axiom 4 (Separation). *Every subclass of a set is a set:*

$$\exists x\, x = \boldsymbol{C} \cap a.$$

Note that this axioms is really a *scheme* of axioms, one for each class.

The collection of axioms that we have so far—Equality, Null Set, Adjunction, and Separation—together with their logical consequences, can be called **General Set Theory,**[7] or GST. Since, as noted at the beginning of the chapter, some set does exist, the Null Set Axiom is a logical consequence of the Separation Axiom.

We can now make the following refinement of Theorem 24.

Theorem 25 (Finite Induction). *The class $\bigcup \omega$ is the smallest of the classes $\boldsymbol{D}$ such that*

1) $0 \in \boldsymbol{D}$,

2) for all sets b in $\boldsymbol{D}$, also $b' \in \boldsymbol{D}$.

That is, $\bigcup \omega$ is such a class, and it is included in every *such class. In particular, $\mathbb{N} \subseteq \bigcup \omega$.*

Proof. We have $\{0\} \in \omega$, so $0 \in \bigcup \omega$. Suppose $n \in \bigcup \omega$. Then $n \in a$ for some a in ω, so $a \cup \{n'\} \in \omega$ by the last lemma, and therefore $n' \in \bigcup \omega$. We have now shown that $\bigcup \omega$ is one of the classes $\boldsymbol{D}$.

Considering any one of these classes $\boldsymbol{D}$, suppose if possible $a \in \bigcup \omega \smallsetminus \boldsymbol{D}$. Then $a \in b$ for some b in ω. The class $b \smallsetminus \boldsymbol{D}$ is a set, by the Separation Axiom. Since $0 \in \boldsymbol{D}$, every element of $b \smallsetminus \boldsymbol{D}$ is c' for some element c of b, and in fact then $c \in b \smallsetminus \boldsymbol{D}$ (since otherwise $c' \in \boldsymbol{D}$). But $c \in c'$. Thus every element of $b \smallsetminus \boldsymbol{D}$ has nonempty intersection with this set. Since b is well-founded, $b \smallsetminus \boldsymbol{D}$ must be empty. Therefore $b \subseteq \boldsymbol{D}$. Consequently, $\bigcup \omega \subseteq \boldsymbol{D}$. □

So $\bigcup \omega$ admits finite induction for classes. Since, as far as our formal set theory is concerned, classes are the only collections of sets that we can talk about, we may assume $\mathbb{N} = \bigcup \omega$; that is, the formal natural numbers are just the natural numbers. We have not *proved* that $\mathbb{N}$ and $\bigcup \omega$ are the same, only that nothing in our theory will enable us to distinguish them. We *cannot* prove that $\mathbb{N}$ and $\bigcup \omega$ are the same. Indeed, let Γ consist of our axioms, together with the sentences $a \in \bigcup \omega$ and $a \neq 0$,

[7] The theory is so called by Boolos [2, p. 196], but is called STZ by Burgess [4, p. 223], for **S**zmielev and **T**arski with **Z**ermelo's Axiom of Separation.

$a \neq 1$, $a \neq 2$, and so on. By the Compactness Theorem in Appendix B, the collection Γ is consistent. Nonetheless, henceforth **natural number** will mean an element of $\bigcup \omega$.

The easiest use of finite induction is perhaps:

Lemma 5. *Every element of $\bigcup \omega$ is either 0 or n' for some n in $\bigcup \omega$.*

Lemma 6. *Every element of $\bigcup \omega$ is transitive.*

Proof. Trivially, 0 is transitive. If n in $\bigcup \omega$ is transitive, and $a \in n'$, then $a \in n$ or $a = n$, so $a \subseteq n$ and therefore $a \subseteq n'$. By finite induction, all elements of $\bigcup \omega$ are transitive. □

Theorem 26. *$\bigcup \omega$ is transitive.*

Proof. Trivially, $0 \subseteq \bigcup \omega$. Suppose $n \in \bigcup \omega$ and $n \subseteq \bigcup \omega$. Then $n' \subseteq \bigcup \omega$. By finite induction, $\bigcup \omega$ includes each of its elements. □

Theorem 27. *$\bigcup \omega \subseteq \omega$.*

Proof. Let $n \in \bigcup \omega$. Then n is transitive and well-founded, by Lemmas 3 and 6. Also, $n \subseteq \bigcup \omega$ by the last theorem, so every element of n is either 0 or m' for some m in $\bigcup \omega$. In the latter case, $m \in m'$ and $m' \in n$; but also $m' \subseteq n$, so $m \in n$. Finally, if $n \neq 0$, then $n = m'$ for some m, and then $m \in n$, but $m' \notin n$ (since n is well-founded). This shows $n \in \omega$. □

The reverse inclusion is in Theorem 43.

3.4. Arithmetic

By an **iterative structure,**[8] I mean a nonempty class, considered together with

1) a distinguished element of the class, and
2) a distinguished singulary operation on the class.

If the class is $\boldsymbol{C}$; the element, e; and the operation, $\boldsymbol{F}$; then we can write $\boldsymbol{C}$ as

$$(\boldsymbol{C}, e, \boldsymbol{F}).$$

Possibly $\boldsymbol{C}$ is a proper subclass of $\operatorname{dom}(\boldsymbol{F})$; but we shall not distinguish between $(\boldsymbol{C}, e, \boldsymbol{F})$ and $(\boldsymbol{C}, e, \boldsymbol{F} \restriction \boldsymbol{C})$. For example, $\bigcup \omega$ is an iterative

[8]This is my terminology; it is not standard.

structure, when considered with 0 and $x \mapsto x'$; in this situation, we may write $\bigcup \omega$ as

$$(\bigcup \omega, 0, {}').$$

If both $(\boldsymbol{C}, e, \boldsymbol{F})$ and $(\boldsymbol{D}, e, \boldsymbol{F})$ are iterative structures, and $\boldsymbol{D} \subseteq \boldsymbol{C}$, then $\boldsymbol{D}$ (or more precisely $(\boldsymbol{D}, e, \boldsymbol{F})$) is an **iterative substructure** of $\boldsymbol{C}$. For example, $(\bigcup \omega, 0, {}')$ is an iterative substructure of $(\mathbf{V}, 0, {}')$.

Generalizing some earlier terminology, we may say that an iterative structure admits **finite induction** if it has no proper iterative substructure. Theorem 25 is that $(\bigcup \omega, 0, {}')$ admits (formal) finite induction.

Theorem 28. *Succession on $\bigcup \omega$ is injective.*

Proof. Suppose $a \neq b$, but $a' = b'$. Then $a \cup \{a\} = b \cup \{b\}$, so in particular $a \in b \cup \{b\}$, and therefore $a \in b$; similarly $b \in a$. Then $\{a, b\}$ is ill-founded. But it is a subset of every transitive set that contains a. Therefore $a \notin \bigcup \omega$. □

In sum, we now have:

Theorem 29.

1. $0 \in \bigcup \omega$.
2. $\bigcup \omega$ *is closed under* $x \mapsto x'$.
3. $0 \neq n'$ *for any* n *in* $\bigcup \omega$.
4. *Succession is injective on* $\bigcup \omega$.
5. $(\bigcup \omega, 0, {}')$ *admits finite induction.*

Proof. The claim is a summary of Theorems 25 and 28, along with the observation that n' is never empty, since it contains n. □

The five conditions in the theorem are called the **Peano Axioms** [28], although Dedekind [9, II: §§ 71, 132] recognized them a bit earlier and understood them better.[9] In any case, for us they are not axioms, but follow from the *definition* of $\bigcup \omega$. A fundamental consequence of the Peano Axioms is the Theorem of Finite Recursion below.

A **homomorphism** from an iterative structure $(\boldsymbol{C}, e, \boldsymbol{F})$ to an iterative structure $(\boldsymbol{D}, f, \boldsymbol{G})$ is a function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that

$$\boldsymbol{H}(e) = f, \qquad \boldsymbol{H} \circ (\boldsymbol{F} \restriction \boldsymbol{C}) = \boldsymbol{G} \circ \boldsymbol{H}.$$

[9] I say this because, unlike Peano, Dedekind stated clearly that induction was not enough for proving the Theorem of Finite Recursion, Theorem 30 below.

This situation is depicted in Figure 3.1. Another way to write the second

$$\begin{tikzcd}
\{e\} \arrow[r] \arrow[d, "\boldsymbol{H}"'] & \boldsymbol{C} \arrow[r, "\boldsymbol{F}"] \arrow[d, "\boldsymbol{H}"'] & \boldsymbol{C} \arrow[d, "\boldsymbol{H}"] \\
\{f\} \arrow[r] & \boldsymbol{D} \arrow[r, "\boldsymbol{G}"'] & \boldsymbol{G}
\end{tikzcd}$$

Figure 3.1. A homomorphism of iterative structures

equation is

$$\forall x\,(x \in \boldsymbol{C} \Rightarrow \boldsymbol{H}(\boldsymbol{F}(x)) = \boldsymbol{G}(\boldsymbol{H}(x))).$$

We use the previous theorem to establish the following.

Theorem 30 (Finite Recursion)**.** *For every iterative structure* $(\boldsymbol{D}, e, \boldsymbol{F})$*, there is a unique homomorphism from* $(\bigcup\omega, 0, {}')$ *to* $(\boldsymbol{D}, e, \boldsymbol{F})$*.*

Proof. Let $\boldsymbol{C}$ be the class of all sets h such that, for some a in ω,

1) h is a function from a to $\boldsymbol{D}$,
2) $h(0) = e$, that is, $(0, e) \in h$, and
3) if $k' \in a$, so that $k \in a$, then $h(k') = \boldsymbol{F}(h(k))$, that is, if $(k, x) \in h$, then
$$(k', \boldsymbol{F}(x)) \in h.$$

Let $\boldsymbol{R} = \bigcup \boldsymbol{C}$. We first prove that, for each n in $\bigcup\omega$, there is b in $\boldsymbol{D}$ such that $n\,\boldsymbol{R}\,b$.

1. Since $\{0\} \in \omega$, and $0 \neq n'$ for any n, we have $\{(0, e)\} \in \boldsymbol{C}$, so $0\ \boldsymbol{R}\ e$.
2. Suppose $k\ \boldsymbol{R}\ a$. Then $a = h(k)$ for some h in $\boldsymbol{C}$. By Lemma 4, $\mathrm{dom}(h) \cup \{k'\} \in \omega$. If $k' = \ell'$, then $k = \ell$, by Theorem 28. Hence $h \cup \{k', \boldsymbol{F}(h(k))\} \in \boldsymbol{C}$, so $k'\ \boldsymbol{R}\ (\boldsymbol{F}(a))$.

Thus, by finite induction, for each n in $\bigcup\omega$ there is b in $\boldsymbol{D}$ such that $n\,\boldsymbol{R}\,b$.

We next prove that there is only one such b.

1. Suppose $0\ \boldsymbol{R}\ a$. Then $h(0) = a$ for some h in $\boldsymbol{C}$, but then also $h(0) = e$, so $e = a$.
2. Suppose, for some k in $\bigcup\omega$, there is just one set b such that $k\ \boldsymbol{R}\ b$. Say $k'\ \boldsymbol{R}\ c$. Then $h(k') = c$ for some h in $\boldsymbol{C}$. But also $h(k') = \boldsymbol{F}(h(k))$, and by our assumption $h(k)$ must be b, so $c = \boldsymbol{F}(b)$.

By finite induction again, $\boldsymbol{R}$ is a function on $\bigcup\omega$ with the desired properties.

By induction yet again, this function is unique. □

If the homomorphism guaranteed by the theorem is called $\boldsymbol{H}$, we may say that it is determined by the requirements

$$\boldsymbol{H}(0) = e, \qquad \boldsymbol{H}(x') = \boldsymbol{F}(\boldsymbol{H}(x)).$$

Now we can obtain the usual operations on $\bigcup\omega$.

Definition 16 (Addition)**.** For each m in $\bigcup\omega$, the operation $x \mapsto m+x$ on $\bigcup\omega$ is the homomorphism from $(\bigcup\omega, 0, ')$ to $(\bigcup\omega, m, ')$ determined by

$$m + 0 = m, \qquad m + n' = (m + n)'.$$

In particular, we have

$$m + 1 = m + 0' = (m + 0)' = m',$$

so we may write $m + 1$ for m'.

Lemma 7. *For all n and m in $\bigcup\omega$,*
1) $0 + n = n$;
2) $m' + n = m + n'$.

Theorem 31. *For all n, m, and k in $\bigcup\omega$,*
1) $n + m = m + n$;
2) $(n + m) + k = n + (m + k)$;

In fact an operation of addition satisfying the theorem can be defined on *any* iterative structure that admits induction.

Definition 17 (Multiplication)**.** For each m in $\bigcup\omega$, the operation $x \mapsto m\cdot x$ on $\bigcup\omega$ is the homomorphism from $(\bigcup\omega, 0, ')$ to $(\bigcup\omega, 0, x \mapsto x+m)$. That is,

$$m \cdot 0 = 0, \qquad m \cdot (n + 1) = m \cdot n + m.$$

Lemma 8. *For all n and m in $\bigcup\omega$,*
1) $0 \cdot n = 0$;

2) $(m+1)\cdot n = m\cdot n + n$.

Theorem 32. *For all n, m, and k in $\bigcup\omega$,*
1) $n\cdot m = m\cdot n$*;*
2) $n\cdot(m+k) = n\cdot m + n\cdot k$*;*
3) $(n\cdot m)\cdot k = n\cdot(m\cdot k)$*;*

As with addition, so with multiplication: an operation satisfying the theorem can be defined on any iterative structure that admits induction. However, the next theorem needs all of the Peano Axioms.

Theorem 33 (Cancellation)**.** *For all n, m, and k in $\bigcup\omega$,*
1) if $n+k=m+k$, then $n=m$;
2) if $n\cdot k = m\cdot k$, then $n=m$.

Definition 18 (Exponentiation)**.** For each m in $\bigcup\omega$, the operation $x\mapsto m^x$ on $\bigcup\omega$ is the homomorphism from $(\bigcup\omega, 0, ')$ to $(\bigcup\omega, 1, x\mapsto x\cdot m)$ determined by

$$m^0 = 1, \qquad\qquad m^{n+1} = m^n\cdot m.$$

Theorem 34. *For all n, m, and k in $\bigcup\omega$,*
1) $n^{m+k} = n^m\cdot n^k$*;*
2) $(n\cdot m)^k = n^k\cdot m^k$*;*
3) $(n^m)^k = n^{m\cdot k}$.

In contrast with addition and multiplication, exponentiation requires more than induction for its existence.

For some operations on $\bigcup\omega$, Theorem 30 as stated is not enough to establish their existence. One needs:

Theorem 35 (Finite Recursion with Parameter)**.** *Suppose $e\in\boldsymbol{D}$, and $\boldsymbol{F}$ is a function from $\bigcup\omega\times\boldsymbol{D}$ to $\boldsymbol{D}$. Then there is a* unique *function $\boldsymbol{G}$ from $\bigcup\omega$ to $\boldsymbol{D}$ such that*
1) $\boldsymbol{G}(0) = e$*, and*
2) $\boldsymbol{G}(n+1) = \boldsymbol{F}(n, \boldsymbol{G}(n))$ *for all n in $\bigcup\omega$.*

Proof. The function $\boldsymbol{G}$ will be such that $x\mapsto(x,\boldsymbol{G}(x))$ is the homomorphism from $(\bigcup\omega, 0, ')$ to $(\bigcup\omega\times\boldsymbol{D}, (0,e), (x,y)\mapsto(x,\boldsymbol{F}(x,y)))$. $\square$

Definition 19 (Factorial)**.** The operation $x \mapsto x!$ on $\bigcup \omega$ is the function $\boldsymbol{G}$ guaranteed by the theorem when $\boldsymbol{D}$ is $\bigcup \omega$ and e is 1 and $\boldsymbol{F}$ is $(x, y) \mapsto (x+1) \cdot y$. That is,

$$0! = 1, \qquad\qquad (n+1)! = (n+1) \cdot n!$$

3.5. Orderings

The relations $\subset$ and $\prec$ on $\mathbf{V}$ are examples of *orderings.*

Definition 20. A binary relation $\boldsymbol{R}$ on a class $\boldsymbol{C}$ is **irreflexive** if, for all a and b in $\boldsymbol{C}$,

$$a \mathrel{\boldsymbol{R}} b \Rightarrow a \neq b;$$

transitive, if

$$a \mathrel{\boldsymbol{R}} b \;\&\; b \mathrel{\boldsymbol{R}} c \Rightarrow a \mathrel{\boldsymbol{R}} c.$$

If $\boldsymbol{R}$ is both irreflexive and transitive on $\boldsymbol{C}$, it is an **ordering** of $\boldsymbol{C}$, and $\boldsymbol{C}$ is an **order** with respect to $\boldsymbol{R}$. So considered, $\boldsymbol{C}$ can be written as

$$(\boldsymbol{C}, \boldsymbol{R}).$$

If in addition $\boldsymbol{R}$ **connects** $\boldsymbol{C}$, that is, for all a and b in $\boldsymbol{C}$,

$$a \neq b \Rightarrow a \mathrel{\boldsymbol{R}} b \lor b \mathrel{\boldsymbol{R}} a,$$

then $\boldsymbol{R}$ **linearly orders** $\boldsymbol{C}$, and $\boldsymbol{R}$ is a **linear ordering** of $\boldsymbol{C}$, and $(\boldsymbol{C}, \boldsymbol{R})$ is a **linear order.** In this case, if $a \mathrel{\boldsymbol{R}} b$, we say that a is **less** than b with respect to $\boldsymbol{R}$. Also a is the **least** element of $\boldsymbol{C}$ if it is less than all other elements.

Note well that a transitive *relation* is not the same thing as a transitive *class* (even though a relation is technically a class). Membership may be transitive on a class that is not transitive: a trivial example is $\{1\}$, a more interesting example is $\{1, 2, 3\}$. Membership may fail to be transitive on a transitive class, for example, $\{0, \{0\}, \{\{0\}\}\}$.

By the definition, $(\mathbf{V}, \subset)$ and $(\mathbf{V}, \prec)$ are orders. They are not linear orders, because neither relation contains either of $(\{0\}, \{1\})$ and $(\{1\}, \{0\})$.

The converse of an ordering of a class is also an ordering of that class; of a linear ordering, a linear ordering. Often a linear ordering is denoted by a symbol like $<$, and then its converse is $>$. Also the relation defined by $x < y \lor x = y$ is denoted by $\leqslant$.[10]

To understand how $\bigcup \omega$ is ordered, we observe:

Theorem 36. *On $\bigcup \omega$, membership is the same as proper inclusion.*

Proof. Since elements of $\bigcup \omega$ are transitive and well-founded by Lemmas 3 and 6, for all k and n in $\bigcup \omega$ we have

$$k \in n \Rightarrow k \subset n.$$

We show the converse, namely

$$k \subset n \Rightarrow k \in n.$$

This is vacuously true when $n = 0$. Suppose it is true when $n = m$. If $m \in k$, then $m \subseteq k$ and hence $m + 1 \subseteq k$. So, supposing $k \subset m + 1$, we have $m \notin k$ and therefore $k \subseteq m$. Either $k = m$, or by inductive hypothesis, $k \in m$; in either case, $k \in m + 1$. $\square$

On $\bigcup \omega$ therefore, we can denote membership and proper inclusion by the same symbol,

$$<;$$

this orders $\bigcup \omega$, since partial inclusion orders all classes. If $m < n$, we may say that m is a **predecessor** of n.

Theorem 37. $(\bigcup \omega, <)$ *is a linear order.*

---

[10] I have chosen terminology so that the relations that we are interested in will have the simplest possible descriptions. We have a standard symbol, namely $\in$, for the relation defined by $x \in y$, but not for the relation defined by $x \in y \lor x = y$. Therefore, even though both $\subset$ and $\subseteq$ are standard symbols, I treat $\subset$ as more basic: it is *this* that I call an ordering (and $\in$ will be an ordering of $\bigcup \omega$). In many references, it is $\subseteq$ that is called an ordering; in that case, $\subset$ would be a *strict* ordering. In fact, $\subseteq$ is often called a *partial* ordering; but then an ordering is still called *linear* or *total* if it connects the class it orders. I have chosen not to require the use of an adjective in every case, but to let $\subset$ be an *ordering,* simply. Note that, while $\prec$ is now also an ordering, the relation $\preccurlyeq$ is not defined by $x \prec y \lor x = y$. Finally, since the words *order* and *ordering* are both available, I have decided to use the latter for the relation, and the former for the class that the relation orders.

Proof. We show, for all m and n in $\bigcup \omega$,

$$m \subseteq n \vee n \subseteq m.$$

This is trivially true when $n = 0$. Suppose it is true when $n = k$. If we do not have $m \subseteq k + 1$, then *a fortiori* we do not have $m \subseteq k$, so, by inductive hypothesis, we have $k \subset m$, that is, $k \in m$, so $k + 1 \subseteq m$. □

The connection between the ordering of $\bigcup \omega$ and the algebraic structure of $\bigcup \omega$ is given by:

Theorem 38. *If n and m are in $\bigcup \omega$, then*

$$m \leqslant n \Leftrightarrow \exists x\,(x \in \bigcup \omega \;\&\; m + x = n).$$

The theorem can be taken as a *definition* of $\leqslant$ on $\bigcup \omega$. Using this definition, one can prove the next two theorems.

Theorem 39. *For all n, m, and k in $\bigcup \omega$,*
1) $0 \leqslant n$*;*
2) $m \leqslant n$ *if and only if* $m + k \leqslant n + k$*;*
3) $m \leqslant n$ *if and only if* $m \cdot (k + 1) \leqslant n \cdot (k + 1)$*.*

Theorem 40. *For all m and n in $\bigcup \omega$,*
1) $m < n$ *if and only if* $m + 1 \leqslant n$*;*
2) $m \leqslant n$ *if and only if* $m < n + 1$*.*

Theorems 38 and 40 can be used to prove Theorem 37. The next theorem introduces a new proof-technique, admitted by certain orders.

Theorem 41 (Transfinite Induction)**.** *Suppose $\boldsymbol{C}$ is a class such that, for all n in $\bigcup \omega$,*

$$n \subseteq \boldsymbol{C} \Rightarrow n \in \boldsymbol{C}. \tag{3.3}$$

Then $\bigcup \omega \subseteq \boldsymbol{C}$.

Proof. Another way to write (3.3) is

$$n \subseteq \boldsymbol{C} \Rightarrow n + 1 \subseteq \boldsymbol{C}.$$

Then by induction, $n \subseteq \boldsymbol{C}$ for all n in $\bigcup \omega$. In particular, for all n in $\bigcup \omega$, we have $n + 1 \subseteq \boldsymbol{C}$, and therefore $n \in \boldsymbol{C}$. □

An application of transfinite induction is the following.

Theorem 42. *Every non-empty subclass of $\bigcup\omega$ has a least element with respect to $\leqslant$.*

Proof. Suppose $\boldsymbol{C}$ is a subclass of $\bigcup\omega$ with no least element. We show $\boldsymbol{C} = 0$, that is, $\bigcup\omega \smallsetminus \boldsymbol{C} = \bigcup\omega$. We use transfinite induction. Suppose $n \subseteq \bigcup\omega \smallsetminus \boldsymbol{C}$. Then $\boldsymbol{C} \subseteq \bigcup\omega \smallsetminus n$. Since n is the least element of $\bigcup\omega \smallsetminus n$, we must have $n \notin \boldsymbol{C}$, so $n \in \bigcup\omega \smallsetminus \boldsymbol{C}$. □

Finally, we can simplify notation with the following, which complements Theorem 27.

Theorem 43. $\omega = \bigcup\omega$.

Proof. It is enough to show $\omega \subseteq \bigcup\omega$. Suppose if possible $\omega \smallsetminus \bigcup\omega$ contains a. Then $a \in \omega$, so $a \subseteq \bigcup\omega$, and a is transitive, but $a \neq 0$. Hence a has an element n, which is in $\bigcup\omega$. Then $n+1 \subseteq a$, but $a \neq n+1$ (since $n+1 \in \bigcup\omega$), so a has an element r that is greater than n. Then $n+1 \leqslant r$ by Theorem 40, and therefore $n+1 \in a$ by transitivity of a. In short, a is nonempty, but closed under succession. This violates the last condition (in Definition 15) of being in ω. □

Now we have $\bigcup\omega = \omega$, so can write ω instead of $\bigcup\omega$.

3.6. Finite sets

Presumably a set is *finite* if and only if it is equipollent with some natural number. But we can define finite sets without referring to natural numbers as such, just by following the pattern of the *definition* of $\mathbb{N}$ (Definition 12):

Definition 21. The **finite** sets are given recursively by two rules:

1. 0 is finite.
2. If a is finite, then so is $a \cup \{b\}$ (for all sets b).

Sets that are not finite are **infinite.**

As with $\mathbb{N}$, so with the collection of finite sets: we should like to understand it as a class. We can try using an analogue of Definition 15, the definition of $\bigcup\omega$, which turns out to be ω itself. This does not work.

Indeed, suppose $\boldsymbol{C}$ is the class of all well-founded sets a such that, for all b in a, either $b = 0$, or $b = c \cup \{d\}$ for some set c in a and some set d. If ω is a set, then $\boldsymbol{C}$ contains $\{0\} \cup \{\omega \smallsetminus x : x \in \omega\}$, and therefore $\omega \in \bigcup \boldsymbol{C}$, although presumably ω is not finite. Well-foundedness does not prevent this problem; something else is needed.

Definition 22. A subset of $\mathscr{P}(a)$ it **inductive** if it contains 0 and is closed under each operation $x \mapsto x \cup \{b\}$, where $b \in a$. A set a is **formally finite** if it belongs to each inductive subset of $\mathscr{P}(a)$.

We aim to prove an analogue of Theorem 25; but for this, we need a characterization of the formally finite sets that involves natural numbers. We develop this now.

Theorem 44. *Every function whose domain is a natural number is a set, and then the range of the function is also a set.*

Proof. Since a function as such is a kind of class, we cannot speak of the 'class of functions with domain n' until we actually prove this theorem. In particular, we can prove this theorem by induction only for one theorem at a time. Given a function $\boldsymbol{F}$ whose domain is a natural number n, we can embed $\boldsymbol{F}$ in $\boldsymbol{F} \cup \{(x, 0) : x \in \bigcup \omega \smallsetminus n\}$, which is a function whose domain is $\bigcup \omega$. Suppose $\boldsymbol{G}$ is a function on $\bigcup \omega$. By induction, for each n in $\bigcup \omega$, the restriction of $\boldsymbol{G}$ to n is a set, and the image of n under $\boldsymbol{G}$ is a set. □

In particular, every class that is equipollent to a natural number is a set.

Lemma 9. *For all m and n in ω, if $m + 1 \approx n + 1$, then $m \approx n$.*

Proof. Suppose f is a bijection from $m + 1$ to $n + 1$. Say $f(k) = n$. Let

$$g = (f \smallsetminus \{(k, n), (m, f(m))\}) \cup \{(k, f(m)), (m, n)\}.$$

(See Figure 3.2. If $k = m$, then $g = f$.) Then $g \restriction m$ is a bijection from m to n. □

Theorem 45. *Every set is equipollent to at most one natural number.*

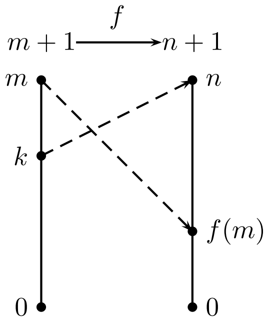

Figure 3.2. A bijection from a natural number to another

Proof. By finite induction, we show that equipollent natural numbers are equal. If $n \in \omega$, and $0 \approx n$, then $n = 0$. Suppose m is a natural number that is equipollent only to itself among natural numbers. If $m + 1$ is equipollent to some natural number, then that number must be $n + 1$ for some n, and therefore $m \approx n$ by the lemma, so $m = n$ and therefore $m + 1 = n + 1$. □

If $n \in \omega$ and $a \approx n$, we can now call n the **size** of a; we denote this size by

$$|a|.$$

Theorem 46. *If $|a| = n$ and $|b| = m$ and $a \cap b = 0$, then $a \cup b$ is a finite set, and*

$$|a \cup b| = n + m.$$

Proof. Use finite induction. Assume $|a| = n$. Then $|a \cup 0| = n = n + 0$. Suppose $|b| = m$ and $|a \cup b| = n + m$. If $c \notin a \cup b$, then $|a \cup b \cup \{c\}| = (n + m) + 1 = n + (m + 1)$. □

Theorem 47. *If $a \approx n$, then $\mathscr{P}(a)$ is a set, and*

$$|\mathscr{P}(a)| = 2^n.$$

Proof. If $|a| = 0$, then $\mathscr{P}(a) = \{0\} = 1 = 2^0$. Suppose $|a| = n$ and $|\mathscr{P}(a)| = 2^n$. If $b \notin a$, then

$$\mathscr{P}(a \cup \{b\}) = \mathscr{P}(a) \cup \{x \cup \{b\} : x \in \mathscr{P}(a)\},$$

and the two sets are disjoint and equipollent, so

$$|\mathscr{P}(a \cup \{b\})| = |\mathscr{P}(a)| + |\mathscr{P}(a)| = 2^n + 2^n = 2^n \cdot 2 = 2^{n+1}. \qquad \square$$

Lemma 10. *If $n \in \omega$ and $a \approx n$, the only inductive subset of $\mathscr{P}(a)$ is itself.*

Proof. Since $\mathscr{P}(0) = \{0\}$, and all inductive subsets of power sets contain 0, the claim is true for sets of size 0. Suppose the claim is true for sets of size n. Say $|a| = n$ and $b \notin a$. If c is an inductive subset of $\mathscr{P}(a \cup \{b\})$, then $c \cap \mathscr{P}(a)$ is an inductive subset of $\mathscr{P}(a)$, so by hypothesis $\mathscr{P}(a) \subseteq c$, and therefore $d \cup \{b\} \in c$ for all d in $\mathscr{P}(a)$,—that is, $\mathscr{P}(a \cup \{b\}) \subseteq c$. $\square$

Theorem 48. *A set is formally finite if and only if it is equipollent to a natural number.*

Proof. If $|a| = n$, then we now know that $\mathscr{P}(a)$ is a set and is the only inductive subset of itself; and it contains a; so a is formally finite.

Conversely, let $\boldsymbol{F}$ be the class of all sets that are equipollent to natural numbers. Say a is formally finite. Let b be an inductive subset of $\mathscr{P}(a)$. Then $b \cap \boldsymbol{F}$ is also an inductive subset of $\mathscr{P}(a)$, so it contains a. In particular, $a \in \boldsymbol{F}$. $\square$

Now finally we have an analogue of Theorem 25:

Theorem 49. *The class of formally finite sets is the smallest of the classes $\boldsymbol{D}$ that contain 0 and are closed under each operation $x \mapsto x \cup \{a\}$.*

Proof. Trivially, 0 is formally finite. Suppose a is formally finite, so that $a \approx n$ for some n in ω. Then $a \cup \{b\}$ is equipollent to n or $n+1$, so it is formally finite. Therefore the class of formally finite sets is one of the classes $\boldsymbol{D}$.

Now let $\boldsymbol{D}$ be any one of those classes, and suppose a is formally finite. Then $\boldsymbol{D} \cap \mathscr{P}(a)$ is an inductive subset of $\mathscr{P}(a)$, so it contains a, and in particular $a \in \boldsymbol{D}$. $\square$

Therefore finite sets are formally finite, and we may assume that all formally finite sets are finite.

Theorem 50. *Subsets of finite sets are finite. Moreover, if $|b| = n$, and $a \subset b$, then $|a| < n$.*

Proof. It is enough to consider subsets of natural numbers. The only subset of 0 is itself. Suppose every subset of n is finite, and every proper subset has less size. Say $a \subset n+1$. If $a = n$, then $|a| < n+1$. If $a \neq n$, then $a \smallsetminus \{n\} \subset n$, so $|a \smallsetminus \{n\}| < n$, and therefore $|a| < n+1$. $\square$

Corollary. *If $\omega \preccurlyeq a$, then a is infinite.*

Proof. By the theorem, if $|a| = n$, then there is not even an injection from $n+1$ into a, much less from ω. $\square$

We cannot now prove the converse of the corollary. We do however have an alternative formulation of the condition $\omega \preccurlyeq a$.

Theorem 51. *Let a be a set with an element b. Then $a \preccurlyeq a \smallsetminus \{b\}$ if and only if $\omega \preccurlyeq a$.*

Proof. Suppose $\boldsymbol{F}$ is an injection from a into $a \smallsetminus \{b\}$. Then $(a, b, \boldsymbol{F})$ is an iterative structure. Let $\boldsymbol{H}$ be the unique homomorphism from $(\omega, 0, {}')$ to $(a, b, \boldsymbol{F})$. Then $\boldsymbol{H}$ is injective. Indeed, if $0 < n$, then $\boldsymbol{H}(n) \neq b$, that is, $\boldsymbol{H}(n) \neq \boldsymbol{H}(0)$. Suppose $m \in \omega$, and for all n in ω, if $m \neq n$, then $\boldsymbol{H}(m) \neq \boldsymbol{H}(n)$. Then $\boldsymbol{F}(\boldsymbol{H}(m)) \neq \boldsymbol{F}(\boldsymbol{H}(n))$, that is, $\boldsymbol{H}(m+1) \neq \boldsymbol{H}(n+1)$. This is enough to show $\boldsymbol{H}$ is injective, and therefore $\omega \preccurlyeq a$.

Suppose conversely $\boldsymbol{H}$ is an injection from ω into a. Let $b = \boldsymbol{H}(0)$. Then $\{(\boldsymbol{H}(x), \boldsymbol{H}(x+1)) \colon x \in \omega\} \cup \{(x, x) \colon x \in a \smallsetminus \boldsymbol{H}[\omega]\}$ is an injection from a into $a \smallsetminus \{b\}$. $\square$

4. Ordinality

4.1. Well-ordered classes

By Theorem 42 (and the equality of $\bigcup \omega$ and ω guaranteed by Theorems 27 and 43), the class ω is *well-ordered* by membership.

Definition 23. Suppose $(\boldsymbol{C}, <)$ is a linear order. If $b \in \boldsymbol{C}$, we define

$$\operatorname{pred}_{(\boldsymbol{C},<)}(b) = \{x \colon x \in \boldsymbol{C} \ \& \ x < b\};$$

this class is called a **section** of $(\boldsymbol{C}, <)$, and its elements are the **predecessors** of b in $(\boldsymbol{C}, <)$. We may denote the section simply by

$$\operatorname{pred}(b),$$

if the linear order is understood. Suppose all sections of $(\boldsymbol{C}, <)$ are sets. Then let us say that $(\boldsymbol{C}, <)$ is a **left-narrow** linear order.[1] Two possibilities are distinguished by name:

1. Suppose every nonempty subclass of $\boldsymbol{C}$ has a least element with respect to $<$. Then $(\boldsymbol{C}, <)$ is a **good order,**[2] and $\boldsymbol{C}$ is **well-ordered** by $<$.
2. Suppose $\boldsymbol{C}$ is the only subclass $\boldsymbol{D}$ of $\boldsymbol{C}$ such that, for every element b of $\boldsymbol{C}$,
$$\operatorname{pred}(b) \subseteq \boldsymbol{D} \Rightarrow b \in \boldsymbol{D}.$$
Then $(\boldsymbol{C}, <)$ admits **transfinite induction.**

When the linear order in question is $(\omega, <)$, the situation is simple: If $n \in \omega$, then

$$\operatorname{pred}_{(\omega,\in)}(n) = n,$$

[1] This terminology is used by Levy [25, p. 33].

[2] One of the irregularities of English is that *well* is the adverb corresponding to the adjective *good.* One does not want technical terminology to have to conform to linguistic irregularities; therefore a *good order* as defined here is often called a *well-order.*

which is of course a set; so $(\omega, <)$ is left-narrow. Again, by Theorem 42, $(\omega, <)$ is well-ordered; it admits transfinite induction, by Theorem 41. Indeed, any left-narrow linear order that has one property has the other:

Theorem 52. *A left-narrow linear order is good if and only if it admits transfinite induction.*

Proof. Suppose $(\boldsymbol{C}, <)$ is a left-narrow linear order, $\boldsymbol{D} \subset \boldsymbol{C}$, and $b \in \boldsymbol{C}$. Then b is the least element of $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ if and only if $\operatorname{pred}(b) \subseteq \boldsymbol{D}$, but $b \notin \boldsymbol{D}$. In other words, $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ has no least member if and only if $\forall x\,(\operatorname{pred}(x) \subseteq \boldsymbol{D} \Rightarrow x \in \boldsymbol{D})$. So $\boldsymbol{C} \smallsetminus \boldsymbol{D}$ is a counterexample, showing that $(\boldsymbol{C}, <)$ is not a good order, if and only if $\boldsymbol{D}$ is a counterexample, showing that $(\boldsymbol{C}, <)$ does not admit transfinite induction. □

In the definitions of linear orders that are good and that admit transfinite induction, only sub*sets* and complements of subsets are considered, respectively. This ensures that each of these properties is expressed by a single sentence. In fact, subclasses and their complements can be considered (as in Theorems 41 and 42):

Theorem 53. *Suppose $(\boldsymbol{C}, <)$ is a left-narrow linear order. For it to be a good order, either of the following conditions is sufficient.*

1. *Every nonempty subset of $\boldsymbol{C}$ has a least element.*
2. *The empty set is the only subset a of $\boldsymbol{C}$ such that, for all b in $\boldsymbol{C}$,*

$$\operatorname{pred}(b) \subseteq \boldsymbol{C} \smallsetminus a \Rightarrow b \in \boldsymbol{C} \smallsetminus a.$$

Proof. 1. Suppose $\boldsymbol{D}$ is a subclass of $\boldsymbol{C}$, and $a \in \boldsymbol{D}$. Then $\boldsymbol{D} \cap \operatorname{pred}(a)$ is a set. If this set is empty, then a is the least element of $\boldsymbol{D}$. If the set is not empty, but has a least element, then this is the least element of $\boldsymbol{D}$.

2. Under the given condition, if a is a nonempty subset of $\boldsymbol{C}$, then a has an element b such that $\operatorname{pred}(b) \subseteq \boldsymbol{C} \smallsetminus a$, so that b is the least element of a. Thus the first condition is met. □

We shall want to know that subclasses of well-ordered classes are well-ordered.

Theorem 54. *If $(\boldsymbol{C}, <)$ is a good order, and $\boldsymbol{D} \subseteq \boldsymbol{C}$, then $(\boldsymbol{D}, <)$ is a good order.*

Proof. All that needs to be checked is that sections of $(\boldsymbol{D}, <)$ are sets; but every such section is $\boldsymbol{D} \cap \operatorname{pred}_{(\boldsymbol{C},<)}(a)$ for some a in $\boldsymbol{D}$, so the section is a set by the Separation Axiom. □

In particular, as ω is well-ordered by membership, so are its subclasses; and among these subclasses are its elements, because ω is transitive. A further connection with what we already know is made by the following.

Theorem 55. *A class is well-ordered by membership if and only if it is well-founded and linearly ordered by membership.*

Proof. Suppose $\boldsymbol{C}$ is linearly ordered by membership. If $a \subseteq \boldsymbol{C}$, and $b \in a$, then $b \cap a = 0$ if and only if b is the least element of a with respect to membership. □

4.2. Ordinals

We now know that ω and its elements are transitive and well-ordered by membership; equivalently, by the last theorem, they are transitive, well-founded, and linearly ordered by membership.

Definition 24. A *set* that is transitive and well-ordered by membership is called an **ordinal.**[3] The class of all ordinals is denoted by

$$\mathbf{ON}.$$

The Greek letters α, β, γ, ... will invariably denote ordinals.

In particular, ω is a class of ordinals, and ω itself is either an ordinal or a proper class of ordinals.

Theorem 56. **ON** *contains* 0 *and is closed under* $x \mapsto x'$*; so* $(\mathbf{ON}, 0, {}')$ *is an iterative structure with substructure* $(\omega, 0, {}')$.

Lemma 11. **ON** *is transitive, that is, every element of an ordinal is an ordinal. Also every ordinal* properly *includes its elements.*

[3]The definition is due to von Neumann [34].

Proof. Suppose $\alpha \in \mathbf{ON}$ and $b \in \alpha$. We want to show b is an ordinal, that is, b is transitive and well-ordered by $\in$. But $b \subseteq \alpha$ by transitivity of α, so b, like α, is well-ordered by membership.

Suppose $c \in b$; we want to show $c \subseteq b$. That is, suppose $d \in c$; we want to show $d \in b$. But $b \subseteq \alpha$, so $c \in \alpha$. Then also $c \subseteq \alpha$, and hence $d \in \alpha$. So b, c, and d are all in α, and $d \in c$, and $c \in b$. Since membership is a transitive relation on α, we have $d \in b$. Thus $c \subseteq b$, so b is transitive. Now we know b is an ordinal. Therefore $\alpha \subseteq \mathbf{ON}$. So $\mathbf{ON}$ is transitive.

Finally, $\alpha \notin \alpha$, since membership, being a linear ordering of α, is irreflexive. But $b \in \alpha$, so $b \neq \alpha$, and therefore $b \subset \alpha$. □

So every element of an element of an ordinal α is an ordinal; and every element of an element of an element of an ordinal is an ordinal, and so on; moreover, all of these elements are elements of α.

Lemma 12. *Every ordinal contains every ordinal that it properly includes.*

Proof. Suppose $\beta \subset \alpha$. Then $\alpha \smallsetminus \beta$ contains some γ, which is an ordinal by the last lemma. We first show $\beta \subseteq \gamma$. Suppose $\delta \in \beta$; we show $\delta \in \gamma$. Since $\gamma \notin \beta$, we have $\gamma \neq \delta$. But also $\delta \subseteq \beta$, so $\gamma \notin \delta$. Since membership on β is a linear ordering, we must have $\delta \in \gamma$.

We now show that, if γ is the *least* member of $\alpha \smallsetminus \beta$, then $\gamma = \beta$. Suppose on the contrary $\beta \subset \gamma$. Then $\gamma \smallsetminus \beta$ contains some δ. In particular, since $\gamma \subseteq \alpha$, we have $\delta \in \alpha \smallsetminus \beta$. So $\alpha \smallsetminus \beta$ contains γ and δ, and $\delta \in \gamma$. In particular, γ is not the least element of $\alpha \smallsetminus \beta$. □

Theorem 57 (Burali-Forti Paradox [3]). $\mathbf{ON}$ *is transitive and well-ordered by membership; so it is not a set.*

Proof. Let α and β be two ordinals such that $\beta \notin \alpha$. By transfinite induction in α, we show $\alpha \subseteq \beta$. Indeed, say $\gamma \in \alpha$ and $\operatorname{pred}_{(\alpha,\in)}(\gamma) \subseteq \beta$, that is, $\gamma \subseteq \beta$. Then $\gamma \neq \beta$, so $\gamma \subset \beta$ and therefore, by the last lemma, $\gamma \in \beta$. By transfinite induction, $\alpha \subseteq \beta$. In particular, if $\alpha \neq \beta$, then $\alpha \subset \beta$, so $\alpha \in \beta$. Therefore $(\mathbf{ON}, \in)$ is a left-narrow linear order.

In particular, if $\alpha \in \mathbf{ON}$, then $\alpha \neq \mathbf{ON}$. So $\mathbf{ON}$ is not a member of itself, even if it is a set; in particular, $\mathbf{ON}$ is not an ordinal.

If a is a set of ordinals with an element β, then the least element of a is the least element of $a \cap \beta$, if this set is nonempty; otherwise it is β.

Thus **ON** is well-ordered by membership. Since however **ON** is not an ordinal, it must not be a set. □

As a consequence of the last two lemmas, we have

$$\alpha \in \beta \Leftrightarrow \alpha \subset \beta.$$

Implicitly, α and β are ordinals; an ordinal β may have a proper subset that is not an ordinal and is not an element of β. But on **ON**, we may use $<$ to denote either membership or proper inclusion. Then

$$\operatorname{pred}_{(\mathbf{ON},<)}(\alpha) = \alpha.$$

Again, if ω is a set, then it is an ordinal. There is only one alternative:

Theorem 58. *If ω is a proper class, then it is* **ON**.

Proof. We know $\omega \subseteq \mathbf{ON}$. If $\omega \subset \mathbf{ON}$, and $\alpha \in \mathbf{ON} \smallsetminus \omega$, then, by the first part of the proof of Lemma 12, we have $\omega \subseteq \alpha$, so ω is a set. □

4.3. Limits

Suppose ω is indeed a set. Then $\omega \in \mathbf{ON}$, and $\omega \neq 0$, but for all ordinals α, if $\alpha < \omega$, then $\alpha' < \omega$. In a word, if it is a set, then ω is a *limit* of $(\mathbf{ON}, <)$.

Definition 25. An element b of an arbitrary well-ordered class $(\boldsymbol{C}, <)$ is a **successor,** and in particular is the successor of the element a, if b is the least element of $\{x \colon x \in \boldsymbol{C} \ \& \ a < x\}$ (which is $\boldsymbol{C} \smallsetminus (\operatorname{pred}(a) \cup \{a\})$). In this case, we may write

$$b = a^+.$$

An element of $\boldsymbol{C}$ is a **limit** if it is neither a successor nor the least element of $\boldsymbol{C}$.

Theorem 59. *In* **ON**, *the successor of α is α'.*

Proof. If $\alpha < \beta$, this means $\alpha \subset \beta$ and $\alpha \in \beta$, hence also $\{\alpha\} \subseteq \beta$; therefore $\alpha \cup \{\alpha\} \subseteq \beta$, that is, $\alpha' \leqslant \beta$. □

In general, a good order $(\boldsymbol{C}, <)$ has at most three distinct kinds of elements:

1) the least element,
2) successors,
3) limits.

If $\boldsymbol{C}$ has a greatest element, then this has no successor; every other element does have a successor, which is unique. An element of $\boldsymbol{C}$ that is not a successor is just an element a such that

$$\forall x\,(x \in \boldsymbol{C} \;\&\; x < a \Rightarrow x^{+} < a).$$

Such an element is either the least element of $\boldsymbol{C}$, or a limit. The least element of $\boldsymbol{C}$ might be thought of as a 'degenerate' limit. Still, by the official definition, 0 is *not* a limit. If it were, we should have to change the wording of the following.

Theorem 60. *ω is the class of ordinals that neither* are *limits nor* contain *limits.*[4]

Now we can make an alternative formulation of transfinite induction:

Theorem 61 (Transfinite induction in two parts)**.** *Suppose $(\boldsymbol{C}, <)$ is a well-ordered class, and $\boldsymbol{D}$ is a subclass meeting the following two conditions.*

1. *If a is not the greatest element of $\boldsymbol{C}$, then*

$$a \in \boldsymbol{D} \Rightarrow a^{+} \in \boldsymbol{D}.$$

2. *If b is not a successor of $\boldsymbol{C}$, then*

$$\operatorname{pred}(b) \subseteq \boldsymbol{D} \Rightarrow b \in \boldsymbol{D}. \tag{4.1}$$

[4]One could say, *The class of ordinals that neither* are *nor* contain *limits is denoted by ω;* but this would violate the grammatical principles laid down by Fowler in [15, Cases] and reaffirmed by his editor Gowers in [14]. In the original sentence of the theorem, the second occurrence of *limits* is the direct object of *contain,* so it is notionally in the 'objective case'; but the first instance of *limits* is is not an object of *are* (which does not take objects), but is in the 'subjective case', like the subject, *that,* of the relative clause, *that neither are limits nor contain limits.* On similar grounds, the common expression, *x is less than or equal to y,* is objectionable, unless *than,* like *to,* is construed as a preposition. However, allowing *than* to be used as a preposition can cause ambiguity: does *She likes tea better than me* mean *She likes tea better than she likes me,* or *She likes tea better than I do*? Therefore it is recommended in [15, Than 6] and (less strongly) in [14] that *than* not be used as a preposition. If we were to follow this recommendation thoroughly, then we should read the inequality $x \leqslant y$ as, *x is less than y or equal to y,* rather than simply, *x is less than or equal to y.* I do not actually propose to make this change.

Then $\boldsymbol{D} = \boldsymbol{C}$.

Proof. Suppose $(\boldsymbol{C}, <)$ is a good order and $\boldsymbol{D} \subset \boldsymbol{C}$, but (4.1) holds whenever b is an element of $\boldsymbol{C}$ that is not a successor. By transfinite induction, there is some b in $\boldsymbol{C}$ such that $\operatorname{pred}(b) \subseteq \boldsymbol{D}$, but $b \notin \boldsymbol{D}$. Then b must be a successor, so $b = a^+$ for some b in $\boldsymbol{C}$. Then $a \in \operatorname{pred}(b)$, so $a \in \boldsymbol{D}$, but $a^+ \notin \boldsymbol{D}$. □

It is sometimes useful to distinguish the least element of a good order from the other non-successors, that is, the limits.

Corollary (Transfinite induction in three parts). *Suppose* $(\boldsymbol{C}, <)$ *is a well-ordered class with least element* ℓ, *and* $\boldsymbol{D}$ *is a subclass meeting the following three conditions.*

1. $\ell \in \boldsymbol{D}$.
2. *If* a *is not the greatest element of* $\boldsymbol{C}$, *then*

$$a \in \boldsymbol{D} \Rightarrow a^+ \in \boldsymbol{D}.$$

3. *If* b *is a limit of* $\boldsymbol{C}$, *then*

$$\operatorname{pred}(b) \subseteq \boldsymbol{D} \Rightarrow b \in \boldsymbol{D}.$$

Then $\boldsymbol{D} = \boldsymbol{C}$.

4.4. Transfinite recursion

The Separation Axiom could be formulated as follows. Given a class $\boldsymbol{C}$, if a is a set a such that $a \cap \boldsymbol{C}$ has an element b, we can let $\boldsymbol{F}$ be the class $\{(x, x) \colon x \in a \cap \boldsymbol{C}\} \cup \{(x, b) \colon x \in a \smallsetminus \boldsymbol{C}\}$. Then $\boldsymbol{F}$ is a function given by

$$\boldsymbol{F}(x) = \begin{cases} x, & \text{if } x \in a \cap \boldsymbol{C}, \\ b, & \text{if } x \in a \smallsetminus \boldsymbol{C}. \end{cases}$$

In particular, $\operatorname{rng}(\boldsymbol{F}) = a \cap \boldsymbol{C}$. The axiom is that this range is a set. We might say that, because a is a set, so is $\boldsymbol{F}[a]$. The latter set is obtained by, so to speak, *replacing* each element x of a with $\boldsymbol{F}(x)$. The Separation Axiom is thus a special case of the following:

Axiom 5 (Replacement). *The image of a set under a function is a set: for all functions $\boldsymbol{F}$, if $a \subseteq \operatorname{dom}(\boldsymbol{F})$, then $\boldsymbol{F}[a]$ is a set:*

$$\forall x\, \exists y\, (x \subseteq \operatorname{dom}(\boldsymbol{F}) \Rightarrow y = \boldsymbol{F}[x]).$$

Like the Separation Axiom, the Replacement Axiom is really a scheme of axioms, one for each function. We need this scheme, to ensure that the following makes sense:

Definition 26. A left-narrow linear order $(\boldsymbol{C}, <)$ admits **transfinite recursion** if, for every class $\boldsymbol{D}$, for every function $\boldsymbol{F}$ from $\mathscr{P}(\boldsymbol{D})$ to $\boldsymbol{D}$, there is a unique function $\boldsymbol{G}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that

$$\forall x\, (x \in \boldsymbol{C} \Rightarrow \boldsymbol{G}(x) = \boldsymbol{F}(\boldsymbol{G}[\operatorname{pred}(x)])). \tag{4.2}$$

Note that this property of a given linear order is not *obviously* expressible with a single formula. It *is* so expressible though, by the following.

Theorem 62 (Transfinite Recursion). *A left-narrow linear order admits transfinite recursion if and only if it is well-ordered.*

Proof. Let $(\boldsymbol{C}, <)$ be a left-narrow linear order. Suppose first that $(\boldsymbol{C}, <)$ is good, $\boldsymbol{D}$ is a class, and $\boldsymbol{F}$ is a function from $\mathscr{P}(\boldsymbol{D})$ to $\boldsymbol{D}$. We show by transfinite induction that, for all a in $\boldsymbol{C}$, there is a unique function g_a with domain $\operatorname{pred}(a) \cup \{a\}$ such that, whenever $c \leqslant a$, then

$$g_a(c) = \boldsymbol{F}(g_a[\,\operatorname{pred}(c)\,]).$$

Suppose the claim holds whenever $a < b$. If $a < d < b$, then $g_d \restriction (\operatorname{pred}(a) \cup \{a\})$ has the defining property of g_a, so it is equal to g_a; in particular, $g_d(a) = g_a(a)$. Therefore we can define g_b by

$$g_b(x) = \begin{cases} g_x(x), & \text{if } x < b; \\ \boldsymbol{F}(\{g_y(y) \colon y < b\}), & \text{if } x = b. \end{cases} \tag{4.3}$$

Moreover, as before, any g_b as desired must agree with g_a on $\operatorname{pred}(a) \cup \{a\}$ when $a < b$, and then $g_b(b)$ must be as in (4.3). By transfinite induction, we have a function g_a as desired for all a in $\boldsymbol{C}$. Then we have (4.2) if and only if $\boldsymbol{G}$ is $x \mapsto g_x(x)$.

Now suppose $(\boldsymbol{C}, <)$ is not good, but $\boldsymbol{D}$ is a nonempty subclass of $\boldsymbol{C}$ with no least element. Let

$$\boldsymbol{E} = \{x \colon x \in \boldsymbol{C} \mathbin{\&} \exists y\,(y \in \boldsymbol{D} \mathbin{\&} y \leqslant x)\},$$

and let $\boldsymbol{F}$ be the function from $\mathscr{P}(2)$ to 2 such that

$$\forall x\,(x \subseteq 2 \Rightarrow (\boldsymbol{F}(x) = 1 \Leftrightarrow 1 \in x)).$$

Then there are two functions $\boldsymbol{G}$ from $\boldsymbol{C}$ to 2 such that (4.2) holds. Indeed, let

$$\boldsymbol{G}_0 = \{(x, 0) \colon x \in \boldsymbol{C}\}, \quad \boldsymbol{G}_1 = \{(x, 0) \colon x \in \boldsymbol{C} \smallsetminus \boldsymbol{E}\} \cup \{(x, 1) \colon x \in \boldsymbol{E}\};$$

that is, if $e \in 2$, let $\boldsymbol{G}_e$ be the function from $\boldsymbol{C}$ into 2 given by

$$\boldsymbol{G}_e(x) = \begin{cases} 0, & \text{if } x \in \boldsymbol{C} \smallsetminus \boldsymbol{E}; \\ e, & \text{if } x \in \boldsymbol{E}. \end{cases}$$

Then $\boldsymbol{G}_e(a) = \boldsymbol{F}(\boldsymbol{G}_e[\operatorname{pred}(a)])$. □

In the notation of Definition 26, if a is an element of $\boldsymbol{C}$ with a successor, then $\boldsymbol{G}(a^+)$ depends on $\{\boldsymbol{G}(x) \colon x \in \boldsymbol{C} \mathbin{\&} x \leqslant a\}$, not just on $\boldsymbol{G}(a)$. We cannot generally recover $\boldsymbol{G}(a)$ from $\{\boldsymbol{G}(x) \colon x \in \boldsymbol{C} \mathbin{\&} x \leqslant a\}$. However, in our applications, we shall *want* to define $\boldsymbol{G}(a^+)$ in terms of $\boldsymbol{G}(a)$ alone. We can do this as follows.

Theorem 63 (Transfinite recursion in two parts)**.** *Suppose $(\boldsymbol{C}, <)$ is a well-ordered class, $\boldsymbol{D}$ is a class, $\boldsymbol{F}$ is a function from $\boldsymbol{D}$ to $\boldsymbol{D}$, and $\boldsymbol{G}$ is a function from $\mathscr{P}(\boldsymbol{D})$ to $\boldsymbol{D}$. Then there is a unique function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that*

1) $\boldsymbol{H}(a^+) = \boldsymbol{F}(\boldsymbol{H}(a))$, if a is not the greatest element of $\boldsymbol{C}$;

2) $\boldsymbol{H}(d) = \boldsymbol{G}(\boldsymbol{H}[\operatorname{pred}(d)])$, if d is not a successor.

Proof. By transfinite induction in two parts, as in Theorem 61, there is at most one such function $\boldsymbol{H}$. Indeed, suppose $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are two such functions.

1. If $\boldsymbol{H}_0(a) = \boldsymbol{H}_1(a)$, then $\boldsymbol{H}_0(a^+) = \boldsymbol{F}(\boldsymbol{H}_0(a)) = \boldsymbol{F}(\boldsymbol{H}_1(a)) = \boldsymbol{H}_1(a^+)$.

2. If a is not a successor, and $\boldsymbol{H}_0 \restriction \operatorname{pred}(a) = \boldsymbol{H}_1 \restriction \operatorname{pred}(a)$, then

$$\boldsymbol{H}_0(a) = \boldsymbol{G}(\boldsymbol{H}_0[\operatorname{pred}(a)]) = \boldsymbol{G}(\boldsymbol{H}_1[\operatorname{pred}(a)]) = \boldsymbol{H}_1(a).$$

Therefore $\boldsymbol{H}_0 = \boldsymbol{H}_1$.

In case $\boldsymbol{C}$ has a greatest element a, so that $\boldsymbol{C} = \operatorname{pred}(a) \cup \{a\}$, then the desired function $\boldsymbol{H}$ is a set, which we may denote by h_a. As in the proof of Theorem 62, but this time using transfinite induction as given by Theorem 61, we have that h_a exists as desired for all a in $\boldsymbol{C}$. Indeed, if a is not the greatest element of $\boldsymbol{C}$, then

$$h_{a^+} = h_a \cup \{(a, \boldsymbol{F}(h_a(a)))\},$$

while if a is not a successor, then

$$h_a = \{(x, h_x(x))\colon x \in \operatorname{pred}(a)\} \cup \{(a, \boldsymbol{G}(\{h_x(x)\colon x \in \operatorname{pred}(a)\}))\}.$$

Then the desired function $\boldsymbol{H}$ on $\boldsymbol{C}$ is $x \mapsto h_x(x)$. □

In applications of the theorem, the function $\boldsymbol{G}$ may be defined by one formula at the empty set, and by another at the non-empty subsets of $\boldsymbol{D}$. That is, we may apply the theorem in the following form:

Corollary (Transfinite recursion in three parts)**.** *Suppose $(\boldsymbol{C}, <)$ is a well-ordered class with least element ℓ, $\boldsymbol{D}$ is a class with element m, $\boldsymbol{F}$ is a function from $\boldsymbol{D}$ to $\boldsymbol{D}$, and $\boldsymbol{G}$ is a function from $\mathscr{P}(\boldsymbol{D}) \smallsetminus \{0\}$ to $\boldsymbol{D}$. Then there is a unique function $\boldsymbol{H}$ from $\boldsymbol{C}$ to $\boldsymbol{D}$ such that*
1) $\boldsymbol{H}(\ell) = m$,
2) $\boldsymbol{H}(a^+) = \boldsymbol{F}(\boldsymbol{H}(a))$, if a is not the greatest element of $\boldsymbol{C}$;
3) $\boldsymbol{H}(d) = \boldsymbol{G}(\boldsymbol{H}[\operatorname{pred}(d)])$, if d is a limit.

An **initial segment** of an order $(\boldsymbol{C}, <)$ is a subclass $\boldsymbol{D}$ of $\boldsymbol{C}$ such that, for all a in $\boldsymbol{D}$,

$$\forall x\,(x \in \boldsymbol{C} \mathbin{\&} x < a \Rightarrow x \in \boldsymbol{D}).$$

Theorem 64. *Every initial segment of a well-ordered class is either the class itself or a section of it.*

Definition 27. An **embedding** of a linearly ordered class $(\boldsymbol{C}, <)$ in another one, $(\boldsymbol{D}, \boldsymbol{R})$, is an injection $\boldsymbol{F}$ of the class $\boldsymbol{C}$ in $\boldsymbol{D}$ that is also **order-preserving** or **increasing** in the sense that, for all a and b in $\boldsymbol{C}$,

$$a < b \Rightarrow \boldsymbol{F}(a)\ \boldsymbol{R}\ \boldsymbol{F}(b).$$

Since the orders in question are linear, this condition implies its converse, so that

$$a < b \Leftrightarrow \boldsymbol{F}(a) \; \boldsymbol{R} \; \boldsymbol{F}(b).$$

If the range of $\boldsymbol{F}$ is $\boldsymbol{D}$, then $\boldsymbol{F}$ is an **isomorphism** from $(\boldsymbol{C}, <)$ to $(\boldsymbol{D}, \boldsymbol{R})$. The least element (if it exists) of a subclass $\boldsymbol{E}$ of $\boldsymbol{C}$ can be called the **minimum** element of $\boldsymbol{E}$ and can accordingly be denoted by

$$\min(\boldsymbol{E}).$$

Theorem 65. *Of any two well-ordered classes, one is uniquely isomorphic to a unique initial segment of the other.*

Proof. Let $(\boldsymbol{C}, <)$ and $(\boldsymbol{D}, \boldsymbol{R})$ be well-ordered classes. Suppose first that there is an isomorphism $\boldsymbol{H}$ from the former to an initial segment of the latter. Then for all a in $\boldsymbol{C}$, we have, by the definitions,

$$\boldsymbol{H}[\operatorname{pred}(a)] \subseteq \operatorname{pred}(\boldsymbol{H}(a)).$$

Also, since $\boldsymbol{H}[\boldsymbol{C}]$ is an initial segment of $\boldsymbol{D}$, we have $\operatorname{pred}(\boldsymbol{H}(a)) \subseteq \boldsymbol{H}[\boldsymbol{C}]$. Since $\boldsymbol{H}$ is order-preserving, we therefore have

$$\boldsymbol{H}[\operatorname{pred}(a)] = \operatorname{pred}(\boldsymbol{H}(a)),$$

and consequently

$$\boldsymbol{H}(a) = \min(\boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)]).$$

Thus $\boldsymbol{H}$ is defined recursively and is therefore unique.

However, the definition fails if, for some a in $\boldsymbol{C}$, it should happen that $\boldsymbol{H}[\operatorname{pred}(a)] = \boldsymbol{D}$. We can then adjust the definition so that, for all a in $\boldsymbol{C}$,

$$\boldsymbol{H}(a) = \begin{cases} \min(\boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)]), & \text{if } \boldsymbol{H}[\operatorname{pred}(a)] \subset \boldsymbol{D}, \\ \min(\boldsymbol{D}), & \text{otherwise.} \end{cases}$$

Then $\operatorname{rng}(\boldsymbol{H})$ is indeed an initial segment of $\boldsymbol{D}$. Indeed, suppose $\boldsymbol{H}(a) = b$, but $c \; \boldsymbol{R} \; b$. Then $\boldsymbol{H}(a) = \min(\boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)])$, so $c \notin \boldsymbol{D} \smallsetminus \boldsymbol{H}[\operatorname{pred}(a)]$, and therefore $c \in \boldsymbol{H}[\operatorname{pred}(a)]$.

If $\boldsymbol{H}$ is injective, then it is an isomorphism from $\boldsymbol{C}$ to its range. If $\boldsymbol{H}$ is not injective, let a be the least of those b in $\boldsymbol{C}$ such that $\boldsymbol{H}(b) = \min(\boldsymbol{D})$, but $b \neq \min(\boldsymbol{C})$. Then $\breve{\boldsymbol{H}}$ is an isomorphism from $\boldsymbol{D}$ to $\operatorname{pred}_{(\boldsymbol{C},<)}(a)$. $\square$

Corollary. *Every well-ordered set is isomorphic to a unique ordinal. Every well-ordered proper class is isomorphic to* **ON**.

Definition 28. If $(a, <)$ is a well-ordered set, then the unique ordinal to which it is isomorphic is its **order-type** or **ordinality;** this can be denoted by

$$\operatorname{ord}(a, <)$$

or simply $\operatorname{ord}(a)$.

4.5. Suprema

Definition 29. Suppose $(\boldsymbol{C}, <)$ is a linear order, $\boldsymbol{D} \subseteq \boldsymbol{C}$, and $a \in \boldsymbol{C}$. Then a is an **upper bound** of $\boldsymbol{D}$ (with respect to $<$) if

$$\forall x\,(x \in \boldsymbol{D} \Rightarrow x \leqslant a);$$

and a is a **strict upper bound** if

$$\forall x\,(x \in \boldsymbol{D} \Rightarrow x < a).$$

If $\boldsymbol{D}$ has a *least* upper bound, then this is unique and is the **supremum** of $\boldsymbol{D}$; it is denoted by

$$\sup(\boldsymbol{D}).$$

If $\boldsymbol{D} = \{\boldsymbol{F}(x) \colon \varphi(x)\}$ for some function $\boldsymbol{F}$ and formula φ, then we may write $\sup(\boldsymbol{D})$ as

$$\sup_{\varphi(x)} \boldsymbol{F}(x).$$

Note in particular that $\sup(0)$ is the least element of $\boldsymbol{C}$, if there is one.

We shall make use of these notions on **ON**.

Theorem 66. *For all ordinals α,*

$$\sup(\alpha') = \alpha.$$

If α is not a successor itself, then

$$\sup(\alpha) = \alpha.$$

Theorem 67. *The union of a set of transitive sets is transitive. In particular, the union of a set of ordinals is either an ordinal or* **ON** *itself.*

To preclude the possibility that the union of a set of ordinals might be **ON**, we have:

Axiom 6 (Union)**.** *The union of a set is a set:*

$$\exists x\, x = \bigcup b.$$

Theorem 68. *For all sets a and b, the union $a \cup b$ is the set $\bigcup\{a, b\}$.*

Theorem 69. *The union of a set of ordinals is an ordinal, which is the supremum of the set:*

$$b \subset \mathbf{ON} \Rightarrow \bigcup b = \sup(b).$$

Proof. Suppose $b \subseteq \mathbf{ON}$. If $\alpha \in b$, then $\alpha \subseteq \bigcup b$; so $\bigcup b$ is an upper bound of b. If $\beta < \bigcup b$, then β belongs to an element α of b; that is, $\beta < \alpha$, so β is not an upper bound of b. □

Theorem 70. *If b is a set of ordinals, then $\bigcup\{x' \colon x \in b\}$ is the least strict upper bound of b.*

4.6. Ordinal addition

We can now extend Definition 16, of addition on ω, to **ON**, using transfinite recursion in three parts (the corollary to Theorem 63).

Definition 30 (Ordinal addition)**.** For each ordinal α, the operation $x \mapsto \alpha + x$ on **ON** is given by

$$\alpha + 0 = \alpha, \qquad \alpha + \beta' = (\alpha + \beta)', \qquad \alpha + \gamma = \sup_{x \in \gamma}(\alpha + x),$$

where γ is a limit. In particular,

$$\alpha + 1 = \alpha',$$

so the second part of the definition can be written as

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1.$$

The operation $(x, y) \mapsto x + y$ on **ON** is **ordinal addition,** and $\alpha + \beta$ is the **ordinal sum** of α and β.

Some of the properties of addition on **ON** are just as on ω. To begin with, we have:

Theorem 71. *For all ordinals α,*

$$0 + \alpha = \alpha.$$

Proof. We use transfinite induction in three parts, as in the corollary to Theorem 61.

1. $0 + 0 = 0$ by definition.
2. If $0 + \alpha = \alpha$, then $0 + (\alpha + 1) = (0 + \alpha) + 1 = \alpha + 1$.
3. If β is a limit, and $0 + \alpha = \alpha$ whenever $\alpha < \beta$, then

$$0 + \beta = \sup_{x \in \beta}(0 + x) = \sup_{x \in \beta} x = \sup(\beta) = \beta$$

by Theorem 66. □

Theorem 72. *For all ordinals α, β, and γ,*

$$\alpha < \beta \Rightarrow \gamma + \alpha < \gamma + \beta.$$

In particular, ordinal addition admits left cancellation:

$$\gamma + \alpha = \gamma + \beta \Rightarrow \alpha = \beta.$$

Proof. We prove the first claim by transfinite induction in three parts on β.

1. The claim is vacuous when $\beta = 0$.
2. If the claim holds when $\beta = \delta$, and now $\alpha < \delta + 1$, then $\alpha \leqslant \delta$, and therefore

$$\gamma + \alpha \leqslant \gamma + \delta < (\gamma + \delta) + 1 = \gamma + (\delta + 1).$$

3. If δ is a limit, and the claim holds when $\beta < \delta$, and now $\alpha < \delta$, then $\alpha < \alpha + 1 < \delta$, and therefore

$$\gamma + \alpha < \gamma + \alpha + 1 \leqslant \sup_{x \in \delta}(\gamma + x) = \gamma + \delta.$$

The second claim is nearly the contrapositive: If $\alpha \neq \beta$, then we may assume $\alpha < \beta$, so $\gamma + \alpha < \gamma + \beta$, and in particular $\gamma + \alpha \neq \gamma + \beta$. ☐

Now we can establish an alternative definition of ordinal addition, using transfinite recursion in the original, one-part form (Definition 26).

Theorem 73. *For all ordinals α and β,*

$$\alpha + \beta = \sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \beta\}).$$

Proof. We consider the three parts of the original definition (Definition 30).

1. $\sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in 0\}) = \sup(\{\alpha\}) = \alpha = \alpha + 0$.
2. If $\delta < \gamma + 1$, then $\delta \leqslant \gamma$, so $\delta + 1 \leqslant \gamma + 1$, and hence

$$\alpha + (\delta + 1) \leqslant \alpha + (\gamma + 1).$$

   Also $\alpha < \alpha + (\gamma + 1)$. Therefore

$$\sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \gamma + 1\}) \leqslant \alpha + (\gamma + 1).$$

   The reverse inequality also holds, because $\gamma < \gamma + 1$, so

$$\alpha + (\gamma + 1) \in \{\alpha + (x + 1) \colon x \in \gamma + 1\}.$$

3. Suppose β is a limit. Then $0 < \beta$, and if $\delta < \beta$, then $\delta + 1 < \beta$. Therefore

$$\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \beta\} \subseteq \{\alpha + x \colon x \in \beta\},$$

   so

$$\sup(\{\alpha\} \cup \{\alpha + (x + 1) \colon x \in \beta\}) \leqslant \sup_{x \in \beta}(\alpha + x) = \alpha + \beta.$$

   The reverse inequality also holds, since $\alpha + \delta < \alpha + (\delta + 1)$. ☐

To establish some additional properties, yet another understanding of ordinal addition will be useful. We develop this now.

Theorem 74. *For all sets a and b, the class $a \times b$ is a set.*

Proof. The class $a \times \{c\}$ is the image of a under the function $x \mapsto (x, c)$, so it is a set. Then $a \times b$ is the set $\bigcup\{a \times \{x\} : x \in b\}$. $\square$

Definition 31. Suppose $(\boldsymbol{C}, \boldsymbol{R})$ and $(\boldsymbol{D}, \boldsymbol{S})$ are linear orders. The **(right) lexicographic ordering** of $\boldsymbol{C} \times \boldsymbol{D}$ is the relation $<$ such that, for all a and b in $\boldsymbol{C}$, and all c and d in $\boldsymbol{D}$,

$$(a, c) < (b, d) \Leftrightarrow c\ \boldsymbol{S}\ d \vee (c = d \ \&\ a\ \boldsymbol{R}\ b).$$

The lexicographic ordering of 4×6 is given in Table 4.1.

$$\begin{tikzcd}
(0,0) \arrow[d] & (0,1) \arrow[d] & (0,2) \arrow[d] & (0,3) \arrow[d] & (0,4) \arrow[d] & (0,5) \arrow[d] \\
(1,0) \arrow[d] & (1,1) \arrow[d] & (1,2) \arrow[d] & (1,3) \arrow[d] & (1,4) \arrow[d] & (1,5) \arrow[d] \\
(2,0) \arrow[d] & (2,1) \arrow[d] & (2,2) \arrow[d] & (2,3) \arrow[d] & (2,4) \arrow[d] & (2,5) \arrow[d] \\
(3,0) \arrow[uuur] & (3,1) \arrow[uuur] & (3,2) \arrow[uuur] & (3,3) \arrow[uuur] & (3,4) \arrow[uuur] & (3,5)
\end{tikzcd}$$

Table 4.1. The lexicographic ordering of 4×6

Theorem 75. *If $\boldsymbol{C}$ and $\boldsymbol{D}$ are linear orders, then the lexicographic ordering of $\boldsymbol{C} \times \boldsymbol{D}$ is a linear ordering. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are* well-ordered, *then $\boldsymbol{C} \times \boldsymbol{D}$ is well-ordered by the lexicographic ordering.*

Proof. In the good case, if $\boldsymbol{E} \subseteq \boldsymbol{C} \times \boldsymbol{D}$ and is nonempty, then its least element is (a, b), where b is the least element of $\{y \colon \exists x\,(x, y) \in \boldsymbol{E}\}$, and a is the least element of $\{x \colon (x, b) \in \boldsymbol{C}\}$. $\square$

Theorem 76. *If $\mathfrak{a}$ is a section of the well-ordered set $\mathfrak{b}$, then*

$$\operatorname{ord}(\mathfrak{a}) < \operatorname{ord}(\mathfrak{b}).$$

Proof. If f is the isomorphism from $\mathfrak{b}$ to $\operatorname{ord}(\mathfrak{b})$ guaranteed by Theorem 65, then, by the proof of that theorem, $f[\mathfrak{a}]$ is a section of $\operatorname{ord}(\mathfrak{b})$, so

$$\operatorname{ord}(\mathfrak{a}) = f[\mathfrak{a}] < \operatorname{ord}(\mathfrak{b}).$$ $\square$

Theorem 77. *For all ordinals α and β,*

$$\alpha + \beta = \operatorname{ord}((\alpha \times \{0\}) \cup (\beta \times \{1\})),$$

where the union has the lexicographic ordering of $\mathbf{ON} \times 2$.

Proof. Fixing α, let us use the notation

$$(\alpha \times \{0\}) \cup (\beta \times \{1\}) = \boldsymbol{F}(\beta).$$

We want to show $\alpha + \beta = \operatorname{ord}(\boldsymbol{F}(\beta))$. We use transfinite induction in three parts.

1. We have $\operatorname{ord}(\boldsymbol{F}(0)) = \operatorname{ord}(\alpha \times \{0\}) = \alpha = \alpha + 0$.
2. Suppose the claim holds when $\beta = \gamma$. Then there is an isomorphism h from $\alpha + \gamma$ to $\boldsymbol{F}(\gamma)$. Then $h \cup \{(\alpha + \gamma, (\gamma, 1))\}$ is an isomorphism from $(\alpha + \gamma) + 1$—which is $\alpha + (\gamma + 1)$—to $\boldsymbol{F}(\gamma + 1)$. So the claim holds when $\beta = \gamma + 1$.
3. Suppose γ is a limit, and the claim holds when $\beta < \gamma$. When $\beta < \gamma$, then $\boldsymbol{F}(\beta)$ is a section of $\boldsymbol{F}(\gamma)$, so by the last theorem,

   $$\alpha + \beta = \operatorname{ord}(\boldsymbol{F}(\beta)) < \operatorname{ord}(\boldsymbol{F}(\gamma)),$$

   and therefore

   $$\alpha + \gamma = \sup_{x \in \gamma}(\alpha + x) \leqslant \operatorname{ord}(\boldsymbol{F}(\gamma)).$$

   For the reverse inequality, suppose $\zeta < \operatorname{ord}(\boldsymbol{F}(\gamma))$. Then ζ is the order type of some section of $\boldsymbol{F}(\gamma)$. This section is either $\boldsymbol{F}(\beta)$ for some β in γ, or $\beta \times \{0\}$ for some β such that $\beta \leqslant \alpha$. In either case, $\zeta \leqslant \alpha + \beta$ for some β in γ. Therefore $\zeta < \alpha + \gamma$. We also have $\beta + 1 < \gamma$, so $\zeta + 1 \leqslant \operatorname{ord}(\boldsymbol{F}(\beta + 1)) < \operatorname{ord}(\boldsymbol{F}(\gamma))$. Therefore $\operatorname{ord}(\boldsymbol{F}(\gamma))$ is not a successor, and so

$$\operatorname{ord}(\boldsymbol{F}(\gamma)) = \sup(\{x \colon x \in \operatorname{ord}(\boldsymbol{F}(\gamma))\}) \leqslant \alpha + \gamma. \qquad \square$$

Theorem 78 (Subtraction). *If $\alpha \leqslant \beta$, then the equation*

$$\alpha + x = \beta$$

has a unique ordinal solution, namely $\operatorname{ord}(\beta \smallsetminus \alpha)$.

Proof. Let $\operatorname{ord}(\beta \smallsetminus \alpha) = \gamma$, and let f be the isomorphism from $\beta \smallsetminus \alpha$ to γ. Then there is an isomorphism g from β to $(\alpha \times \{0\}) \cup (\gamma \times \{1\})$ given by

$$g(x) = \begin{cases} (x, 0), & \text{if } x \in \alpha, \\ (f(x), 1), & \text{if } \alpha \in x \in \beta. \end{cases}$$

Therefore $\beta = \alpha + \gamma$, by the last theorem. If also $\beta = \alpha + \delta$, then $\gamma = \delta$ by Theorem 72. $\square$

The theorem can be proved by transfinite induction. However, this method does not give insight into what the solution of the equation *is;* and we can use that insight for the following.

Theorem 79. *For all ordinals α, β, and γ,*

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

Proof. By the last theorem,

$$\operatorname{ord}((\alpha + \beta) \smallsetminus \alpha) = \beta, \qquad \operatorname{ord}(((\alpha + \beta) + \gamma) \smallsetminus (\alpha + \beta)) = \gamma,$$

so by Theorem 77,

$$\operatorname{ord}(((\alpha + \beta) + \gamma) \smallsetminus \alpha) = \beta + \gamma.$$

By the last theorem again, the claim follows. $\square$

In Theorem 76, it is important that $\mathfrak{a}$ is a *section* of $\mathfrak{b}$. In a more general situation, we have the following.

Lemma 13. *Suppose* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are well-ordered sets, and* $\mathfrak{a}$ *embeds in* $\mathfrak{b}$. *Then*

$$\operatorname{ord}(\mathfrak{a}) \leqslant \operatorname{ord}(\mathfrak{b}).$$

Proof. An embedding of $\mathfrak{a}$ in $\mathfrak{b}$ induces an embedding g of $\operatorname{ord}(\mathfrak{a})$ in $\operatorname{ord}(\mathfrak{b})$. Then

$$\alpha < \operatorname{ord}(\mathfrak{a}) \Rightarrow g(\alpha) < \operatorname{ord}(\mathfrak{b}).$$

We shall show by transfinite induction (in one part) that

$$\alpha < \operatorname{ord}(\mathfrak{a}) \Rightarrow \alpha \leqslant g(\alpha).$$

Suppose this is so when $\alpha < \beta$, and suppose $\beta < \operatorname{ord}(\mathfrak{a})$. If $\alpha < \beta$, then $\alpha \leqslant g(\alpha) < g(\beta)$, so $\alpha < g(\beta)$. Briefly, $\alpha < \beta \Rightarrow \alpha < g(\beta)$. Therefore $\beta \leqslant g(\beta)$. This completes the induction. We conclude

$$\alpha < \operatorname{ord}(\mathfrak{a}) \Rightarrow \alpha < \operatorname{ord}(\mathfrak{b}),$$

and hence $\operatorname{ord}(\mathfrak{a}) \leqslant \operatorname{ord}(\mathfrak{b})$. □

The following should be contrasted with Theorem 72.

Theorem 80. *For all ordinals* α *and* β,

$$\alpha \leqslant \beta \Rightarrow \alpha + \gamma \leqslant \beta + \gamma.$$

Proof. If $\alpha \leqslant \beta$, then $(\alpha \times \{0\}) \cup (\gamma \times \{1\}) \subseteq (\beta \times \{0\}) \cup (\gamma \times \{1\})$; this inclusion is an embedding of the well-ordered sets, so $\alpha + \gamma \leqslant \beta + \gamma$ by the lemma. □

We already knew the foregoing in case $\omega = \mathbf{ON}$. Moreover, we cannot now *prove* that $\omega \neq \mathbf{ON}$. Indeed, by finite induction, and Theorem 47,—ultimately, by GST alone (p. 61)—, we can define a function $\boldsymbol{F}$ on ω by

$$\boldsymbol{F}(0) = 0, \qquad\qquad \boldsymbol{F}(n+1) = \mathscr{P}(\boldsymbol{F}(n)).$$

Let $\boldsymbol{C}$ be the class $\{x \colon \exists y\,(y \in \omega \;\&\; x \in \boldsymbol{F}(y)\}$. Then all of our axioms so far are true in $\boldsymbol{C}$; that is, they are true when we assume $\mathbf{V} = \boldsymbol{C}$ and $\in$ is the relation $\{(x, y) \colon y \in \boldsymbol{C} \;\&\; x \in y\}$. Even the Power Set Axiom (Axiom 8, p. 111) is true in $\boldsymbol{C}$. However, all elements of $\boldsymbol{C}$ are finite.

Nonetheless, there is no obvious way to prove that $\mathbf{V} \smallsetminus \boldsymbol{C}$ is empty. For the sake of developing some interesting possibilities, we assume $\omega \in \mathbf{ON}$:

Axiom 7 (Infinity). *The class of natural numbers is a set:*

$$\exists x\, x = \omega.$$

Theorem 81. *If $n < \omega \leqslant \alpha$, then*

$$n + \alpha = \alpha.$$

Proof. We have $\alpha = \omega + \beta$ for some β, so $n + \alpha = n + \omega + \beta$. By Theorem 38, there is an isomorphism f from ω into $(n \times \{0\}) \cup (\omega \times \{1\})$ given by

$$f(x) = \begin{cases} (x, 0), & \text{if } x < n; \\ (y, 1), & \text{if } x = n + y. \end{cases}$$

So $\omega = n + \omega$, and therefore $n + \alpha = \omega + \beta = \alpha$. □

For example, we have

$$1 + \omega = \omega \neq \omega + 1;$$

so ordinal addition is not commutative. Also,

$$0 < 1, \qquad\qquad 0 + \omega = \omega = 1 + \omega,$$

so the ordering in Theorem 80 cannot be made strict.

By Theorem 72, along with the Axiom of Infinity, we have the following initial segment of **ON**:

$$\{0, 1, 2, \dots; \omega, \omega + 1, \omega + 2, \dots; \omega + \omega, \omega + \omega + 1, \dots; \omega + \omega + \omega, \dots\}.$$

Here the ordinals following the semicolons (;) are limits.

4.7. Ordinal multiplication

Following the pattern of the previous section, we can extend Definition 17, of multiplication on ω, to **ON**. This time, the recursion needs only two parts.

Definition 32 (Ordinal multiplication)**.** For each ordinal α, the operation $x \mapsto \alpha \cdot x$ on **ON** is given by

$$\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha, \qquad \alpha \cdot \gamma = \sup(\{\alpha \cdot x \colon x \in \gamma\}),$$

where γ is not a successor. In particular,

$$\alpha \cdot 0 = 0, \qquad \alpha \cdot 1 = \alpha.$$

The operation $(x, y) \mapsto x \cdot y$ on **ON** is **ordinal multiplication,** and $\alpha \cdot \beta$ (or simply $\alpha\beta$) is the **ordinal product** of α and β. Notationally, multiplication is more binding than addition, so that $\alpha \cdot \beta + \gamma$ means $(\alpha \cdot \beta) + \gamma$.

Theorem 82. *For all ordinals α,*

$$0 \cdot \alpha = 0, \qquad 1 \cdot \alpha = \alpha.$$

Theorem 83. *For all ordinals α, β, and γ, where $\gamma > 0$,*

$$\alpha < \beta \Rightarrow \gamma \cdot \alpha < \gamma \cdot \beta.$$

Theorem 84. *For all ordinals α and β,*

$$\alpha \cdot \beta = \sup_{x \in \beta}(\alpha \cdot x + \alpha).$$

Theorem 85. *For all ordinals α and β,*

$$\alpha \cdot \beta = \operatorname{ord}(\alpha \times \beta),$$

where $\alpha \times \beta$ has the lexicographic ordering.

Lemma 14. *For all well-ordered classes $\boldsymbol{C}$, $\boldsymbol{D}$, and $\boldsymbol{E}$, the classes $(\boldsymbol{C} \times \boldsymbol{D}) \times \boldsymbol{E}$ and $\boldsymbol{C} \times (\boldsymbol{D} \times \boldsymbol{E})$, with the lexicographic orderings, are isomorphic.*

Theorem 86. *For all ordinals α, β, and γ,*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma, \qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Proof. Since isomorphic good orders have the same order-type, we have

$$\begin{aligned}
\alpha \cdot (\beta + \gamma) &= \operatorname{ord}(\alpha \times (\beta + \gamma)) \\
&= \operatorname{ord}(\alpha \times ((\beta \times \{0\}) \cup (\gamma \times \{1\}))) \\
&= \operatorname{ord}((\alpha \times \beta \times \{0\}) \cup (\alpha \times \gamma \times \{1\})) \\
&= \operatorname{ord}((\alpha \cdot \beta \times \{0\}) \cup (\alpha \cdot \gamma \times \{1\})) \\
&= \alpha \cdot \beta + \alpha \cdot \gamma,
\end{aligned}$$

and also $(\alpha \cdot \beta) \cdot \gamma = \operatorname{ord}((\alpha \times \beta) \times \gamma) = \operatorname{ord}(\alpha \times (\beta \times \gamma)) = \alpha \cdot (\beta \cdot \gamma)$. □

Theorem 87. *For all ordinals α, β, and γ,*

$$\alpha \leqslant \beta \Rightarrow \alpha \cdot \gamma \leqslant \beta \cdot \gamma.$$

Theorem 88 (Division)**.** *If $0 < \alpha$, then the system*

$$\alpha \cdot x + y = \beta, \qquad\qquad y < \alpha$$

has a unique ordinal solution, namely (γ, δ), where

$$\gamma = \sup(\{x \colon x \in \mathbf{ON} \ \& \ \alpha \cdot x \leqslant \beta\}), \qquad \delta = \operatorname{ord}(\beta \smallsetminus (\alpha \cdot \gamma)).$$

Proof. We have $\beta = 1 \cdot \beta \leqslant \alpha \cdot \beta$, so γ does exist, and $\gamma \leqslant \beta$. Then

$$\alpha \cdot \gamma \leqslant \beta < \alpha \cdot (\gamma + 1) = \alpha \cdot \gamma + \alpha.$$

Then $\alpha \cdot \gamma + \delta = \beta$, by Theorem 78, and $\delta < \alpha$. If $\alpha \cdot \zeta + \eta = \beta$ and $\eta < \alpha$, then $\alpha \cdot \zeta \leqslant \beta < \alpha \cdot (\zeta + 1)$, so $\zeta = \gamma$, and then $\eta = \delta$. □

An alternative way to solve the system is to note that, if $\beta < \alpha \cdot \beta$, then β is isomorphic to a section of $\alpha \times \beta$, by Theorems 64 and 65. This section is $\operatorname{pred}((\delta, \gamma))$ for some γ and δ; but

$$\operatorname{pred}((\delta, \gamma)) = (\alpha \times \gamma) \cup (\delta \times \{\gamma\}),$$

whose order type is $\alpha \cdot \gamma + \delta$.

There is a partial analogue of Theorem 81:

Theorem 89. *If $0 < n < \omega$, then $n \cdot \omega = \omega$.*

Proof. The function $(x, y) \mapsto n \cdot y + x$ from $n \times \omega$ to ω is an isomorphism, by the last theorem. $\square$

We cannot replace ω here with an arbitrary infinite ordinal, since we have

$$n \cdot (\omega + 1) = n \cdot \omega + n \cdot 1 = \omega + n.$$

We do have

$$2 \cdot \omega = \omega \neq \omega \cdot 2,$$

so ordinal multiplication is not commutative. Also,

$$1 < 2, \qquad 1 \cdot \omega = \omega = 2 \cdot \omega,$$

so the ordering in Theorem 87 cannot be made strict. Finally,

$$(1 + 1) \cdot \omega = 2 \cdot \omega = \omega \neq \omega + \omega = 1 \cdot \omega + 1 \cdot \omega,$$

so ordinal multiplication does not distribute from the right over addition.

We can extend the initial segment of **ON** given at the end of the last section:

$$\{0, 1, \ldots; \omega, \omega + 1, \ldots; \omega \cdot 2, \ldots; \omega \cdot 3, \ldots; \omega \cdot \omega, \ldots; \omega \cdot \omega \cdot \omega, \ldots\}.$$

4.8. Ordinal exponentiation

We now extend Definition 18, of exponentiation on ω.

Definition 33 (Ordinal exponentiation)**.** For each ordinal α, where $\alpha > 0$, the operation $x \mapsto \alpha^x$ on **ON** is given by

$$\alpha^0 = 1, \qquad \alpha^{\beta+1} = \alpha^\beta \cdot \alpha, \qquad \alpha^\gamma = \sup_{x \in \gamma} \alpha^x,$$

where γ is a limit. In particular,

$$\alpha^1 = \alpha.$$

We also define

$$0^0 = 1, \qquad 0^\alpha = 0,$$

where again $\alpha > 0$. The binary operation $(x, y) \mapsto x^y$ on **ON** is **ordinal exponentiation,** and α^β is the β-th **ordinal power** of α.

Theorem 90. *For all ordinals* α,

$$1^\alpha = 1.$$

Theorem 91. *If* $\alpha > 1$ *and* $\beta < \gamma$, *then*

$$\alpha^\beta < \alpha^\gamma.$$

Theorem 92. *If* $\alpha > 0$ *and* $\beta > 0$, *then*

$$\alpha^\beta = \sup_{x \in \beta}(\alpha^x \cdot \alpha).$$

It is possible to understand α^β as the ordinality of a certain well-ordered set obtained directly from α and β. Meanwhile, we *can* obtain the basic properties of exponentiation directly from Definition 33. The process is simplified by the following notions.

Definition 34. An embedding of a linear order in itself is an **endomorphism.** An endomorphism $\boldsymbol{F}$ of **ON** is called **normal** if

$$\boldsymbol{F}(\alpha) = \sup(\boldsymbol{F}[\,\alpha\,])$$

whenever α is a limit. (There is no requirement on $\boldsymbol{F}(0)$.)

Theorem 93. *The following operations on* **ON** *are normal:*
1) $x \mapsto \alpha + x$;
2) $x \mapsto \alpha \cdot x$, *when* $\alpha > 0$;
3) $x \mapsto \alpha^x$, *when* $\alpha > 1$.

Proof. They are endomorphisms of **ON**, by Theorems 72, 83, and 91. Then they are normal by the original definitions (Definitions 30, 32, and 33). □

Lemma 15. *If* $\boldsymbol{F}$ *is normal and* $0 \subset c \subset$ **ON**, *then*

$$\boldsymbol{F}(\sup(c)) = \sup(\boldsymbol{F}[\,c\,]).$$

Proof. Let $\alpha = \sup(c)$. There are two cases to consider.

1. If $\alpha \in c$, then α is the greatest element of c, so $\sup(\boldsymbol{F}[\,c\,]) = \boldsymbol{F}(\alpha)$ since $\boldsymbol{F}$ preserves order.

2. Suppose $\alpha \notin c$. Then $c \subseteq \alpha$, so $\sup(\boldsymbol{F}[\,c\,]) \leqslant \sup(\boldsymbol{F}[\,\alpha\,])$. Also, if $\beta < \alpha$, then $\beta < \gamma < \alpha$ for some γ in c, so $\sup(\boldsymbol{F}[\,\alpha\,]) \leqslant \sup(\boldsymbol{F}[\,c\,])$. Therefore $\sup(\boldsymbol{F}[\,\alpha\,]) = \sup(\boldsymbol{F}[\,c\,])$. But α must be a limit, and hence $\sup(\boldsymbol{F}[\,\alpha\,]) = \boldsymbol{F}(\alpha)$ by normality of $\boldsymbol{F}$. □

Theorem 94. *For all ordinals α, β, and γ,*

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}, \qquad \alpha^{\beta\cdot\gamma} = (\alpha^{\beta})^{\gamma}.$$

Proof. By the lemma, if $1 < \alpha$ and δ is a limit, if the first equation holds when $\gamma < \delta$, then

$$\alpha^{\beta} \cdot \alpha^{\delta} = \alpha^{\beta} \cdot \sup_{x\in\delta} \alpha^{x} = \sup_{x\in\delta}(\alpha^{\beta} \cdot \alpha^{x}) = \sup_{x\in\delta} \alpha^{\beta+x} = \alpha^{\sup_{x\in\delta}(\beta+x)} = \alpha^{\beta+\delta}.$$

□

Just as natural numbers can be written in base ten, so the next theorem below allows ordinals to be written in base α whenever $\alpha > 1$. We shall work out the details for base ω in the next section. Meanwhile, the theorem needs the following:

Lemma 16. *If $\alpha > 1$, then for all ordinals β,*

$$\alpha^{\beta} \geqslant \beta.$$

Proof. Since

$$\beta > \alpha^{\beta} \Rightarrow \alpha^{\beta} > \alpha^{\alpha^{\beta}},$$

the class $\{x\colon x \in \mathbf{ON} \mathbin{\&} \alpha^{x} < x\}$ has no least element, so it is empty. □

Theorem 95. *For all ordinals α and β, where $\alpha > 1$ and $\beta > 0$, the system*

$$\alpha^{x} \cdot y + z = \beta, \qquad 0 < y < \alpha, \qquad z < \alpha^{x}$$

has a unique ordinal solution.

Proof. The system implies

$$\alpha^{x} \cdot y \leqslant \beta < \alpha^{x} \cdot (y+1), \qquad \alpha^{x} \leqslant \beta < \alpha^{x+1}.$$

Since the functions $y \mapsto \alpha^{x} \cdot y$ and $x \mapsto \alpha^{x}$ are increasing, the original system has at most one solution. The class $\{x\colon x \in \mathbf{ON} \mathbin{\&} \alpha^{x} \leqslant \beta\}$ has the upper bound β, so it has a supremum, γ, which belongs to the class by normality of $x \mapsto \alpha^x$. Then the class $\{y\colon \alpha^\gamma \cdot y \leqslant \beta\}$ has the strict upper bound α, so the supremum δ of the class is less than α, but it belongs to the class. Then $(\gamma, \delta, \operatorname{ord}(\beta \smallsetminus \alpha^\gamma \cdot \delta))$ is the desired solution. □

We now have the following initial segment of **ON**:

$$\{0, 1, \dots; \omega, \omega + 1, \dots, \omega \cdot 2, \dots; \omega^2, \omega^2 + 1, \dots; \omega^2 + \omega, \dots; \omega^2 \cdot 2, \dots; \omega^3, \dots; \omega^\omega, \dots; \omega^{\omega \cdot 2}, \dots; \omega^{\omega^2}, \dots; \omega^{\omega^\omega}, \dots; \omega^{\omega^{\omega^\omega}}, \dots\}.$$

This set is $\sup_{x \in \omega} \boldsymbol{F}(x)$, where

$$\boldsymbol{F}(0) = \omega, \qquad \boldsymbol{F}(n+1) = \omega^{\boldsymbol{F}(n)}.$$

We may use for $\sup_{x \in \omega} \boldsymbol{F}(x)$ the notation

$$\epsilon_0.$$

This set is closed under the operations that we have defined so far. The reason for the subscript 0 in ϵ_0 is the following.

Theorem 96. ϵ_0 *is the least solution of the equation*

$$\omega^x = x.$$

We end the section by developing an alternative definition of exponentiation, parallel to Theorems 77 and 85.

Definition 35. The set of functions from a set b to a set a is denoted by

$${}^b a.$$

It is indeed a set, since it is included in $\mathscr{P}(b \times a)$. Suppose a is an ordinal α. If $f \in {}^b\alpha$, the **support** of f is the set $\{x\colon x \in b \;\&\; f(x) \neq 0\}$; this can be denoted by

$$\operatorname{supp}(f).$$

Let $\operatorname{fs}({}^b\alpha)$ be the set of elements of ${}^b\alpha$ with finite support, that is,

$$\operatorname{fs}({}^b\alpha) = \{x\colon x \in {}^b\alpha \;\&\; |\operatorname{supp}(x)| < \omega\}.$$

Suppose now b is an ordinal β. Then $\operatorname{fs}({}^\beta\alpha)$ can be given the **right lexicographic ordering,** whereby $f < g$, provided $f(\gamma) < g(\gamma)$, where $\gamma = \sup(\{x\colon f(x) \neq g(x)\})$. See Table 4.2.

$$\begin{array}{l}
(0,0,0,0,\dots) \\
(1,0,0,0,\dots) \\
(2,0,0,0,\dots) \\
\dots\dots\dots\dots \\
(0,1,0,0,\dots) \\
(1,1,0,0,\dots) \\
(2,1,0,0,\dots) \\
\dots\dots\dots\dots \\
(0,0,1,0,\dots) \\
(1,0,1,0,\dots) \\
\dots\dots\dots\dots \\
(0,1,1,0,\dots) \\
\dots\dots\dots\dots
\end{array}$$

Table 4.2. The right lexicographic ordering of $\mathrm{fs}({}^{\beta}\alpha)$.

Theorem 97. *The right lexicographic ordering well-orders* $\mathrm{fs}({}^{\beta}\alpha)$.

Theorem 98. *For all ordinals α and β,*

$$\alpha^{\beta} = \mathrm{ord}(\mathrm{fs}({}^{\beta}\alpha)),$$

where $\mathrm{fs}({}^{\beta}\alpha)$ *has the right lexicographic ordering,*

Proof. Suppose $\alpha > 0$. Then

$$0^{\alpha} = 0 = \mathrm{ord}(0) = \mathrm{ord}(\mathrm{fs}({}^{\alpha}0)), \quad \alpha^{0} = \{0\} = \mathrm{ord}(\{0\}) = \mathrm{ord}(\mathrm{fs}({}^{0}\alpha)).$$

Suppose $\alpha^{\beta} = \mathrm{ord}(\mathrm{fs}({}^{\beta}\alpha))$. Then

$$\alpha^{\beta+1} = \alpha^{\beta} \cdot \alpha = \mathrm{ord}(\alpha^{\beta} \times \alpha) = \mathrm{ord}(\mathrm{fs}({}^{\beta}\alpha) \times \alpha) = \mathrm{ord}(\mathrm{fs}({}^{\beta+1}\alpha)),$$

since the function $(x, y) \mapsto x \cup \{(\beta, y)\}$ is an order-preserving bijection from $\mathrm{fs}({}^{\beta}\alpha) \times \alpha$ to $\mathrm{fs}({}^{\beta+1}\alpha)$.

Suppose finally γ is a limit, and $\alpha^\beta = \operatorname{ord}(\operatorname{fs}({}^\beta\alpha))$ whenever $\beta < \gamma$. We have

$$\operatorname{fs}({}^\gamma\alpha) = \bigcup\{\operatorname{fs}({}^x\alpha)\colon x \in \gamma\}.$$

Also, if $\beta < \gamma$, then $\operatorname{fs}({}^\beta\alpha)$ is a section of $\operatorname{fs}({}^\gamma\alpha)$, so the isomorphism from $\operatorname{fs}({}^\gamma\alpha)$ to $\operatorname{ord}(\operatorname{fs}({}^\gamma\alpha))$ restricts to an isomorphism from $\operatorname{fs}({}^\beta\alpha)$ to $\operatorname{ord}(\operatorname{fs}({}^\beta\alpha))$. Therefore

$$\operatorname{ord}(\operatorname{fs}({}^\gamma\alpha)) = \bigcup\{\operatorname{ord}(\operatorname{fs}({}^x\alpha))\colon x \in \gamma\} = \sup\{\alpha^x : x \in \gamma\} = \alpha^\gamma. \qquad \square$$

4.9. Base omega

If $1 < b < \omega$, then every element n of ω can be written uniquely as a sum

$$b^m \cdot a_0 + b^{m-1} \cdot a_1 + \cdots + b \cdot a_{m-1} + a_m, \tag{4.4}$$

where $m \in \omega$ and $a_k \in b$, and $a_0 > 0$ unless $n = 0$. The sum is the *base-b representation* of n. The notation in (4.4) can be defined precisely as follows.

Definition 36. Given a function $x \mapsto \alpha_x$ from ω (or one of its elements) into $\mathbf{ON}$, and an element c of ω, we define the function $x \mapsto \sum_{i=c}^{x} \alpha_i$ on ω recursively by

$$k < c \Rightarrow \sum_{i=c}^{k} \alpha_i = 0, \qquad \sum_{i=c}^{c} \alpha_i = \alpha_c, \qquad \sum_{i=c}^{c+n+1} \alpha_i = \sum_{i=c}^{c+n} \alpha_i + \alpha_{c+n+1}.$$

We may write

$$\sum_{i=c}^{c+n} \alpha_i = \alpha_c + \cdots + \alpha_{c+n}.$$

An ordinal is **positive** if it is greater than 0. By Theorem 95, for every positive ordinal α, there is a unique ordinal β such that

$$\omega^\beta \leqslant \alpha < \omega^{\beta+1};$$

we may refer to β as the **degree** of α, writing

$$\deg(\alpha) = \beta.$$

In particular, the ordinals of degree 0 are just the natural numbers that are successors. We may suppose

$$\deg(0) < 0.$$

Lemma 17. *For all functions $x \mapsto \alpha_x$ from ω to $\mathbf{ON}$, for all c, k, and n in ω,*

$$\sum_{i=c}^{k+n} \alpha_i = \sum_{i=c}^{k} \alpha_i + \sum_{i=k+1}^{k+n} \alpha_i.$$

Theorem 99. *For every positive ordinal α, there are, uniquely,*

1) *a function $x \mapsto (\alpha_x, a_x)$ from ω to $\mathbf{ON} \times \omega$,*
2) *an element $\ell(\alpha)$ of ω,*

such that

$$\alpha_0 > \cdots > \alpha_{\ell(\alpha)}, \qquad \ell(\alpha) < i \Rightarrow \alpha_i = 0, \qquad a_i > 0 \Leftrightarrow i \leqslant \ell(\alpha),$$

and

$$\alpha = \sum_{k=0}^{\ell(\alpha)} \omega^{\alpha_k} \cdot a_k = \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_{\ell(\alpha)}} \cdot a_{\ell(\alpha)}. \tag{4.5}$$

Here

$$\alpha_0 = \deg(\alpha).$$

Proof. Given α, and using Theorems 78 and 95, by finite recursion on ω we define $x \mapsto (\alpha_x, a_x)$ by requiring $a_k \in \omega$, and $a_k > 0$ unless $\alpha_k = 0$, and

$$\omega^{\alpha_0} \cdot a_0 \leqslant \alpha < \omega^{\alpha_0} \cdot (a_0 + 1),$$

$$\omega^{\alpha_{k+1}} \cdot a_{k+1} \leqslant \operatorname{ord}(\alpha \smallsetminus \sum_{i=0}^{k} \omega^{\alpha_i} \cdot a_i) < \omega^{\alpha_{k+1}} \cdot (a_{k+1} + 1).$$

Then $\alpha_0 = \deg(\alpha)$. Also, for all k in ω,

$$\operatorname{ord}(\alpha \smallsetminus \sum_{i=0}^{k} \omega^{\alpha_i} \cdot a_i) < \omega^{\alpha_k},$$

so we have $\alpha_{k+1} < \alpha_k$, unless $a_{k+1} = 0$. Therefore $a_{k+1} = 0$ for some k. The least such k is $\ell(\alpha)$, and then we have (4.5).

Uniqueness is by the lemma and Theorem 95. In detail: Suppose we have also

$$\alpha = \sum_{i=0}^{\ell(\beta)} \omega^{\beta_i} \cdot b_i,$$

where

$$\beta_0 > \cdots > \beta_{\ell(\beta)}, \qquad \ell(\alpha) < i \Rightarrow \beta_i = 0, \qquad a_i > 0 \Leftrightarrow i \leqslant \ell(\beta).$$

If $(\alpha_i, a_i) = (\beta_i, b_i)$ when $i < n$, then, by the lemma, we have

$$\begin{aligned}\omega^{\alpha_n} \cdot a_n + \sum_{i=n+1}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i &= \sum_{i=n}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i \\ &= \sum_{i=n}^{\ell(\beta)} \omega^{\beta_i} \cdot b_i = \omega^{\beta_n} \cdot b_n + \sum_{i=n+1}^{\ell(\beta)} \omega^{\beta_i} \cdot b_i.\end{aligned}$$

If either side is 0, then $a_n = 0 = b_n$, so $\alpha_n = 0 = \beta_n$ by definition. If one side is not 0, then $(\alpha_n, a_n) = (\beta_n, b_n)$ by Theorem 95. □

When α is written as in (4.5), it is said to be in **normal form.**[5] We can add ordinals in normal form by using:

Theorem 100. *If $\beta < \alpha$, and $n \in \omega$, and $0 < m < \omega$, then*

$$\omega^\beta \cdot n + \omega^\alpha \cdot m = \omega^\alpha \cdot m.$$

Proof. For some positive γ we have $\beta + \gamma = \alpha$, and then

$$\omega^\beta \cdot n + \omega^\alpha \cdot m = \omega^\beta \cdot (n + \omega^\gamma \cdot m) = \omega^\beta \cdot \omega^\gamma \cdot m = \omega^\alpha \cdot m$$

by Theorem 81. □

For example,

$$\omega^3 + \omega \cdot 8 + \omega^2 \cdot 5 = \omega^3 + \omega^2 \cdot 5.$$

[5]The terminology is due to Cantor, as are most of the results of this section and, indeed, this chapter.

Corollary. *If* $\deg(\alpha) > \deg(\beta)$*, then*

$$\beta + \alpha = \alpha.$$

To *multiply,* we need a generalization of Theorem 89:

Theorem 101. *For all positive natural numbers n and positive ordinals α,*

$$n \cdot \omega^{\alpha} = \omega^{\alpha}.$$

Theorem 102. *For all infinite ordinals α and positive natural numbers n,*

$$\alpha \cdot n = \omega^{\alpha_0} \cdot a_0 \cdot n + \sum_{i=1}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i.$$

Theorem 103. *For all infinite ordinals α and positive ordinals β,*

$$\alpha \cdot \omega^{\beta} = \omega^{\alpha_0 + \beta}.$$

Proof. For the case $\beta = 1$, we note that, by the last theorem,

$$\alpha \cdot \omega = \sup_{x \in \omega} \alpha \cdot x = \sup_{x \in \omega} \omega^{\alpha_0} \cdot a_0 \cdot x = \sup_{x \in \omega} \omega^{\alpha_0} \cdot x = \omega^{\alpha_0} \cdot \omega = \omega^{\alpha_0 + 1}.$$

Similarly, if the claim holds for some β, then

$$\alpha \cdot \omega^{\beta+1} = \sup_{x \in \omega} \alpha \cdot \omega^{\beta} \cdot x = \sup_{x \in \omega} \omega^{\alpha_0 + \beta} \cdot x = \omega^{\alpha_0 + \beta + 1}.$$

Finally, if γ is a limit, and the claim holds when $\beta < \gamma$, then

$$\alpha \cdot \omega^{\gamma} = \sup_{x \in \gamma} \alpha \cdot \omega^{x} = \sup_{x \in \gamma} \omega^{\alpha_0 + x} = \omega^{\alpha_0 + \gamma}. \qquad \square$$

For example, we can now compute:

$$\begin{aligned}
&(\omega^{\omega+1} \cdot 3 + \omega^{6} \cdot 4 + 1) \cdot (\omega^{\omega^2} \cdot 2 + 3) \\
&= (\omega^{\omega+1} \cdot 3 + \omega^{6} \cdot 4 + 1) \cdot (\omega^{\omega^2} \cdot 2) + (\omega^{\omega+1} \cdot 3 + \omega^{6} \cdot 4 + 1) \cdot 3 \\
&= \omega^{\omega+1+\omega^2} \cdot 2 + \omega^{\omega+1} \cdot 3 \cdot 3 + \omega^{6} \cdot 4 + 1 \\
&= \omega^{\omega^2} \cdot 2 + \omega^{\omega+1} \cdot 9 + \omega^{6} \cdot 4 + 1.
\end{aligned}$$

Finally, for *exponentiation,* we have:

Theorem 104. *For all positive natural numbers n and positive ordinals α,*

$$n^{\omega^\alpha} = \begin{cases} \omega^{\omega^{\alpha-1}}, & \text{if } 0 < \alpha < \omega, \\ \omega^{\omega^\alpha}, & \text{if } \omega < \alpha. \end{cases}$$

Theorem 105. *For all infinite ordinals α and natural numbers n,*

1) if α is a limit, that is, $\alpha_{\ell(\alpha)} > 0$, then

$$\alpha^{n+1} = \omega^{\alpha_0 \cdot n} \cdot \alpha,$$

2) if α is a successor, that is, $\alpha_{\ell(\alpha)} = 0$, then

$$\alpha^{n+1} = \omega^{\alpha_0 \cdot (n+1)} \cdot a_0 + \sum_{i=1}^{n} \omega^{\alpha_0 \cdot (n+1-i)} \cdot (\beta + \alpha_0 \cdot \alpha_{\ell(\alpha)}) + \beta + \alpha_{\ell(\alpha)},$$

where

$$\alpha = \sum_{i=0}^{\ell(\alpha)} \omega^{\alpha_i} \cdot a_i = \omega^{\alpha_0} \cdot a_0 + \beta + \alpha_{\ell(\alpha)}.$$

Proof. 1. The claim holds trivially when $n = 0$. If it holds when $n = m$, then it holds when $n = m + 1$ by Theorem 103.

2. The claim holds trivially when $n = 0$. If it holds when $n = m$, then

$$\begin{aligned} \alpha^{m+2} &= \alpha^{m+1} \cdot (\omega^{\alpha_0} \cdot a_0 + \beta + a_{\ell(\alpha)}) \\ &= \omega^{\alpha_0 \cdot (m+2)} \cdot a_0 + \omega^{\alpha_0 \cdot (m+1)} \cdot \beta + \omega^{\alpha_0 \cdot (m+1)} \cdot a_0 \cdot a_{\ell(\alpha)} \\ &\qquad + \sum_{i=1}^{m} \omega^{\alpha_0 \cdot (m+1-i)} \cdot (\beta + \alpha_0 \cdot \alpha_{\ell(\alpha)}) + \beta + \alpha_{\ell(\alpha)} \\ &= \omega^{\alpha_0 \cdot (m+2)} \cdot a_0 + \sum_{i=1}^{m+1} \omega^{\alpha_0 \cdot (m+2-i)} \cdot (\beta + \alpha_0 \cdot \alpha_{\ell(\alpha)}) + \beta + \alpha_{\ell(\alpha)}, \end{aligned}$$

so it holds when $n = m + 1$. □

Theorem 106. *For all infinite ordinals α and positive ordinals β,*

$$\alpha^{\omega^\beta} = \omega^{\alpha_0 \cdot \omega^\beta}.$$

Proof. By the previous theorem, we have

$$\alpha^{\omega} = \sup_{x \in \omega} \alpha^{x} = \sup_{x \in \omega} \omega^{\alpha_0 \cdot x} = \omega^{\alpha_0 \cdot \omega},$$

so the claim holds when $\gamma = 1$. Suppose it holds for some β; then

$$\alpha^{\omega^{\beta+1}} = (\alpha^{\omega^{\beta}})^{\omega} = (\omega^{\alpha_0 \cdot \omega^{\beta}})^{\omega} = \omega^{\alpha_0 \cdot \omega^{\beta+1}}.$$

Finally, if γ is a limit, and the claim holds when $0 < \beta < \gamma$, then

$$\alpha^{\omega^{\gamma}} = \alpha^{\sup_{x \in \gamma} \omega^{x}} = \sup_{x \in \gamma} \alpha^{\omega^{x}} = \sup_{x \in \gamma} \omega^{\alpha_0 \cdot \omega^{x}} = \omega^{\alpha_0 \cdot \omega^{\gamma}}. \qquad \square$$

Summing up, we have

Theorem 107. *For all infinite ordinals α, limit ordinals β, and positive natural numbers n,*

$$\alpha^{\beta} = \omega^{\alpha_0 \cdot \beta}, \qquad \alpha^{\beta+n} = \omega^{\alpha_0 \cdot \beta} \cdot \alpha^{n}.$$

For example,

$$
\begin{aligned}
&(\omega^{\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13}\cdot 11+\omega^{5}\cdot 17+19)^{\omega^{37}+\omega\cdot 2+3}\\
&=\omega^{(\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13)\cdot(\omega^{37}+\omega\cdot 2)}\cdot(\omega^{\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13}\cdot 11+\omega^{5}\cdot 17+19)^{3}\\
&=\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2}\cdot(\,\omega^{(\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13)\cdot 3}\cdot 11+\\
&\qquad\omega^{(\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13)\cdot 2}\cdot(\omega^{5}\cdot 17+209)+\\
&\qquad\omega^{\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13}\cdot(\omega^{5}\cdot 17+209)+\\
&\qquad\omega^{5}\cdot 17+19)\\
&=\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2}\cdot(\,\omega^{\omega^{\omega}\cdot 21+\omega^{31}\cdot 29+13}\cdot 11+\\
&\qquad\omega^{\omega^{\omega}\cdot 14+\omega^{31}\cdot 29+13}\cdot(\omega^{5}\cdot 17+209)+\\
&\qquad\omega^{\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13}\cdot(\omega^{5}\cdot 17+209)+\\
&\qquad\omega^{5}\cdot 17+19)\\
&=\ \omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^{\omega}\cdot 21+\omega^{31}\cdot 29+13}\cdot 11+\\
&\quad\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^{\omega}\cdot 14+\omega^{31}\cdot 29+18}\cdot 17+\\
&\quad\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^{\omega}\cdot 14+\omega^{31}\cdot 29+13}\cdot 209)+\\
&\quad\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+18}\cdot 17+\\
&\quad\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+\omega^{\omega}\cdot 7+\omega^{31}\cdot 29+13}\cdot 209)+\\
&\quad\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2+5}\cdot 17+\\
&\quad\omega^{\omega^{\omega+37}+\omega^{\omega+1}\cdot 2}\cdot 19.
\end{aligned}
$$

5. Cardinality

5.1. Cardinality

Definition 37. If a set a is equipollent to some ordinal, then the *least* ordinal to which a is equipollent is called the **cardinality** of a. We may denote the cardinality of a by

$$\mathrm{card}(a).$$

In particular, every ordinal has a cardinality, so every well-ordered set has a cardinality. The converse holds too: every set with a cardinality can be well-ordered. If a is finite, we have

$$\mathrm{card}(a) = |a|.$$

If two sets a and b have cardinalities, then

$$a \approx b \Leftrightarrow \mathrm{card}(a) = \mathrm{card}(b), \qquad a \prec b \Leftrightarrow \mathrm{card}(a) < \mathrm{card}(b).$$

So the following theorem is easy when a and b have cardinalities. However, we do not yet *know* that all sets have cardinalities.

Theorem 108 (Schroeder–Bernstein[1]). *For all sets a and b,*

$$a \preccurlyeq b \;\&\; b \preccurlyeq a \Rightarrow a \approx b.$$

Proof. Suppose f embeds a in b, and g embeds b in a. Define $x \mapsto (a_x, b_x)$ recursively on ω by

$$(a_0, b_0) = (a, b), \qquad (a_{n+1}, b_{n+1}) = (g[f[a_n]], f[g[b_n]]).$$

Then, for each n in ω, the relation defined by

$$x \in a_n \smallsetminus a_{n+1} \;\&\; y \in b_n \smallsetminus b_{n+1} \;\&\; (f(x) = y \vee x = g(y))$$

[1] The theorem is also called the Cantor–Bernstein Theorem, as for example by Levy [25, III.2.8, p. 85], who nonetheless observes that Dedekind gave the first proof in 1887.

is a bijection from $a_n \smallsetminus a_{n+1}$ to $b_n \smallsetminus b_{n+1}$. Since f is injective,

$$f[\bigcap\{a_x : x \in \omega\}] = \bigcap\{b_{x+1} : x \in \omega\} = \bigcap\{b_x : x \in \omega\}.$$

Therefore $a \approx b$. □

With a bit more work, the theorem can be shown to hold for arbitrary classes:

Porism. *For all classes* $\boldsymbol{C}$ *and* $\boldsymbol{D}$*,*

$$\boldsymbol{C} \preccurlyeq \boldsymbol{D} \mathbin{\&} \boldsymbol{D} \preccurlyeq \boldsymbol{C} \Rightarrow \boldsymbol{C} \approx \boldsymbol{D}.$$

Proof. Suppose $\boldsymbol{F}$ is an embedding of $\boldsymbol{C}$ in $\boldsymbol{D}$, and $\boldsymbol{G}$ is an embedding of $\boldsymbol{D}$ in $\boldsymbol{C}$. We can adjust the proof of the Recursion Theorem (Theorem 30) to show that there is a relation $\boldsymbol{R}$ such that, for all a in $\boldsymbol{C}$ and all n in ω,

$$\begin{gathered} 0 \mathbin{\boldsymbol{R}} a \Leftrightarrow a \in \boldsymbol{C}, \\ n' \mathbin{\boldsymbol{R}} a \Leftrightarrow \exists x\,(n \mathbin{\boldsymbol{R}} x \mathbin{\&} a = \boldsymbol{G}(\boldsymbol{F}(x))). \end{gathered}$$

Likewise, there is a relation $\boldsymbol{S}$ such that, for all b in $\boldsymbol{D}$ and all n in ω,

$$\begin{gathered} 0 \mathbin{\boldsymbol{S}} b \Leftrightarrow b \in \boldsymbol{D}, \\ n' \mathbin{\boldsymbol{S}} b \Leftrightarrow \exists y\,(n \mathbin{\boldsymbol{S}} y \mathbin{\&} b = \boldsymbol{F}(\boldsymbol{G}(y))). \end{gathered}$$

Denote the class $\{x : n \mathbin{\boldsymbol{R}} x\}$ by $\boldsymbol{C}_n$, and $\{y : n \mathbin{\boldsymbol{S}} y\}$ by $\boldsymbol{D}_n$. As before,

$$\boldsymbol{C}_n \smallsetminus \boldsymbol{C}_{n+1} \approx \boldsymbol{D}_n \smallsetminus \boldsymbol{D}_{n+1}, \qquad \bigcap_{n\in\omega} \boldsymbol{C}_n \approx \bigcap_{n\in\omega} \boldsymbol{D}_n$$

(where $\bigcap_{n\in\omega} \boldsymbol{C}_n = \{x : \forall z\,(z \in \omega \Rightarrow z \mathbin{\boldsymbol{R}} x\})$). Therefore $\boldsymbol{C} \approx \boldsymbol{D}$. □

However, the following has no generalization to classes:

Theorem 109 (Cantor)**.** *For all sets* a*,*

$$a \prec \mathscr{P}(a).$$

Proof. The function $x \mapsto \{x\}$ shows $a \preccurlyeq \mathscr{P}(a)$. Suppose f is an embedding of a in $\mathscr{P}(a)$. Let b be the set $\{x\colon x \in a \mathbin{\&} x \notin f(x)\}$. Then

$$c \in b \Rightarrow c \in b \smallsetminus f(c), \qquad\qquad c \in a \smallsetminus b \Rightarrow c \in f(c) \smallsetminus b$$

Thus, if $c \in a$, then $f(c) \neq b$. So $b \notin \operatorname{rng}(f)$. Therefore, there is no bijection from a to $\mathscr{P}(a)$; so $a \prec \mathscr{P}(a)$. □

The theorem may be *false* when applied to proper classes. Indeed,

$$\mathscr{P}(\mathbf{V}) = \mathbf{V}.$$

As noted on page 93, the following is consistent with GST, and with the rest of our axioms so far, except Infinity:

Axiom 8 (Power Set)**.** *The power class of a set is a set: that is,*

$$\exists x\, x = \mathscr{P}(a).$$

We may now refer to the power class of a set as its **power set.** An infinite set that is not equipollent to ω is called **uncountable;** all other sets are **countable.** So $\mathscr{P}(\omega)$ is uncountable. However, we do not yet know whether it, or any other uncountable set, has a cardinality.

Theorem 110. *For all sets a and b, the class ${}^b a$ is a set.*

5.2. Cardinals

Definition 38. An ordinal that is the cardinality of some ordinal is a **cardinal.** The cardinals compose the class denoted by

$$\mathbf{CN};$$

this is a subclass of $\mathbf{ON}$. Cardinals are denoted by minuscule Greek letters like κ, λ, μ, and so on.

The class $\mathbf{CN}$ inherits the ordering $<$ of $\mathbf{ON}$, which is $\in$ and $\subset$; on $\mathbf{CN}$ the ordering is also $\prec$. The finite ordinals are cardinals. Also, ω is a cardinal. But $\omega + 1$ is not a cardinal, since $\omega < \omega + 1$, but $\omega + 1 \approx 1 + \omega = \omega$.

Theorem 111. *Every cardinal is a limit ordinal.*

The converse fails: $\omega \cdot 2$ is a limit ordinal, but $\omega < \omega \cdot 2$, and $\omega \cdot 2 \approx 2 \cdot \omega = \omega$. There are uncountable cardinals:

Lemma 18 (Hartogs)**.** *For every set, there is an ordinal that does not embed in it.*

Proof. Supposing a is a set, let b be the subset of $\mathscr{P}(a) \times \mathscr{P}(a \times a)$ comprising those well-ordered sets $(c, <)$ such that $c \subseteq a$. If $\operatorname{ord}(c, <) = \beta$, and $\gamma < \beta$, then $\operatorname{ord}(d, <) = \gamma$ for some section d of c. This shows that $\{\operatorname{ord}(\mathfrak{c})\colon \mathfrak{c} \in b\}$ is a transitive subset of $\mathbf{ON}$; so it is an ordinal α. If f is an embedding of β in a, then f determines an element of b whose ordinality is β; so $\beta \in \alpha$. Since $\alpha \notin \alpha$, there is no injection of α in a. □

In the proof, α is the class of ordinals that embed in a. If α were simply defined this way, it would not obviously be a set. In any case, we can now make the following:

Definition 39. For every cardinal κ, by Hartogs's Lemma, there is an ordinal α such that $\kappa < \alpha$, but $\kappa \not\approx \alpha$. Therefore $\kappa < \operatorname{card}(\alpha)$. Thus κ has a **cardinal successor,** denoted by

$$\kappa^+;$$

this is the *least* of the cardinals that are greater than κ.

Theorem 112. *The supremum of a set of cardinals is a cardinal.*

Proof. Let a be a set of cardinals. If $\kappa < \sup(a)$, then $\kappa < \lambda$ for some λ in a, and therefore $\kappa \neq \operatorname{card}(\sup(a))$. Therefore $\sup(a)$ must be a cardinal (namely its own cardinality). □

Definition 40. The function

$$x \mapsto \aleph_x$$

from $\mathbf{ON}$ into $\mathbf{CN}$ is given recursively by

$$\aleph_0 = \omega, \qquad \aleph_{\alpha'} = (\aleph_\alpha)^+, \qquad \aleph_\beta = \sup_{x \in \beta} \aleph_x,$$

where β is a limit ordinal. Here $\aleph$ is *aleph,* the first letter of the Hebrew alphabet.

Theorem 113. *The function $x \mapsto \aleph_x$ is an isomorphism between* **ON** *and the class of infinite cardinals.*

5.3. Cardinal addition and multiplication

Definition 41. The **cardinal sum** of two cardinals is the cardinality of their ordinal sum. The **cardinal product** of two cardinals is the cardinality of their ordinal product. The operations of finding cardinal sums and products are **cardinal addition** and **cardinal multiplication,** respectively, and are denoted by $+$ and $\cdot$ (as are ordinal addition and multiplication; context must indicate which operations are meant).

Theorem 114. *For all cardinals κ and λ,*

$$\kappa+\lambda=\operatorname{card}((\kappa\times\{0\})\cup(\lambda\times\{1\})),\qquad \kappa\cdot\lambda=\operatorname{card}(\kappa\times\lambda).$$

Theorem 115. *For all cardinals κ, λ, and μ,*

$$\begin{gathered}
\kappa+\lambda=\lambda+\kappa,\\
\kappa+0=\kappa,\\
(\kappa+\lambda)+\mu=\kappa+(\lambda+\mu),\\
\kappa\cdot\lambda=\lambda\cdot\kappa,\\
\kappa\cdot 1=\kappa,\\
(\kappa\cdot\lambda)\cdot\mu=\kappa\cdot(\lambda\cdot\mu),\\
\kappa\cdot(\lambda+\mu)=\kappa\cdot\lambda+\kappa\cdot\mu,\\
\kappa\leqslant\lambda\Rightarrow\kappa+\mu\leqslant\lambda+\mu,\\
\kappa\leqslant\lambda\Rightarrow\kappa\cdot\mu\leqslant\lambda\cdot\mu.
\end{gathered}$$

The cardinal operations agree with the ordinal operations on ω.

Lemma 19. *The Cartesian product $a\times b$ is always a set.*

Proof. We have $a\times b=\bigcup\{a\times\{x\}\colon x\in b\}$. □

Lemma 20. *The class* $\mathbf{ON}\times\mathbf{ON}$ *is well-ordered by $<$, where*

$$\begin{aligned}(\alpha,\beta)<(\gamma,\delta)\Leftrightarrow{}&\max(\alpha,\beta)<\max(\gamma,\delta)\vee\\ &\Big(\max(\alpha,\beta)=\max(\gamma,\delta)\mathbin{\&}\big(\alpha<\gamma\vee(\alpha=\gamma\mathbin{\&}\beta<\delta)\big)\Big).\end{aligned}$$

(See Figure 5.1.) With respect to this ordering, $\mathbf{ON}\times\mathbf{ON}$ *is isomorphic to* $\mathbf{ON}$ *with its usual ordering.*

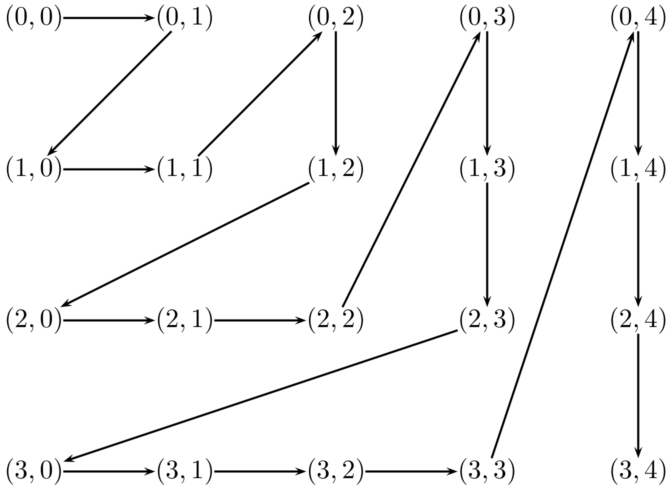

Figure 5.1. **ON** $\times$ **ON**, well-ordered

Proof. It is straightforward to show that the given relation is a linear ordering of **ON** $\times$ **ON**. If a is a nonempty subset of **ON** $\times$ **ON**, we can define

$$\begin{aligned}
\alpha &= \min\{\max(x, y)\colon (x, y) \in a\},\\
\beta &= \min\{x\colon \exists y\,(x, y) \in a \;\&\; \max(x, y) = \alpha)\},\\
\gamma &= \min\{y\colon (\beta, y) \in a\}.
\end{aligned}$$

Then (β, γ) is the least element of a. The linear ordering is left-narrow, since every section is a subset of $\delta \times \delta$ for some δ. So **ON** $\times$ **ON** is well-ordered. Since it is a proper class, it is isomorphic to **ON**, by the corollary to Theorem 65. $\square$

Lemma 21. *For all infinite cardinals κ,*

$$\kappa \cdot \kappa = \kappa.$$

Proof. We establish the claim by induction on the infinite cardinals. Suppose λ is an infinite cardinal, and the equation holds whenever $\omega \leqslant \kappa < \lambda$. Let $\boldsymbol{F}$ be the isomorphism from **ON** $\times$ **ON** onto **ON** guaranteed by the last lemma. For every ordinal α, the section $\operatorname{pred}(0,\alpha)$ of $\mathbf{ON}\times\mathbf{ON}$ is just $\alpha\times\alpha$. Then $\boldsymbol{F}[\,\alpha\times\alpha\,]$ must be a section of $\mathbf{ON}$: that is, $\boldsymbol{F}[\,\alpha\times\alpha\,]$ is an ordinal. Suppose $\boldsymbol{F}[\,\lambda\times\lambda\,]=\beta$. Then

$$\lambda=\lambda\cdot 1\leqslant\lambda\cdot\lambda=\operatorname{card}(\lambda\times\lambda)=\operatorname{card}(\beta)\leqslant\beta.$$

So $\lambda\leqslant\beta$. We shall show $\beta\leqslant\lambda$. For this, it is enough to show that, for all infinite cardinals μ,

$$\mu<\beta\Rightarrow\mu<\lambda.$$

Suppose μ is an infinite cardinal, and $\mu<\beta$. Then $\mu=\boldsymbol{F}(\gamma,\delta)$ for some ordinals γ and δ such that $(\gamma,\delta)\in\lambda\times\lambda$. Since λ is a limit ordinal by Theorem 111, the successor ζ of $\max(\beta,\gamma)$ is also less than λ. Hence

$$\mu\in\boldsymbol{F}[\,\zeta\times\zeta\,],\qquad\qquad \mu\subset\boldsymbol{F}[\,\zeta\times\zeta\,],$$

and so

$$\mu\leqslant\operatorname{card}(\zeta\times\zeta)=\operatorname{card}(\zeta)\cdot\operatorname{card}(\zeta)=\operatorname{card}(\zeta)<\lambda$$

by inductive hypothesis. □

Theorem 116. *For all cardinals κ and λ such that $0<\kappa\leqslant\lambda$ and λ is infinite,*

$$\kappa+\lambda=\kappa\cdot\lambda=\lambda.$$

In particular,

$$\aleph_\alpha+\aleph_\beta=\aleph_\alpha\cdot\aleph_\beta=\aleph_{\max(\alpha,\beta)}.$$

Proof. By the lemma, we need only observe

$$\lambda\leqslant\kappa+\lambda\leqslant\lambda+\lambda=\lambda\cdot 2\leqslant\lambda\cdot\lambda,\qquad\qquad \lambda\leqslant\kappa\cdot\lambda\leqslant\lambda\cdot\lambda.$$ □

Suppose we have a set a of cardinality κ, and a function $x\mapsto b_x$ on a such that, for all c in a, the set b_c has a cardinality. Let $\lambda=\sup\{\operatorname{card}(b_x)\colon x\in a\}$. It may appear that $\bigcup_{x\in a}b_x$ has a cardinality, which is bounded above by $\kappa\cdot\lambda$. For, if $c\in a$, let f_c be an embedding of b_c in λ. We may assume that $b_c\cap b_d=\varnothing$ when $c\neq d$. Then we should have an embedding of $\bigcup_{x\in a}b_x$ in $\kappa\times\lambda$, namely

$$\bigcup_{x\in a}\{(y,(x,f_x(y)))\colon y\in b_x\}.$$

The problem with this argument is that it assumes the existence of the function $x\mapsto f_x$ on a, and we do not yet have a way to ensure the existence of this function.

5.4. Cardinalities of ordinal powers

Definition 42. If $n \in \omega$ and $\alpha \in \mathbf{ON}$, an element a of ${}^n\alpha$ can be understood as the function $x \mapsto a_x$ on n. This function can also be written as

$$(a_0, \dots, a_{n-1})$$

and called an n**-tuple.** In case $n = 0$, this n-tuple is the empty set (which is indeed the only function from 0 to α).

Recalling the notation of Definition 35, we have:

Lemma 22. *For all infinite cardinals* κ*,*

$$\operatorname{card}(\operatorname{fs}({}^\omega\kappa)) = \kappa.$$

Proof. We have

$$\operatorname{fs}({}^\omega\kappa) = \bigcup\{{}^x\kappa \colon x \in \omega\}.$$

Let f be an embedding of $\kappa \times \kappa$ in κ. By finite recursion, we have a function $x \mapsto g_x$ on ω such that

1. g_0 is the embedding $\{(0,0)\}$ of ${}^0\kappa$ in κ.
2. if g_n is an embedding of ${}^n\kappa$ in κ, then g_{n+1} is the embedding of ${}^{n+1}\kappa$ in κ given by

$$g_{n+1}(a_0, \dots, a_n) = f(g_n(a_0, \dots, a_{n-1}), a_n).$$

Then we have an embedding h of $\operatorname{fs}({}^\omega\kappa)$ in $\omega \times \kappa$ given by

$$h(a_0, \dots, a_{n-1}) = (n, f_n(a_0, \dots, a_{n-1})).$$

So we have

$$\kappa \approx {}^1\kappa \preccurlyeq \operatorname{fs}({}^\omega\kappa) \preccurlyeq \omega \times \kappa \preccurlyeq \kappa.$$

By the Schröder–Bernstein Theorem, we are done. □

Lemma 23. *For all infinite cardinals* κ*, the set of finite subsets of* κ *has a cardinality, which is* κ*.*

Proof. The given set embeds in $\operatorname{fs}({}^\omega\kappa)$ under the map that takes every set $\{\alpha_0, \dots, \alpha_{n-1}\}$, where $\alpha_0 < \dots < \alpha_{n-1}$, to $(\alpha_0, \dots, \alpha_{n-1})$. □

Lemma 24. *For all infinite cardinals κ,*

$$\operatorname{card}(\operatorname{fs}({}^{\kappa}\kappa)) = \kappa.$$

Proof. Let f be the function from $\operatorname{fs}({}^{\kappa}\kappa)$ to $\operatorname{fs}({}^{\omega}\kappa)$ such that, if $g \in \operatorname{fs}({}^{\kappa}\kappa)$, and $\operatorname{dom}(g) = \{\alpha_0, \dots, \alpha_{n-1}\}$, where $\alpha_0 < \dots < \alpha_{n-1}$, then $f(g)$ is the function $x \mapsto g(\alpha_x)$ from n to κ. Let b be the set of finite subsets of κ. Then the function $x \mapsto (\operatorname{dom}(x), f(x))$ is an embedding of $\operatorname{fs}({}^{\kappa}\kappa)$ in $b \times \operatorname{fs}({}^{\omega}\kappa)$. Hence

$$\kappa \preccurlyeq \operatorname{fs}({}^{\kappa}\kappa) \preccurlyeq b \times \operatorname{fs}({}^{\omega}\kappa) \approx \kappa \times \kappa \approx \kappa. \qquad \square$$

Theorem 117. *For all cardinals κ and λ such that $\kappa > 0$, and $\lambda > 1$, and at least one of the two is infinite,*

$$\operatorname{card}(\operatorname{fs}({}^{\lambda}\kappa)) = \kappa \cdot \lambda = \kappa + \lambda = \max(\kappa, \lambda).$$

Proof. Let $\mu = \max(\kappa, \lambda)$. Then

$$\mu \preccurlyeq \operatorname{fs}({}^{\lambda}\kappa) \preccurlyeq \operatorname{fs}({}^{\mu}\mu) \approx \mu. \qquad \square$$

Therefore it would not be of great interest to define a cardinal power as the cardinality of an ordinal power. We should like to define κ^{λ} as the cardinality of ${}^{\lambda}\kappa$. The problem is that this set has no obvious good ordering. We are just going to declare that one exists, in the next section.

5.5. The Axiom of Choice

Theorem 118. *Every natural number embeds in every infinite set.*

Proof. Let a be an infinite set. Trivially, 0 embeds in a. Suppose n embeds in a under a function f. Since a is infinite, we have $f[n] \neq a$. Therefore $a \smallsetminus f[n]$ has an element b, and so $f \cup \{n, b\}$ is an embedding of $n + 1$ in a. By finite induction, every natural number embeds in a. $\square$

The theorem is *not* that ω embeds in every infinite set. One might try to adapt the proof of the theorem so as to give a recursive definition of an embedding of ω in a. The problem is that we have no way to select a particular element of $a \smallsetminus f[n]$. A technical term for what we need is given by the following.

Definition 43. A **choice-function** for a set a is a function f on $\mathscr{P}(a) \smallsetminus \{0\}$ such that $f(b) \in b$ for each nonempty subset b of a.

Theorem 119. *The set of natural numbers embeds in every infinite set that has a choice-function.*

Proof. Suppose f is a choice-function for an infinite set a. By transfinite recursion in one part, there is an embedding g of ω in a given by

$$g(n) = f(a \smallsetminus g[\,n\,]). \qquad \square$$

Theorem 120. *A set has a choice-function if and only if the set can be well-ordered.*

Proof. Suppose a set a has the choice-function f. We may assume $f(\varnothing)$ is defined, but is not in a. There is a function $\boldsymbol{G}$ on $\mathbf{ON}$ defined recursively by

$$\boldsymbol{G}(\alpha) = f(a \smallsetminus \boldsymbol{G}[\,\alpha\,]).$$

Suppose $\boldsymbol{G}[\alpha] \subseteq a$. If $\gamma < \beta < \alpha$, then $\boldsymbol{G}(\beta) \in a \smallsetminus \boldsymbol{G}[\beta]$, so in particular $\boldsymbol{G}(\beta) \in a \smallsetminus \{\boldsymbol{G}(\gamma)\}$, and therefore $\boldsymbol{G}(\beta) \neq \boldsymbol{G}(\gamma)$. Thus $\boldsymbol{G}$ embeds α in a. The same proof shows that $\boldsymbol{G}$ embeds $\mathbf{ON}$ in a, if $\boldsymbol{G}[\mathbf{ON}] \subseteq a$. Therefore $\boldsymbol{G}(\alpha) \notin a$ for some α. Let β be the least such α. Then $\boldsymbol{G}$ is a bijection from β to a. This induces a good ordering of a.

Now suppose conversely that a is well-ordered. Then there is a choice-function for a that assigns to each non-empty subset of a its least element. $\square$

Theorem 121 (Zorn's Lemma)**.** *Assume a has a choice-function. If $(a, <)$ is an order such that every linearly ordered subset of a has an upper bound in a, then a has a maximal element.*

Proof. Let f be the operation on $\mathscr{P}(a)$ taking each element b to the set (possibly empty) of strict upper bounds of b. So f is order-reversing, in the sense that

$$c \subseteq b \Rightarrow f(c) \supseteq f(b).$$

Let g be a choice-function for a, extended so that $g(\varnothing) \notin a$. Then define $\boldsymbol{H}$ on $\mathbf{ON}$ by

$$\boldsymbol{H}(\alpha) = g(f(a \cap \boldsymbol{H}[\,\alpha\,])).$$

As in the proof of the previous theorem, if $\boldsymbol{H}[\alpha] \subseteq a$, then $\boldsymbol{H}$ embeds α in a; in fact it embeds $(\alpha, \in)$ in $(a, <)$, so in particular $\boldsymbol{H}[\alpha]$ is linearly ordered and has an upper bound. Also as in the previous proof, $\boldsymbol{H}(\alpha) \notin a$ for some α. Let β be the least such α. Then $f(\boldsymbol{H}[\beta]) = \varnothing$, so the upper bound of $\boldsymbol{H}[\beta]$ is not strict. This upper bound is therefore a maximal element of a. □

A sort of converse to the last theorem is the following.

Theorem 122. *For a set a, let b be the set of functions f such that the domain of f is a subset of $\mathscr{P}(a) \smallsetminus \{0\}$ and, for all x in the domain, $f(x) \in x$. Every maximal element of b with respect to proper inclusion is a choice-function for a.*

Proof. Suppose $f \in b$, but is a not a choice-function for a. Then some nonempty subset c of a is not in the domain of f. But c has an element d, and then $f \cup \{(c, d)\}$ is in b. Thus f is not a maximal element of b. □

In the notation of the theorem, not only is b ordered by proper inclusion, but every linearly ordered subset c of b has the upper bound $\bigcup c$. So the following are equivalent statements about $\mathbf{V}$:

1. Every set has a choice-function.
2. Every set can be well-ordered, so it has a cardinality.
3. Every order has a maximal element, provided every linearly ordered subset of the order has an upper bound.

Axiom 9 (Choice)**.** *Every set has a choice-function.*

The Axiom of Choice, or AC, is a completely new kind of axiom, since it asserts the existence of certain sets (namely, choice-functions) that we do not already have as classes. However, as with the Generalized Continuum Hypothesis (Definition 46 below), so with AC, we shall see in Chapter 6 that we *can* assume it without contradicting our other axioms. The Axiom of Choice is convenient for mathematics in that it allows many theorems to be proved, such as the following, alluded to at the end of § 5.3.

Lemma 25. *For all sets a and functions $x \mapsto c_x$ on a, there is a function f on a such that $f(d) \in c_d$ for all d in a such that c_d is nonempty.*

Proof. Let f be a choice-function for $\bigcup_{x\in a} c_x$. Then we can let $f(x) = g(c_x)$. □

Theorem 123. *For all sets a and functions $x \mapsto b_x$ on a,*

$$\operatorname{card}(\bigcup_{x\in a} b_x) \leqslant \operatorname{card}(a) \cdot \sup_{x\in a} \operatorname{card}(b_x).$$

Proof. We can use the argument at the end of §5.3, since the function $x \mapsto f_x$ there does exist, by the lemma. □

5.6. Exponentiation

Cardinal exponentiation is quite different from ordinal exponentiation.

Definition 44. If κ and λ are cardinals, then

$$\kappa^\lambda = \operatorname{card}({}^\lambda\kappa);$$

this is the λ-th **cardinal power** of κ. The operation $(x, y) \mapsto x^y$ on **CN** is **cardinal exponentiation.**

Theorem 124. *For all cardinals κ, λ, μ and ν,*

$$\begin{gathered}
\kappa^0 = 1,\\
0 < \lambda \Rightarrow 0^\lambda = 0,\\
1^\lambda = 1,\\
\kappa^1 = \kappa,\\
\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu,\\
\kappa^{\lambda\cdot\mu} = (\kappa^\lambda)^\mu,\\
\kappa \leqslant \mu \mathbin{\&} \lambda \leqslant \nu \Rightarrow \kappa^\lambda \leqslant \mu^\nu.
\end{gathered}$$

Theorem 125. *For all sets a,*

$$\mathscr{P}(a) \approx 2^a.$$

Proof. There is a bijection between ${}^a2$ and $\mathscr{P}(a)$ that takes the function f to the set $\{x \in a \colon f(x) = 1\}$. □

Corollary. *For all cardinals κ,*

$$\kappa < 2^\kappa.$$

Theorem 126. *If κ and λ are cardinals such that $2 \leqslant \kappa \leqslant 2^\lambda$ and λ is infinite, then*

$$\kappa^\lambda = 2^\lambda.$$

Proof. $2^\lambda \leqslant \kappa^\lambda \leqslant \lambda^\lambda \leqslant (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$ by Theorem 116. □

Definition 45. The function

$$x \mapsto \beth_x$$

from **ON** into **CN** is given recursively by

$$\beth_0 = \omega, \qquad \beth_{\alpha'} = 2^{\beth_\alpha}, \qquad \beth_\beta = \sup_{x\in\beta} \beth_x,$$

where β is a limit. Here $\beth$ is *beth,* the second letter of the Hebrew alphabet.

Theorem 127. *The function $x \mapsto \beth_x$ is an embedding of* **ON** *in* **CN**, *and*

$$\aleph_\alpha \leqslant \beth_\alpha. \tag{5.1}$$

Proof. Theorem 125 and induction. □

Definition 46. The **Continuum Hypothesis,** or CH, is

$$\aleph_1 = \beth_1;$$

the **Generalized Continuum Hypothesis,** or GCH, is

$$\aleph_\alpha = \beth_\alpha$$

for all ordinals α.

We shall see in Chapter 6 that we *can* make these hypotheses without contradicting our other axioms about sets. We shall not see what is also the case, that these hypotheses are not *implied* by our axioms [8].

The Continuum Hypothesis is so called because it is that $\aleph_1$ is the cardinality of the **continuum,** namely the set of *real numbers.* Indeed, full details are a lengthy exercise in ordered algebra; but one approach to the real numbers can be sketched out as follows.

Suppose a relation $\sim$ on a set s is an **equivalence relation,** that is,

$$a \sim a, \qquad a \sim b \Rightarrow b \sim a, \qquad a \sim b \mathbin{\&} b \sim c \Rightarrow a \sim c,$$

for all a, b, and c in s. Then we let

$$[a] = [a]_\sim = \{x \colon x \in s \mathbin{\&} x \sim a\},$$

and we let

$$s/{\sim} = \{[x] \colon x \in s\}.$$

Let $\mathbb{Z}^+$ denote $\omega \smallsetminus \{0\}$. There is an equivalence-relation $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ given by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Then $(\mathbb{Z}^+ \times \mathbb{Z}^+)/{\sim}$ is denoted by

$$\mathbb{Q}^+,$$

and its element $[(a, b)]$ is denoted by

$$\frac{a}{b}$$

or a/b. The set $\mathbb{Q}^+$ has binary operations $+$ and $\cdot$ such that

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}, \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d},$$

and a linear ordering $<$ so that

$$\frac{a}{b} < \frac{c}{d} \Leftrightarrow a \cdot d < b \cdot c,$$

and the function $x \mapsto x/1$ embeds $\mathbb{Z}^+$ in $\mathbb{Q}^+$. In the present context we identify a in $\mathbb{Z}^+$ with $a/1$ in $\mathbb{Q}^+$.

A **cut** of $\mathbb{Q}^+$ is a proper nonempty subset c of $\mathbb{Q}^+$ with no greatest element that contains all predecessors of its elements. We denote the set of cuts of $\mathbb{Q}^+$ by

$$\mathbb{R}^+;$$

this is the set of **positive** real numbers. The set is linearly ordered by proper inclusion, and $\mathbb{Q}^+$ embeds in it under $x \mapsto \mathrm{pred}(x)$. The binary operations $+$ and $\cdot$ are defined on $\mathbb{R}^+$ by

$$a+b=\{x+y\colon x\in a\mathbin{\&} y\in b\},\qquad a\cdot b=\{x\cdot y\colon x\in a\mathbin{\&} y\in b\}.$$

A nonempty subset of $\mathbb{R}^+$ with an upper bound has a supremum, namely the union of the subset. We identify an element a of $\mathbb{Q}^+$ with its image $\mathrm{pred}(a)$ in $\mathbb{R}^+$.

On $\mathbb{R}^+ \times \mathbb{R}^+$ there is an equivalence relation $\sim$ given by

$$(a,b)\sim(c,d)\Leftrightarrow a+d=b+c.$$

Then $(\mathbb{R}^+ \times \mathbb{R}^+)/{\sim}$ is denoted by

$$\mathbb{R},$$

and $\mathbb{R}^+$ embeds in this under $x \mapsto [(x+1,1)]$. The element $[(1,1)]$ of $\mathbb{R}$ is also denoted by

$$0.$$

Then

$$\mathbb{R}=\{[(1,1+x)]\colon x\in\mathbb{R}^+\}\cup\{0\}\cup\{[(x+1,1)]\colon x\in\mathbb{R}^+\},$$

and the three sets of the union are disjoint. We identify a in $\mathbb{R}^+$ with $[(a+1,1)]$.

We have $\mathbb{R} \approx \mathbb{R}^+$, and $\mathbb{R}^+ \subseteq \mathscr{P}(\mathbb{Q}^+)$, and $\mathrm{card}(\mathbb{Q}^+) = \aleph_0$, so

$$\mathrm{card}(\mathbb{R})\leqslant 2^{\aleph_0}.$$

The reverse inequality follows as well, because there is an embedding f of ${}^{\omega}2$ into $\mathbb{R}$ defined by

$$f(\sigma)=\sup\Bigl\{\sum_{k=0}^{x}\frac{2\cdot\sigma(k)}{3^{k+1}}\colon x\in\omega\Bigr\}.$$

The function f is indeed well-defined, since, by induction,

$$\sum_{k=0}^{n}\frac{2\cdot\sigma(k)}{3^{k+1}}\leqslant 1-\frac{1}{3^{n+1}}<1.$$

Also, f is injective, since, if $\sigma \restriction n = \tau \restriction n$, but $\sigma(n) = 0 < 1 = \tau(n)$, then

$$f(\sigma) \leqslant \sum_{k=0}^{n-1} \frac{2 \cdot \sigma(k)}{3^{k+1}} + \frac{1}{3^n} < \sum_{k=0}^{n-1} \frac{2 \cdot \sigma(k)}{3^{k+1}} + \frac{2}{3^n} \leqslant f(\tau).$$

So $2^{\aleph_0} \leqslant \operatorname{card}(\mathbb{R})$; by the Schröder–Bernstein Theorem,

$$\operatorname{card}(\mathbb{R}) = 2^{\aleph_0}.$$

Here $f[{}^{\omega}2]$ is called the **Cantor set;** it is the intersection of the sets depicted in Figure 5.2.

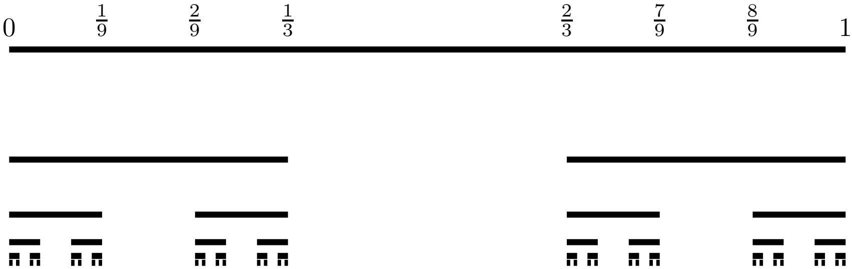

Figure 5.2. Towards the Cantor set

6. Models

6.1. Consistency and models

We defined *well-* and *ill-founded* sets in Definition 13 in §3.3. In that section, the possible existence of ill-founded sets caused a difficulty in the formulation of the class of natural numbers. We might have made things easier for ourselves by assuming the **Axiom of Foundation** (Axiom 10 below), namely that all sets are well-founded; equivalently, every nonempty set has an element that is disjoint from it.

However, it is good that we have not yet assumed the Foundation Axiom. The notion of *consistency* was introduced in §2.1: a collection of sentences is consistent if it no contradiction is derivable from it. The axioms that we have now are:

1) Equality,	4) Separation,	7) Infinity,
2) Null Set,	5) Replacement,	8) Power Set,
3) Adjunction,	6) Union,	9) Choice.

(The list is in Appendix E.) If these axioms are indeed consistent, then they will still be consistent when the Foundation Axiom is added: this is Theorem 154 below.

As axioms of set theory, Zermelo [35] proposed

1) Extension,	4) Power Set,	7) Infinity.
2) Pairing,	5) Union,	
3) Separation,	6) Choice,	

However, Zermelo did not have a formal logic. If he had had a formal logic, it might have had the equals sign $=$ as one of its official symbols, as described at the end of §2.8. Extension is the axiom mentioned there. The existence of sets is a logical truth, so our Null Set Axiom is a consequence of Zermelo's (and now our) Separation Axiom. Zermelo's **Pairing Axiom** corresponds to our Theorem 17; this and Union entail our Adjunction Axiom.

Fraenkel (see [16, p. 50, n. 3]), and independently Skolem [31], proposed the Replacement Axiom (which makes Separation redundant). Skolem and more definitely von Neumann [33] proposed Foundation. The collection of all of these axioms, besides Choice, is called

$$\mathrm{ZF}$$

for Zermelo and Fraenkel. When Choice is added, the collection is called

$$\mathrm{ZFC}\,.$$

Our own axioms so far entail exactly the same sentences as ZFC without Foundation. We shall establish in this chapter the following **relative consistency** results:

1. If ZFC without Foundation is consistent, then so is ZFC itself (Theorem 154).
2. If ZFC without Infinity is consistent, then so is ZFC with Infinity replaced by its negation (Theorem 152). Thus ZFC without Infinity does not entail Infinity.
3. If ZFC without Replacement is consistent, then so is ZFC with Replacement replaced by its negation (Theorem 153). Thus ZFC without Replacement does not entail Replacement.
4. If ZF is consistent, then so is ZFC with GCH (Theorems 163 and 166).

We shall use the following notions:

Definition 47. Given a class $\boldsymbol{M}$ and a sentence σ, we can ask whether σ is **true in $\boldsymbol{M}$**, that is, true under the assumption that every set belongs to $\boldsymbol{M}$. Of course, this will require that any constants occurring in σ name elements of $\boldsymbol{M}$. If σ is true in $\boldsymbol{M}$, we may express this by writing

$$\models_{\boldsymbol{M}} \sigma.$$

Each formula φ has a **relativization** to the class $\boldsymbol{M}$. This relativization is denoted by

$$\varphi_{\boldsymbol{M}}.$$

The definition is recursive:

1. $\varphi_{\boldsymbol{M}}$ is φ, if φ is atomic.
2. $(\neg\varphi)_{\boldsymbol{M}}$ is $\neg(\varphi_{\boldsymbol{M}})$.

3. $(\varphi \Rightarrow \psi)_{\boldsymbol{M}}$ is $\varphi_{\boldsymbol{M}} \Rightarrow \psi_{\boldsymbol{M}}$.
4. $(\exists x\, \varphi)_{\boldsymbol{M}}$ is
$$\exists x\, (x \in \boldsymbol{M} \;\&\; \varphi_{\boldsymbol{M}}).$$

Note then that $(\forall x\, \varphi)_{\boldsymbol{M}}$ can be understood as
$$\forall x\, (x \in \boldsymbol{M} \Rightarrow \varphi_{\boldsymbol{M}}).$$

A sentence σ with constants from $\boldsymbol{M}$ is true in $\boldsymbol{M}$ if and only if $\sigma_{\boldsymbol{M}}$ is true simply—that is, true in $\mathbf{V}$. That is, the following are equivalent:
1. σ is true in $\boldsymbol{M}$.
2. $\models_{\boldsymbol{M}} \sigma$.
3. $\sigma_{\boldsymbol{M}}$ is true.

If Δ is a collection of sentences, each of which is true in $\boldsymbol{M}$, then $\boldsymbol{M}$ is a **model**[1] of Δ, and we may write
$$\models_{\boldsymbol{M}} \Delta.$$

Theorem 128. *Every collection of sentences with a model is consistent.*

Proof. If $\models_{\boldsymbol{M}} \Delta$, and $\Delta \vdash \sigma$, then σ is true in $\boldsymbol{M}$ (by Theorem 10 of §2.5), so σ cannot be a contradiction. $\square$

We shall establish our relative consistency results by constructing models of the appropriate collections of sentences. The results will be only relative, since our constructions will naturally make use of certain axioms: thus the constructions will assume the consistency of those axioms (just as we have been doing all along).

6.2. The well-founded universe

Using our current axioms, essentially ZFC without Foundation, we shall find a class $\mathbf{WF}$, the *well-founded universe,* that is a model of ZFC. Note that we cannot just let $\mathbf{WF}$ be the class of well-founded sets. For, consider again some examples in §3.3:
1. Suppose there are sets a and b such that $a = \{b\}$ and $b = \{a\}$. If $a \neq b$, then a and b are well-founded, but $\{a, b\}$ is not. Thus, Zermelo's Pairing Axiom is false in the class of well-founded sets. Note that a and b are not transitive. However, $\{a, b\} = \{b\} \cup b = a \cup \bigcup a$, and this is transitive, but ill-founded.

[1] The term *model* is defined more generally at the end of Appendix B.

2. Similarly, if a, b, and c are distinct sets such that $a \in b$, $b \in c$, and $c \in a$, then:
   a) a is well-founded, but not transitive;
   b) $\{a, b\}$, which is $a \cup \bigcup a$, is well-founded, but not transitive;
   c) $\{a, b, c\}$, which is $a \cup \bigcup a \cup \bigcup \bigcup a$, is transitive, but ill-founded.

Again, such sets were a difficulty that had to be met in the formulation of the class of natural numbers. Now that we do have this class, we can make the following:

Definition 48. Given a set a, we define the function $x \mapsto \bigcup^x a$ on ω recursively by

$$\bigcup^0 a = a, \qquad \bigcup^{n+1} a = \bigcup \bigcup^n a.$$

Then we let

$$\mathrm{tc}(a) = \bigcup \Big\{ \bigcup^x a \colon x \in \omega \Big\}.$$

We call this set the **transitive closure** of a, because of the following.

Theorem 129. *For all sets a, the set* $\mathrm{tc}(a)$
1) includes a,
2) is transitive,
3) is included in every class that includes a and is transitive.

By definition, a class is well-founded if every nonempty sub*set* has an element that is disjoint from it. By using transitive closures, we can show that the same is true for arbitrary sub*classes* of well-founded classes:

Theorem 130. *Every non-empty well-founded class* $\boldsymbol{C}$ *has an element a such that*

$$\boldsymbol{C} \cap a = 0.$$

Proof. Suppose $\boldsymbol{C}$ is well-founded, and $b \in \boldsymbol{C}$, but $\boldsymbol{C} \cap b \neq 0$. Since $\boldsymbol{C}$ is well-founded, its nonempty subset $\boldsymbol{C} \cap b$ has an element a such that $\boldsymbol{C} \cap b \cap a = 0$. We would be done if we could show $\boldsymbol{C} \cap a = 0$. We could conclude this if b were transitive, so that $a \subseteq b$ and therefore $\boldsymbol{C} \cap a = \boldsymbol{C} \cap b \cap a$. So we start over, replacing b with $\mathrm{tc}(b)$:

Since $b \subseteq \mathrm{tc}(b)$, the subclass $\boldsymbol{C} \cap \mathrm{tc}(b)$ of $\boldsymbol{C}$ is nonempty, so it has an element a such that $\boldsymbol{C} \cap \mathrm{tc}(b) \cap a = 0$. But $a \subseteq \mathrm{tc}(b)$, so $\boldsymbol{C} \cap a = 0$, as desired. □

Above we described examples of a well-founded set a such that $a \cup \bigcup a$ or $a \cup \bigcup a \cup \bigcup\bigcup a$ was ill-founded. The latter set is $\operatorname{tc}(a)$ in each case. This suggests that the following will be useful:

Definition 49. We denote by

$$\mathbf{WF}$$

the class of all sets whose transitive closures are well-founded.[2]

Theorem 131. *Every set is well-founded if and only if every set belongs to* $\mathbf{WF}$*; so the Foundation Axiom can be expressed as* $\mathbf{V} = \mathbf{WF}$*.*

Proof. If every set is well-founded, then in particular every transitive closure of a set is well-founded, so every set belongs to $\mathbf{WF}$. Conversely, if $a \in \mathbf{WF}$, then $\operatorname{tc}(a)$ is well-founded; but $a \subseteq \operatorname{tc}(a)$, so a is well-founded. □

We shall establish another characterization of $\mathbf{WF}$, which will be more useful and suggestive. We start with the following.

Definition 50. The function $\mathbf{R}$ on $\mathbf{ON}$ is defined recursively by

$$\mathbf{R}(0) = 0, \qquad \mathbf{R}(\alpha + 1) = \mathscr{P}(\mathbf{R}(\alpha)), \qquad \mathbf{R}(\beta) = \bigcup \mathbf{R}[\beta],$$

where β is a limit. If $c \in \bigcup \mathbf{R}[\mathbf{ON}]$, then the least ordinal α such that $c \in \mathbf{R}(\alpha)$ must be a successor, $\beta + 1$. In this case, β is called the **rank** of c and is denoted by $\operatorname{rank}(c)$. That is,

$$\begin{aligned} \operatorname{rank}(c) &= \min\{x \colon x \in \mathbf{ON} \mathbin{\&} c \in \mathbf{R}(x+1)\} \\ &= \min\{x \colon x \in \mathbf{ON} \mathbin{\&} c \subseteq \mathbf{R}(x)\}. \end{aligned}$$

We may refer to $\bigcup \mathbf{R}[\mathbf{ON}]$ as the class of **ranked sets.**

We shall show in Theorem 137 that the class of ranked sets is precisely $\mathbf{WF}$.

Theorem 132. *For all ordinals* α*,*

$$\operatorname{card}(\mathbf{R}(\omega + \alpha)) = \beth_\alpha.$$

[2] It is the elements of this class that are called *well-founded* in [16].

Lemma 26.

1. *The empty set is transitive.*
2. *The power set of a transitive set is transitive.*
3. *The union of a set of transitive sets is transitive.*

Theorem 133. *Each set $\mathbf{R}(\alpha)$ is transitive, and so is the whole class of ranked sets.*

Corollary. *Every element of a ranked set is ranked and has a lower rank than that set.*

Theorem 134. *For all ordinals α and β,*

$$\alpha < \beta \Leftrightarrow \mathbf{R}(\alpha) \subset \mathbf{R}(\beta).$$

Proof. We show $\alpha < \beta \Rightarrow \mathbf{R}(\alpha) \subset \mathbf{R}(\beta)$ by induction on β. (The converse follows from the linearity of the ordering of $\mathbf{ON}$.)

1. The claim holds vacuously when $\beta = 0$.

2. Suppose the claim holds when $\beta = \gamma$. If $\alpha < \gamma + 1$, then $\alpha \leqslant \gamma$, so $\mathbf{R}(\alpha) \subseteq \mathbf{R}(\gamma)$ and therefore $\mathbf{R}(\alpha) \in \mathscr{P}(\mathbf{R}(\gamma))$, which is $\mathbf{R}(\gamma + 1)$. By the last theorem then, $\mathbf{R}(\alpha) \subseteq \mathbf{R}(\gamma+1)$. Since a set is never equal to its power set (by Cantor's Theorem, Theorem 109 in §5.1), $\mathbf{R}(\alpha) \subset \mathbf{R}(\gamma+1)$.

3. Suppose γ is a limit, and the claim is true when $\beta < \gamma$. If $\alpha < \gamma$, then $\alpha + 1 < \gamma$, so $\mathbf{R}(\alpha) \subset \mathbf{R}(\alpha + 1) \subseteq \mathbf{R}(\gamma)$. □

A partial converse of Theorem 133 is the following.

Theorem 135. *Every subset of an element of $\mathbf{R}(\alpha)$ is an element of $\mathbf{R}(\alpha)$. Every set of ranked sets is itself a ranked set.*

Proof. Suppose $b \in \mathbf{R}(\alpha)$. Then $b \subseteq \mathbf{R}(\beta)$ for some β in α. If $a \subseteq b$, then $a \in \mathbf{R}(\alpha + 1)$, so $a \in \mathbf{R}(\beta)$.

Suppose c is a set of ranked sets. Let $\beta = \sup\{\mathrm{rank}(x)\colon x \in c\}$; then $c \subseteq \mathbf{R}(\beta + 1)$, so $c \in \mathbf{R}(\beta + 2)$. □

Theorem 136. *Every ranked set is well-founded.*

Proof. Suppose a is ranked, and b is a nonempty subset of a. The elements of b are ranked. Let c be an element of b of minimal rank. Since any element of $b \cap c$ would have lower rank than c, this intersection must be empty. □

Theorem 137. *The class of ranked sets is just* $\mathbf{WF}$.

Proof. Suppose a is ranked. Then a is *included* in the class of ranked sets, since this class is transitive. But then this class must then include $\mathrm{tc}(a)$, by Theorem 129. Then $\mathrm{tc}(a)$ itself is ranked by Theorem 135, so $\mathrm{tc}(a)$ is well-founded by Theorem 136.

Now suppose a is not ranked. Then a has unranked elements, so $\mathrm{tc}(a)$ has unranked elements. Let b be the set of unranked elements of $\mathrm{tc}(a)$, and let c be an arbitrary element of b. Then $c \subseteq \mathrm{tc}(a)$, and since c is unranked, it has unranked elements. But then these are also elements of b. Thus $b \cap c \neq 0$. Therefore $\mathrm{tc}(a)$ is not well-founded. □

Since each ordinal is transitive and well-founded, we have $\mathbf{ON} \subseteq \mathbf{WF}$. Moreover:

Theorem 138. *For all ordinals* α,

$$\mathrm{rank}(\alpha) = \alpha.$$

Proof. By induction, $\alpha \subseteq \mathbf{R}(\alpha)$, so $\mathrm{rank}(\alpha) \leqslant \alpha$. If $\mathrm{rank}(\alpha) = \beta < \alpha$, then $\mathrm{rank}(\beta) < \mathrm{rank}(\alpha) = \beta$ by the Corollary to Theorem 133. Thus the class of α such that $\mathrm{rank}(\alpha) < \alpha$ has no least element; so it must be empty. □

One might picture $\mathbf{WF}$ as in Figure 6.1.[3]

6.3. Absoluteness

We shall establish that $\mathbf{WF}$ is a model of ZFC, along with similar results. How can we do this? We can establish one part of this result directly, along the lines of Theorem 136:

Theorem 139. *The Foundation Axiom is true in every subclass of* $\mathbf{WF}$.

Proof. Suppose $\boldsymbol{M} \subseteq \mathbf{WF}$, and $\boldsymbol{M}$ has elements a and b such that a is a nonempty subset of b in $\boldsymbol{M}$, that is,

$$\exists x\,(x \in \boldsymbol{M} \;\&\; x \in a) \;\&\; \forall x\,(x \in \boldsymbol{M} \Rightarrow x \in a \Rightarrow x \in b).$$

[3] Like much of the mathematics in this section, the picture is adapted from Kunen [23, Ch. 3, §4, p. 101].

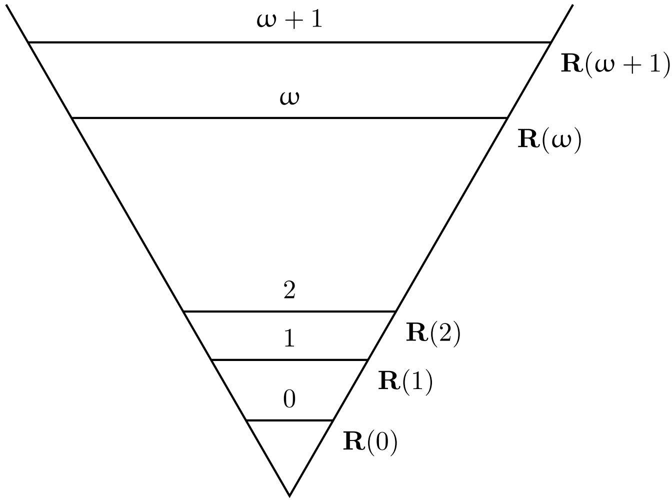

Figure 6.1. The well-founded universe

Let c be an element of $a \cap \boldsymbol{M}$ of minimal rank. Then $a \cap \boldsymbol{M} \cap c$ must be empty, since any element would have lesser rank than c. In particular,

$$\forall y \, (y \in \boldsymbol{M} \Rightarrow y \notin a \cap c).$$

So the sentence $\exists x \, (x \in a \,\&\, a \cap x = 0)$ is true in $\boldsymbol{M}$. $\square$

Consider next the Equality Axiom,

$$a = b \Rightarrow \forall x \, (a \in x \Leftrightarrow b \in x).$$

Here, by definition of equality,

$$a = b \Leftrightarrow \forall x \, (x \in a \Leftrightarrow x \in b).$$

To decide whether the Equality Axiom is true in a particular class $\boldsymbol{M}$, we should first check whether equality *in* $\boldsymbol{M}$ is the same as 'real' equality, or equality in $\mathbf{V}$. If $\boldsymbol{M}$ is not transitive, then it may have a nonempty element a, none of whose elements is in $\boldsymbol{M}$. Then $a \neq 0$, but

$$\models_{\boldsymbol{M}} a = 0$$

($a = 0$ in $\boldsymbol{M}$). In particular, equality may fail to be *absolute* for $\boldsymbol{M}$. The precise definition of this notion is as follows.

Definition 51. If $\boldsymbol{M}$ is a class, then every formula whose constants are from $\boldsymbol{M}$ can be said to be **over $\boldsymbol{M}$**. Suppose φ is such a formula.

1. We first consider the case when φ is singulary. We define

$$\varphi^{\boldsymbol{M}} = \{x\colon x \in \boldsymbol{M} \mathbin{\&} \varphi_{\boldsymbol{M}}(x)\}.$$

That is, $\varphi^{\boldsymbol{M}}$ is the class of all a in $\boldsymbol{M}$ such that $\varphi(a)$ is true in $\boldsymbol{M}$. In particular, $\varphi^{\mathbf{V}}$ is just $\{x\colon \varphi(x)\}$, the class defined by φ. We may say then that $\varphi^{\boldsymbol{M}}$ is the class **defined in $\boldsymbol{M}$** by φ. The formula φ is **absolute** for $\boldsymbol{M}$ if

$$\varphi^{\boldsymbol{M}} = \boldsymbol{M} \cap \varphi^{\mathbf{V}}.$$

2. Now suppose φ is not necessarily singulary. We may assume that each variable in our formulas is x_k for some k in ω. Then there is a finite subset p of ω comprising those k such that x_k is a free variable of φ. We may then refer to φ as a **p-ary formula,** and we may refer to p as the **arity** of φ.[4] An element of ${}^{p}\boldsymbol{M}$ can be called a **p-tuple;** such an element can be denoted by $\vec{a}$ or $x \mapsto a_x$ or $\{(x, a_x)\colon x \in p\}$. Then we can denote by

$$\varphi(\vec{a})$$

the result of replacing each free occurrence of x_k in φ with a_k, for each k in p. Correspondingly, we may denote φ itself by

$$\varphi(\vec{x}).$$

We denote by

$$\varphi^{\boldsymbol{M}}$$

the class of all $\vec{a}$ in ${}^{p}\boldsymbol{M}$ such that $\varphi(\vec{a})$ is true in $\boldsymbol{M}$, that is, $\varphi_{\boldsymbol{M}}(\vec{a})$ is true. This is the class **defined in $\boldsymbol{M}$** by φ. The formula φ is **absolute for $\boldsymbol{M}$** if

$$\varphi^{\boldsymbol{M}} = {}^{p}\boldsymbol{M} \cap \varphi^{\mathbf{V}}.$$

[4] We could restrict ourselves to n-ary formulas, where $n \in \omega$. That is, we could require the free variables of a formula to be indexed by an initial segment of ω. However, this restriction causes its own complications. I have decided here not to consider the variables as being ordered, but to consider any set of n variables to be as good as any other.

Every subset of ${}^p\boldsymbol{M}$ can be referred to as a **p-ary relation on $\boldsymbol{M}$**. If $\boldsymbol{A}$ is such a relation, and $\boldsymbol{K} \subseteq \boldsymbol{M}$, then $\boldsymbol{A}$ is **definable over $\boldsymbol{K}$** if there is a formula ψ over $\boldsymbol{K}$ such that $\boldsymbol{A} = \psi^{\boldsymbol{M}}$. Then $\boldsymbol{A}$ is **definable,** simply, if it is definable over $\boldsymbol{M}$.

3. Note the following special case. For a sentence σ with constants from $\boldsymbol{M}$, we have

$$\sigma^{\boldsymbol{M}} = \{0\colon \sigma_{\boldsymbol{M}}\},$$

that is, $\sigma^{\boldsymbol{M}}$ is 1 if σ is true in $\boldsymbol{M}$, and otherwise $\sigma^{\boldsymbol{M}} = 0$. So σ is absolute for $\boldsymbol{M}$ if and only if σ is true in $\boldsymbol{M}$.

Theorem 140.

1. *The formula $x \in y$ and all other quantifier-free formulas are absolute for all classes.*
2. *If φ is absolute for $\boldsymbol{M}$, then so is $\neg\varphi$.*
3. *If φ and ψ are absolute for $\boldsymbol{M}$, then so is $(\varphi \Rightarrow \psi)$.*

Definition 52. If p and q are disjoint finite subsets of ω, we may denote an element of ${}^{p\cup q}\boldsymbol{M}$ by

$$(\vec{a}, \vec{b}),$$

where $\vec{a} \in {}^p\boldsymbol{M}$ and $\vec{b} \in {}^q\boldsymbol{M}$; correspondingly, we may denote a $p \cup q$-ary formula by

$$\varphi(\vec{x}, \vec{y}).$$

Then $\varphi(\vec{a}, \vec{y})$ and $\varphi(\vec{x}, \vec{b})$ are the obvious formulas.

Theorem 141. *If $\varphi(\vec{x}, \vec{y})$ is absolute for $\boldsymbol{M}$, and $\vec{a}$ is from $\boldsymbol{M}$, then $\varphi(\vec{a}, \vec{y})$ is absolute for $\boldsymbol{M}$.*

Definition 53. The Δ_0 formulas are defined by:

1. Atomic formulas are Δ_0.
2. If φ is Δ_0, then so is $\neg\varphi$.
3. If φ and ψ are Δ_0, then so is $(\varphi \Rightarrow \psi)$.
4. If φ is Δ_0, then so is $\exists x\,(x \in y \mathbin{\&} \varphi)$.

Theorem 142. *All Δ_0 formulas are absolute for all transitive classes.*

We may have to analyze the abbreviations that we use in formulas in order to see that the underlying formula is Δ_0:

Theorem 143. *If φ is Δ_0, then so is*

$$\forall x\,(x \in y \Rightarrow \varphi).$$

Proof. The given formula is $\neg\exists x\,(x \in y \mathbin{\&} \neg\varphi)$. □

Theorem 144. *The following formulas are Δ_0:*

1. $x = y$.
2. *The Equality Axiom.*
3. $x = 0$.
4. $x = y \cup \{z\}$.
5. $x = \bigcup y$.

Proof. The formula $x = y$ is $\forall z\,(z \in x \Leftrightarrow z \in y)$, which can be understood as

$$\forall z\,(z \in x \Rightarrow z \in y) \mathbin{\&} \forall z\,(z \in y \Rightarrow z \in x).$$

Similarly for the rest. □

Theorem 145.

1. *The Null Set Axiom is true in every class that contains* 0.
2. *The Adjunction Axiom is true in every transitive class that is closed under the operation $(x, y) \mapsto x \cup \{y\}$.*
3. *The Union Axiom is true in every transitive class that is closed under the operation $x \mapsto \bigcup x$.*

Theorem 146. *The Null Set, Adjunction, and Union Axioms are true in every class $\mathbf{R}(\alpha)$ such that α is a limit, and in* **WF**.

Proof. Since $0 \subseteq \mathbf{R}(0)$, we have $0 \in \mathbf{R}(1)$, so 0 is in **WF** and in all $\mathbf{R}(\alpha)$ such that $\alpha > 0$.

If a and b are in $\mathbf{R}(\alpha)$, then, by transitivity of this class, the sets $a \cup \{b\}$ and $\bigcup a$ are subsets of $\mathbf{R}(\alpha)$, so they are elements of $\mathbf{R}(\alpha + 1)$. □

Theorem 147. *The Power Set Axiom is true in every transitive class $\boldsymbol{M}$ that contains $\boldsymbol{M} \cap \mathscr{P}(a)$ for every element a of $\boldsymbol{M}$.*

Proof. The formula $x \in \mathscr{P}(y)$ is Δ_0. Therefore the formula $x = \mathscr{P}(a)$, relativized to $\boldsymbol{M}$, is

$$\forall y\,(y \in x \Rightarrow y \in \mathscr{P}(a)) \mathbin{\&} \forall y\,(y \in \boldsymbol{M} \cap \mathscr{P}(a) \Rightarrow y \in x).$$

Hence the sentence $b = \mathscr{P}(a)$ is true in $\boldsymbol{M}$ if $b = \boldsymbol{M} \cap \mathscr{P}(a)$. □

Theorem 148. *The Power Set Axiom is true in* $\mathbf{WF}$ *and in* $\mathbf{R}(\alpha)$ *when* α *is a limit.*

Proof. Each of these classes both contains and includes the power set of each of its elements. □

Theorem 149. *The Separation Axiom is true in every set* $\mathbf{R}(\alpha)$ *and in the class* $\mathbf{WF}$.

Proof. The Separation Axiom consists of a sentence $\exists x\, x = a \cap \varphi^{\mathbf{V}}$ for every set a and every singulary formula φ. This sentence can be understood as $\exists x\, \psi$, where ψ is

$$\forall y\, (y \in x \Leftrightarrow y \in a \;\&\; \varphi(y)).$$

This is Δ_0, if φ is. In any case, the relativization of ψ to a transitive class $\boldsymbol{M}$ that contains a can be understood as

$$\forall y\, (y \in x \Leftrightarrow y \in a \;\&\; \varphi_{\boldsymbol{M}}(y)),$$

that is, $x = a \cap \varphi^{\boldsymbol{M}}$. But $a \cap \varphi^{\boldsymbol{M}}$ is just a subset of a. Therefore, if $\boldsymbol{M}$ contains all subsets of all of its elements, then $\exists x\, \psi$ is true in $\boldsymbol{M}$. We are done by Theorem 135. □

Theorem 150. *The following formulas are absolute for all transitive subclasses of* $\mathbf{WF}$*:*

1. x *is an ordinal:* $x \in \mathbf{ON}$.
2. x *is a successor ordinal.*
3. x *is a limit ordinal.*
4. $x = \omega$.

In particular, the Axiom of Infinity is true in all transitive subclasses of $\mathbf{WF}$ *that contain* ω.

Proof. In a subclass $\boldsymbol{M}$ of $\mathbf{WF}$, since the Foundation Axiom is true, $\mathbf{ON}$ is the class of transitive sets that are linearly ordered by membership. Then all of the needed formulas are Δ_0. □

Lemma 27. *The following are absolute for all transitive classes:*

1. x *is a singleton:* $\exists y\, x = \{y\}$.
2. x *is a pair:* $\exists y\, \exists z\, x = \{y, z\}$.
3. x *is an ordered pair:* $\exists y\, \exists z\, x = (y, z)$.

4. *x is a binary relation:* $\forall y\,(y \in x \Rightarrow \exists z\,\exists w\; y = (z, w))$.
5. *x is a function: x is a relation, and*

$$\forall y\,\forall z\,\forall w\,((y, z) \in x \;\&\; (y, w) \in x \Rightarrow z = w).$$

Theorem 151. *The Axiom of Choice is true in* $\mathbf{WF}$ *and in* $\mathbf{R}(\alpha)$ *for each limit ordinal* α*.*

Proof. Suppose α is a limit ordinal, and $a \in \mathbf{R}(\alpha)$. Then $a \in \mathbf{R}(\beta)$ for some β in α. Then $\mathscr{P}(a) \subseteq \mathbf{R}(\beta)$, so $\mathscr{P}(a) \cup a \subseteq \mathbf{R}(\beta)$, and both $\mathscr{P}(a)$ and $\mathscr{P}(a) \cup a$ are elements of $\mathbf{R}(\beta + 1)$.

Let f be a choice-function for a. Then $f \subseteq \mathscr{P}(a) \times a$. Each element (b, c) of $\mathscr{P}(a) \times a$ is $\{\{b\}, \{b, c\}\}$, where b and c are in $\mathscr{P}(a) \cup a$ and hence are in $\mathbf{R}(\beta)$; so $(b, c) \in \mathbf{R}(\beta+2)$. Thus $f \in \mathbf{R}(\beta+3)$, so $f \in \mathbf{R}(\alpha)$. By the last lemma and transitivity of $\mathbf{R}(\alpha)$, the sentence 'f is a choice-function for a' is true in $\mathbf{R}(\alpha)$. $\square$

Theorem 152. *In* $\mathbf{R}(\omega)$*, the axioms of* ZFC *besides Infinity are true, but Infinity is false.*

Proof. Every element of $\mathbf{R}(\omega)$ is finite; in particular, $\omega \notin \mathbf{R}(\omega)$. Therefore, because of all of the foregoing theorems, we need only show that Replacement is true in $\mathbf{R}(\omega)$. Every subset of $\mathbf{R}(\omega)$ is an element of some $\mathbf{R}(n)$, where $n \in \omega$. Therefore, if a formula defines in $\mathbf{R}(\omega)$ a function, then the image of a finite set under this function is again an element of $\mathbf{R}(\omega)$. $\square$

Theorem 153. *In* $\mathbf{R}(\omega \cdot 2)$*, the axioms of* ZFC*, besides Replacement, are true, but Replacement is false.*

Proof. Since GST is true in $\mathbf{R}(\omega \cdot 2)$, we can define in this set, by finite induction, the function $x \mapsto \omega + x$. But $\mathbf{R}(\omega \cdot 2)$ contains ω. The image of ω under $x \mapsto \omega + x$ is $\{\omega + x \colon x \in \omega\}$, whose union is $\omega \cdot 2$, which is not in $\mathbf{R}(\omega \cdot 2)$. Therefore the image itself is not in $\mathbf{R}(\omega \cdot 2)$. $\square$

Theorem 154. *In* $\mathbf{WF}$*, the axioms of* ZFC *are true.*

Proof. The Replacement Axiom is true in $\mathbf{WF}$ by Theorem 135. $\square$

6.4. Collections of equivalence classes

Let us finally complete our list of axioms with the following.

Axiom 10 (Foundation). *All sets are well-founded:*

$$a \neq 0 \Rightarrow \exists y \, (y \in a \mathbin{\&} y \cap a = 0).$$

By Theorem 131, this axiom is expressed by the equation

$$\mathbf{V} = \mathbf{WF}.$$

For the purposes of this chapter, the rest of this section is merely a curiosity.

Definition 54. For every nonempty class $\boldsymbol{C}$, if α is the least rank of an element of $\boldsymbol{C}$, we let

$$\tau(\boldsymbol{C}) = \mathbf{R}(\alpha + 1) \cap \boldsymbol{C}.$$

We also let

$$\tau(0) = 0.$$

So $\tau(\boldsymbol{C})$ is always a set, and

$$\tau(\boldsymbol{C}) \subseteq \boldsymbol{C}, \qquad\qquad \boldsymbol{C} \neq 0 \Rightarrow \tau(\boldsymbol{C}) \neq 0.$$

The following is now immediate:

Theorem 155. *If $\boldsymbol{E}$ is an equivalence-relation on $\boldsymbol{C}$, and for all a in $\boldsymbol{C}$ we write*

$$[a] = \{x \colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{E}} a\},$$

then $\tau([a])$ is the set $\{x \colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{E}} a \mathbin{\&} \operatorname{rank}(x) = \operatorname{rank}(a)\}$, and the function $x \mapsto \tau([x])$ is a function $\boldsymbol{F}$ on $\boldsymbol{C}$ such that, for all a and b in $\boldsymbol{C}$,

$$\boldsymbol{F}(a) = \boldsymbol{F}(b) \Leftrightarrow [a] = [b].$$

In the notation of the theorem then, the collection of equivalence classes $[a]$ can be identified with the class $\boldsymbol{F}[\boldsymbol{C}]$.

6.5. Constructible sets

In showing that the Axiom of Choice was true in **WF**, we assumed the Axiom of Choice was true simply. Now, using ZF alone, we define a subclass of **WF** in which all of ZFC is true.

Theorem 156. *For all finite subsets p of ω, the collection of p-ary definable relations on a set is a set.*

Proof. We first work with an arbitrary class $\boldsymbol{M}$. Let $\boldsymbol{A}$ be a p-ary definable relation on $\boldsymbol{M}$. Then $\boldsymbol{A} = \varphi^{\boldsymbol{M}}$ for some p-ary formula φ whose constants are from $\boldsymbol{M}$. We can put the subformulas of φ in a string,

$$\varphi_0 \cdots \varphi_v,$$

where φ_v is φ, and for each u in $v+1$, for some i and j in ω, for some a and b in $\boldsymbol{M}$, and for some s and t in u, the formula φ_u is one of:

1) $x_i \in x_j$,
2) $x_i \in a$,
3) $a \in x_i$,
4) $a \in b$,
5) $\neg\varphi_s$,
6) $(\varphi_s \Rightarrow \varphi_t)$,
7) $\exists x_i\, \varphi_s$.

Letting p_s be the arity of φ_s, and writing $\boldsymbol{A}_s$ for ${\varphi_s}^{\boldsymbol{M}}$, we have correspondingly that $\boldsymbol{A}_u$ is:

1) $\{\{(i,x),(j,y)\}\colon x \in \boldsymbol{M} \;\&\; y \in \boldsymbol{M} \;\&\; x \in y\}$,
2) $\{\{(i,x)\}\colon x \in \boldsymbol{M} \;\&\; x \in a\}$,
3) $\{\{(i,x)\}\colon x \in \boldsymbol{M} \;\&\; a \in x\}$,
4) $\{0\colon a \in b\}$ (which is 1 if $a \in b$, and otherwise 0),
5) ${}^{p_u}\boldsymbol{M} \smallsetminus \boldsymbol{A}_s$,
6) $\{\vec{x}\colon \vec{x} \in {}^{p_u}\boldsymbol{M} \;\&\; (\vec{x} \restriction p_s \in \boldsymbol{A}_s \Rightarrow \vec{x} \restriction p_t \in \boldsymbol{A}_t)\}$,
7) $\pi[\boldsymbol{A}_s]$, where π is $\vec{x} \mapsto \vec{x} \restriction p_u$ on ${}^{p_u}\boldsymbol{M}$.

Conversely, $\boldsymbol{A}$ is definable if there exists such a string $\boldsymbol{A}_0 \cdots \boldsymbol{A}_v$ of relations on $\boldsymbol{M}$. When $\boldsymbol{M}$ is a set, then so are the relations $\boldsymbol{A}_s$, and then the collection of definable p-ary relations on $\boldsymbol{M}$ is the subclass—in fact, subset—of $\mathscr{P}({}^{p}\boldsymbol{M})$ comprising those subsets $\boldsymbol{A}$ of ${}^{p}\boldsymbol{M}$ for which there is, for some v in ω, a v-tuple $(\boldsymbol{A}_0, \dots, \boldsymbol{A}_v)$ of sets meeting the appropriate conditions. $\square$

Definition 55. For all finite subsets p of ω, the set of p-ary definable relations on a set a is denoted by

$$\mathscr{D}_p(a).$$

Then $\mathscr{D}_p(M) \subseteq \mathscr{P}({}^pM)$. The function $x \mapsto \mathbf{L}(x)$ on $\mathbf{ON}$ is defined recursively by

$$\mathbf{L}(0) = 0, \qquad \mathbf{L}(\alpha + 1) = \mathscr{D}_1(\mathbf{L}(\alpha)), \qquad \mathbf{L}(\beta) = \bigcup \mathbf{L}[\beta]$$

where β is a limit. We now denote $\bigcup\{\mathbf{L}(x) \colon x \in \mathbf{ON}\}$ by

$$\mathbf{L}.$$

(So this letter, by itself, will not denote the function $x \mapsto \mathbf{L}(x)$.) The elements of $\mathbf{L}$ are called the **constructible sets.**

The following supplements Lemma 26 in §6.2.

Lemma 28. *If a is a transitive set, then so is $\mathscr{D}_1(a)$.*

Proof. If $b \in \mathscr{D}_1(a)$, then $b \subseteq a$. If also $c \in b$, then $c \in a$, and also c is the subset of a defined by $x \in c$, since a is transitive; so $c \in \mathscr{D}_1(a)$. Thus $b \subseteq \mathscr{D}_1(a)$. □

Then the following are analogous to Theorems 133 and 134.

Theorem 157. *Each set $\mathbf{L}(\alpha)$ is transitive, and so is the whole class $\mathbf{L}$.*

Theorem 158. *For all ordinals α and β,*

$$\alpha < \beta \Leftrightarrow \mathbf{L}(\alpha) \subset \mathbf{L}(\beta).$$

We now generalize Definition 51.

Definition 56. If $\boldsymbol{M} \subseteq \boldsymbol{N}$, and the p-ary formula φ takes its constants from $\boldsymbol{M}$, then φ is **absolute for** $(\boldsymbol{M}, \boldsymbol{N})$ if

$$\varphi^{\boldsymbol{M}} = {}^p\boldsymbol{M} \cap \varphi^{\boldsymbol{N}}.$$

So absoluteness for $\boldsymbol{M}$ is just absoluteness for $(\boldsymbol{M}, \mathbf{V})$.

Theorem 144 is still true with the more general sense of absoluteness. Hence we obtain the following test for absoluteness.

Theorem 159 (Tarski–Vaught Test). *If $\boldsymbol{M} \subseteq \boldsymbol{N}$, and the formula φ takes its constants from $\boldsymbol{M}$, then φ is absolute for $(\boldsymbol{M}, \boldsymbol{N})$, provided that, for every subformula of φ of the form $\exists x\,\psi(x, \vec{y})$, for all $\vec{b}$ from $\boldsymbol{M}$, if $\exists x\,\psi(x, \vec{b})$ is true in $\boldsymbol{N}$, then $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in $\boldsymbol{M}$.*

Proof. We argue by induction on a string of subformulas of φ such as was considered in the proof of Theorem 156. It is enough to suppose that ψ is absolute for $(\boldsymbol{M}, \boldsymbol{N})$ and prove the same for $\exists x\,\psi$, assuming that, for all $\vec{b}$ from $\boldsymbol{M}$, if $\exists x\,\psi(x, \vec{b})$ is true in $\boldsymbol{N}$, then $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in $\boldsymbol{M}$. But in this case the following are equivalent:

1. $\exists x\,\psi(x, \vec{b})$ is true in $\boldsymbol{N}$;
2. $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in $\boldsymbol{M}$;
3. $\psi(a, \vec{b})$ is true in $\boldsymbol{M}$ for some a in $\boldsymbol{M}$;
4. $\exists x\,\psi(x, \vec{b})$ is true in $\boldsymbol{M}$. □

In the notation of the theorem, even if φ is *not* necessarily absolute for $(\boldsymbol{M}, \boldsymbol{N})$, the theorem suggests a method of finding a class $\boldsymbol{M}^\dagger$ such that

$$\boldsymbol{M} \subseteq \boldsymbol{M}^\dagger \subseteq \boldsymbol{N}$$

and φ is absolute for $(\boldsymbol{M}^\dagger, \boldsymbol{N})$. The method is given given by the following corollary in case $\boldsymbol{M}$ is a set. We shall make a couple of applications of the method.

Corollary. *Suppose φ is a formula over $\boldsymbol{N}$, and there is a function $x \mapsto x^*$ such that, if $m \subseteq \boldsymbol{N}$, and φ is over m, then*

$$m \subseteq m^* \subseteq \boldsymbol{N},$$

and for every subformula of φ of the form $\exists x\,\psi(x, \vec{y})$, for all $\vec{b}$ from m, if $\exists x\,\psi(x, \vec{b})$ is true in $\boldsymbol{N}$, then $\psi(a, \vec{b})$ is true in $\boldsymbol{N}$ for some a in m^*. Then there is a set $m^\dagger$ such that

$$m \subseteq m^\dagger \subseteq \boldsymbol{N}$$

and φ is absolute for $(m^\dagger, \boldsymbol{N})$. Namely,

$$m^\dagger = \bigcup\{m_k \colon k \in \omega\},$$

where $x \mapsto m_x$ is defined by

$$m_0 = m, \qquad\qquad m_{k+1} = {m_k}^*.$$

Theorem 160. *For every formula with constants from* $\mathbf{L}$*, there is* β *such that the formula has its constants in* $\mathbf{L}(\beta)$ *and is absolute for* $(\mathbf{L}(\beta), \mathbf{L})$*.*

Proof. In the notation of the corollary of the Tarski–Vaught Test, if $m = \mathbf{L}(\alpha)$, we let m^* be $\mathbf{L}(\alpha^*)$, where α^* is the least ordinal γ such that $\alpha \leqslant \gamma$ and, for all subformulas of φ of the form $\exists x\, \psi(x, \vec{y})$, for all tuples $\vec{b}$ from $\mathbf{L}(\alpha)$, if $\exists x\, \psi(x, \vec{b})$ is true in $\mathbf{L}$, then $\psi(a, \vec{b})$ is true in $\mathbf{L}$ for some a in $\mathbf{L}(\gamma)$. We choose α_0 so that φ is over $\mathbf{L}(\alpha_0)$; and then $m^\dagger$ is $\mathbf{L}(\beta)$, where $\beta = \sup\{\alpha_k \colon k \in \omega\}$, where $\alpha_{k+1} = {\alpha_k}^*$. $\square$

By analogy with Definition 50:

Definition 57. If $a \in \mathbf{L}$, we define

$$\operatorname{rank}_{\mathbf{L}}(a) = \min\{x \colon a \in \mathbf{L}(x + 1)\}.$$

Note however that possibly $a \in \mathbf{L}$, and $a \subseteq \mathbf{L}(\beta)$, but a is not a *definable* subset of $\mathbf{L}(\beta)$, so $a \notin \mathbf{L}(\beta + 1)$, and so $\beta < \operatorname{rank}_{\mathbf{L}}(a)$.

Theorem 161. $\mathbf{ON} \subseteq \mathbf{L}$*, and for all ordinals* α*,*

$$\operatorname{rank}_{\mathbf{L}}(\alpha) = \alpha.$$

Theorem 162. ZF *is true in* $\mathbf{L}$*.*

Proof. By the theorems in §6.3 and the last theorem, equality is absolute for $\mathbf{L}$, and the Equality, Null Set, Adjunction, Union, Foundation, and Infinity Axioms are true in $\mathbf{L}$. Indeed, if a and b are in $\mathbf{L}(\alpha)$, then, since this set is transitive, $a \cup \{b\}$ and $\bigcup a$ are definable subsets of $\mathbf{L}(\alpha)$, so they are in $\mathbf{L}(\alpha + 1)$.

Suppose $a \in \mathbf{L}$. Let $\beta = \sup\{\operatorname{rank}_{\mathbf{L}}(x) \colon x \in \mathbf{L} \mathrel{\&} x \subseteq a\}$. Then $\mathbf{L} \cap \mathscr{P}(a) \in \mathbf{L}(\beta + 2)$. Thus Power Set is true in $\mathbf{L}$.

Suppose $\varphi(x, y)$ defines in $\mathbf{L}$ a function $\boldsymbol{F}$, and $a \in \mathbf{L}$. Some $\mathbf{L}(\alpha)$ contains all constants in φ and elements of $\boldsymbol{F}[\,a\,]$. Then $\varphi(a, y)^{\mathbf{L}} \subseteq \mathbf{L}(\alpha)$. By Theorem 160, there is β such that $\varphi(a, y)^{\mathbf{L}} = \varphi(a, y)^{\mathbf{L}(\beta)}$. Thus $\boldsymbol{F}[\,a\,] \in \mathbf{L}(\beta + 1)$. Therefore Replacement is true in $\mathbf{L}$. $\square$

Theorem 163. *The Axiom of Choice is true in* $\mathbf{L}$*; indeed,* $\mathbf{L}$ *itself is well-ordered. In particular,* ZFC *is consistent (assuming* ZF *is consistent).*

Proof. There is a binary formula φ such that $\mathbf{L}$ is well-ordered by $\varphi_{\mathbf{L}}$. Indeed, because of the recursive construction of the sets $\mathscr{D}_1(a)$, there is a ternary formula ψ such that, if $\mathbf{L}(\alpha)$ is well-ordered by r, then $\mathbf{L}(\alpha+1)$ is well-ordered by $\psi(r,x,y)^{\mathbf{L}}$, and this ordering agrees with r on $\mathbf{L}(\alpha)$. By transfinite recursion, there is a function $x \mapsto r_x$ on $\mathbf{ON}$ such that $\mathbf{L}(\alpha)$ is well-ordered by r_α for each ordinal α, and r_α agrees with r_β on $\mathbf{L}(\alpha)$ if $\alpha < \beta$. Then $\mathbf{L}$ is well-ordered by $\bigcup\{r_x : x \in \mathbf{ON}\}$. This argument does not use the Axiom of Choice. $\square$

6.6. The Generalized Continuum Hypothesis

Our second application of the corollary of the Tarski–Vaught Test is the following.

Theorem 164 (Löwenheim–Skolem)**.** *For every set* m_0 *and every formula* φ *over* m_0*, there is a set* $m^\dagger$ *such that* $m_0 \subseteq m^\dagger$*, and*

$$\operatorname{card}(m^\dagger) \leqslant \operatorname{card}(m_0) + \aleph_0,$$

and φ *is absolute for* $m^\dagger$*.*

Proof. Given m, we let a be the set of all *nonempty* sets $\psi(x,\vec{b})^{\mathbf{V}}$, where $\exists x\, \psi(x,\vec{y})$ is a subformula of φ, and $\vec{b}$ is from m. Then we let

$$\alpha = \sup\{\min\{\operatorname{rank}(z) \colon z \in x\} \colon x \in a\}.$$

This ensures that $\mathbf{R}(\alpha+1) \cap c \neq 0$ for all c in a. The set $\mathbf{R}(\alpha+1)$ is well-ordered by some binary relation r. We now define

$$m^*_r = m \cup \{\min(\mathbf{R}(\alpha+1) \cap x) \colon x \in a\}.$$

Note that $\operatorname{card}(m^*_r) \leqslant \operatorname{card}(m) + \aleph_0$.

Since m^*_r depends on r as well as m, we do not yet have the desired function $x \mapsto x^*$. We can get it by considering the set of appropriate sequences $((m_x, r_x) \colon x \in \alpha)$, where $\alpha \leqslant \omega$, and $m_{k+1} = (m_k)^*_{r_k}$. By Zorn's Lemma, there is such a sequence where $\alpha = \omega$. Then we can define $m_k{}^* = m_{k+1}$. $\square$

A class is well-founded if and only if the relation of membership on the class is *well-founded* in the following sense.

Definition 58. A binary relation $\boldsymbol{R}$ is **well-founded** on a class $\boldsymbol{C}$ if
1) for all a in $\boldsymbol{C}$, the class $\{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\}$ is a set;
2) every nonempty subset b of $\boldsymbol{C}$ has an element c such that $\{x\colon x \in b \mathbin{\&} x \mathrel{\boldsymbol{R}} c\}$ is empty.

In particular, a linear ordering that meets the first part of this definition is just a left-narrow linear ordering, and a well-founded linear ordering is just a good ordering. (See Definition 23 in §4.1.) The relation $\boldsymbol{R}$ is **extensional** on $\boldsymbol{C}$ if the Extension Axiom is true in the structure $(\boldsymbol{C}, \boldsymbol{R})$ in the sense that

$$\{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\} = \{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} b\} \Rightarrow a = b.$$

Theorem 165 (Mostowski Collapsing)**.** *Let $\boldsymbol{R}$ be a well-founded relation on $\boldsymbol{C}$. There is a unique function $\boldsymbol{F}$ on $\boldsymbol{C}$ given by*

$$\boldsymbol{F}(a) = \boldsymbol{F}[\,\{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\}\,].$$

Then $\boldsymbol{F}[\,\boldsymbol{C}\,]$ is transitive. If $\boldsymbol{R}$ is extensional, then $\boldsymbol{F}$ is an isomorphism from $(\boldsymbol{C}, \boldsymbol{R})$ to $(\boldsymbol{F}[\,\boldsymbol{C}\,], \in)$.

Proof. We follow the proof of Theorem 62, though since $\boldsymbol{R}$ need not be transitive, we shall need the following. If $a \in \boldsymbol{C}$, then by recursion we define

$$\operatorname{cl}_0(a) = \{a\}, \qquad \operatorname{cl}_{n+1}(a) = \{x\colon \exists y\,(y \in \operatorname{cl}_n(a) \mathbin{\&} x \mathrel{\boldsymbol{R}} y)\},$$

Now we let

$$\operatorname{cl}(a) = \bigcup\{\operatorname{cl}_n(a)\colon n \in \omega\}.$$

We shall also use the notation

$$\boldsymbol{R}a = \{x\colon x \in \boldsymbol{C} \mathbin{\&} x \mathrel{\boldsymbol{R}} a\}.$$

We first show that the structure $(\boldsymbol{C}, \boldsymbol{R})$ admits induction in the sense that, if $\boldsymbol{C}_0 \subseteq \boldsymbol{C}$ and, for all a in $\boldsymbol{C}$, we have $a \in \boldsymbol{C}_0$ whenever $\boldsymbol{R}a \subseteq \boldsymbol{C}_0$, then $\boldsymbol{C}_0 = \boldsymbol{C}$. Indeed, suppose $\boldsymbol{C}_0 \subset \boldsymbol{C}$, and $a \in \boldsymbol{C} \smallsetminus \boldsymbol{C}_0$. By definition, the set $(\boldsymbol{C} \smallsetminus \boldsymbol{C}_0) \cap \operatorname{cl}(a)$ has an element b such that

$$(\boldsymbol{C} \smallsetminus \boldsymbol{C}_0) \cap \operatorname{cl}(a) \cap \boldsymbol{R}b = 0.$$

But $\boldsymbol{R}b \subseteq \mathrm{cl}(a)$ (since $b \in \mathrm{cl}_n(a)$ for some n, and then $\boldsymbol{R}b \subseteq \mathrm{cl}_{n+1}(a)$). Hence $(\boldsymbol{C} \smallsetminus \boldsymbol{C}_0) \cap \boldsymbol{R}b = 0$, so $\boldsymbol{R}b \subseteq \boldsymbol{C}_0$.

Now we can show by induction that, for all a in $\boldsymbol{C}$, there is a unique function f_a on $\mathrm{cl}(a)$ such that

$$f_a(c) = f_a[\,\boldsymbol{R}c\,].$$

Indeed, suppose the claim holds when $a \mathrel{\boldsymbol{R}} b$. If a and d are in $\boldsymbol{R}b$, then f_a and f_d must agree on $\mathrm{cl}(a) \cap \mathrm{cl}(d)$ since, if f_a and g disagree on $\mathrm{cl}(a) \cap \mathrm{cl}(d)$, then by well-foundedness this set has an element e such that f_a and g agree on $\boldsymbol{R}e$, but

$$g(e) \neq f_a(e) = f_a[\,\boldsymbol{R}e\,] = g[\,\boldsymbol{R}e\,].$$

Now we can define f_b on $\mathrm{cl}(b)$ so that, if $c \in \mathrm{cl}(b) \smallsetminus \{b\}$, then $f_b(c) = f_a(c)$, where a is such that $a \mathrel{\boldsymbol{R}} b$ and $c \in \mathrm{cl}(a)$; and $f_b(b) = f_b[\,\boldsymbol{R}b\,]$.

The desired function $\boldsymbol{F}$ is now $\bigcup\{f_a \colon a \in \boldsymbol{C}\}$. Indeed, this is a function, since any two functions f_a agree as before on the intersection of their domains. Likewise, $\boldsymbol{F}$ itself is unique. Since $\boldsymbol{F}(a) = \boldsymbol{F}[\,\boldsymbol{R}a\,] \subseteq \boldsymbol{F}[\,\boldsymbol{C}\,]$, it follows that $\boldsymbol{F}[\,\boldsymbol{C}\,]$ is transitive.

We have $a \mathrel{\boldsymbol{R}} b \Rightarrow \boldsymbol{F}(a) \in \boldsymbol{F}(b)$. If $\boldsymbol{F}$ is injective, then $\boldsymbol{F}(a) \in \boldsymbol{F}(b) \Rightarrow a \mathrel{\boldsymbol{R}} b$, so $\boldsymbol{F}$ is an isomorphism. Suppose $\boldsymbol{F}$ is not injective. Let $\boldsymbol{C}_0$ comprise those a in $\boldsymbol{C}$ for which there is no distinct b such that $\boldsymbol{F}(a) = \boldsymbol{F}(b)$. As in the proof that $(\boldsymbol{C}, \boldsymbol{R})$ admits induction, there is an element a of $\boldsymbol{C} \smallsetminus \boldsymbol{C}_0$ such that $\boldsymbol{R}a \subseteq \boldsymbol{C}_0$. Then $\boldsymbol{F}(a) = \boldsymbol{F}(b)$ for some distinct b. This means

$$\{\boldsymbol{F}(x) \colon x \in \boldsymbol{R}a\} = \{\boldsymbol{F}(y) \colon y \in \boldsymbol{R}b\}.$$

Since $\boldsymbol{R}a \subseteq \boldsymbol{C}_0$, we conclude $\boldsymbol{R}b = \boldsymbol{R}a$. Thus $\boldsymbol{R}$ is not extensional. □

Lemma 29. *For all ordinals α, in* $\mathbf{V}$ *and in* $\mathbf{L}$.

$$\mathrm{card}(\mathbf{L}(\beta)) = \mathrm{card}(\beta),$$

Theorem 166. *The Generalized Continuum Hypothesis is true in* $\mathbf{L}$. *Thus* GCH *is consistent with* ZFC *(assuming* ZF *is consistent).*

Proof. Suppose $a \in \mathscr{P}(\mathbf{L}(\alpha)) \cap \mathbf{L}$, where α is infinite. We shall show

$$a \in \mathbf{L}(\mathrm{card}(\alpha)^+).$$

By the last lemma, it will follow that, in $\mathbf{L}$,

$$\text{card}(\mathscr{P}(\kappa)) = \kappa^+.$$

Apply the Löwenheim–Skolem Theorem to the set $\mathbf{L}(\alpha) \cup \{a\}$ and the formula

$$x = y \mathbin{\&} \exists z\, a \in \mathbf{L}(z).$$

We get a set $m^\dagger$ such that $\mathbf{L}(\alpha) \subseteq m^\dagger$, $a \in m^\dagger$, $\text{card}(m^\dagger) = \text{card}(\alpha)$, $(m^\dagger, \in)$ is extensional, and

$$\models_{m^\dagger} \exists x\, a \in \mathbf{L}(x).$$

By the Mostowski Collapsing Theorem, we may assume further that $m^\dagger$ is *transitive.* In particular, an element β of $m^\dagger$ such that $\models_{m^\dagger} a \in \mathbf{L}(\beta)$ really is an ordinal. Then $a \in \mathbf{L}(\beta)$, but $\text{card}(\beta) = \text{card}(\alpha)$, so $a \in \mathbf{L}(\text{card}(\alpha)^+)$. $\square$

About a quarter century after Gödel proved that AC and GCH are consistent with ZF, Cohen (see [8]) proved the same of their negations.

A. The Greek alphabet

capital	minuscule	transliteration	name
Α	α	a	alpha
Β	β	b	beta
Γ	γ	g	gamma
Δ	δ	d	delta
Ε	ε	e	epsilon
Ζ	ζ	z	zeta
Η	η	ê	eta
Θ	ϑ	th	theta
Ι	ι	i	iota
Κ	ϰ	k	kappa
Λ	λ	l	lambda
Μ	μ	m	mu
Ν	ν	n	nu
Ξ	ξ	x	xi
Ο	ο	o	omicron
Π	π	p	pi
Ρ	ρ	r	rho
Σ	σ, ς	s	sigma
Τ	τ	t	tau
Υ	υ	y, u	upsilon
Φ	φ	ph	phi
Χ	χ	ch	chi
Ψ	ψ	ps	psi
Ω	ω	ô	omega

The following remarks pertain to *ancient* Greek. The vowels are α, ε, η, ι, ο, υ, ω, where η is a long ε, and ω is a long ο; the other vowels (α, ι, υ) can be long or short. Some vowels may be given tonal accents (ά, ᾶ, ὰ). An initial vowel takes either a rough-breathing mark (as in ἁ) or a smooth-breathing mark (ἀ): the former mark is transliterated by a preceding h, and the latter can be ignored, as in ὑπερβολή hyperbolê *hyperbola,* ὀρθογώνιον orthogônion *rectangle.* Likewise, ῥ is transliterated as rh, as in ῥόμβος rhombos *rhombus.* A long vowel may have an iota subscript (ᾳ, ῃ, ῳ), especially in case-endings of nouns. Of the two forms of minuscule sigma, the ς appears at the ends of words; elsewhere, σ appears, as in βάσις basis *base.*

B. Completeness

We show that the syntactical notion of derivation is the same in extension as the semantic notion of logical entailment: the syntactic turnstile $\vdash$ is interchangeable with the semantic turnstile $\models$. We shall use the convention (given also at the end of § 2.5) that $\sigma \Rightarrow \tau$ means $(\sigma \Rightarrow \tau)$, and then $\rho \Rightarrow \sigma \Rightarrow \tau$ means $\rho \Rightarrow (\sigma \Rightarrow \tau)$.

Lemma 30 (Deduction). *If* $\Gamma \cup \{\sigma\} \vdash \tau$, *then*

$$\Gamma \vdash \sigma \Rightarrow \tau.$$

Proof. Suppose $\Gamma \cup \{\sigma\} \vdash \tau$. We prove $\Gamma \vdash \sigma \Rightarrow \tau$ by induction.

1. If $\vdash \tau$, then, since $\tau \Rightarrow \sigma \Rightarrow \tau$ is a tautology, we have $\vdash \sigma \Rightarrow \tau$, and therefore $\Gamma \vdash \sigma \Rightarrow \tau$.

2. Suppose τ is in $\Gamma \cup \{\sigma\}$. There are two cases to consider.

a) If τ is in Γ, then $\Gamma \vdash \tau$, but then $\Gamma \vdash \sigma \Rightarrow \tau$ as before.

b) If τ is σ, then $\sigma \Rightarrow \tau$ is a tautology, so $\Gamma \vdash \sigma \Rightarrow \tau$.

3. Suppose both ρ and $\rho \Rightarrow \tau$ are derivable from $\Gamma \cup \{\sigma\}$, and $\sigma \Rightarrow \rho$ and $\sigma \Rightarrow \rho \Rightarrow \tau$ are derivable from Γ. Since the sentence

$$(\sigma \Rightarrow \rho) \Rightarrow (\sigma \Rightarrow \rho \Rightarrow \tau) \Rightarrow \sigma \Rightarrow \tau$$

is a tautology, $\Gamma \vdash \sigma \Rightarrow \tau$. □

A sentence or collection of sentences is **consistent** if no contradiction is derivable from it. If σ and τ are contradictions, then $\sigma \Rightarrow \tau$ is a tautology; therefore every contradiction is derivable from every contradiction. We may use the symbol

$$\perp$$

to denote an arbitrary contradiction.

Lemma 31. *If* $\neg\sigma$ *is not a logical theorem, then* σ *is consistent.*

Proof. Suppose σ is not consistent. Then $\sigma \vdash \perp$, so $\vdash \sigma \Rightarrow \perp$ by the Deduction Lemma. But $(\sigma \Rightarrow \perp) \Rightarrow \neg\sigma$ is a tautology, so $\vdash \neg\sigma$. □

Lemma 32. *If every* finite *collection of sentences from* Γ *is consistent, then* Γ *must be consistent.*

Proof. If Γ is inconsistent, there is a formal proof of this, and this proof uses only finitely many formulas from Γ. $\square$

In a word, the lemma is that consistency is **finitary.**

Lemma 33. *If* Γ *is a consistent collection of sentences, then* $\Gamma \cup \{\sigma\}$ *or* $\Gamma \cup \{\neg\sigma\}$ *is consistent.*

Proof. Suppose both $\Gamma \cup \{\sigma\}$ and $\Gamma \cup \{\neg\sigma\}$ are inconsistent. Since consistency is finitary, there is a finite collection $\{\tau_0, \dots, \tau_n\}$ of sentences from Γ such that both $\{\tau_0, \dots, \tau_n, \sigma\}$ and $\{\tau_0, \dots, \tau_n, \neg\sigma\}$ are inconsistent. By Deduction, we have

$$\vdash \sigma \Rightarrow \tau_0 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \bot,$$
$$\vdash \neg\sigma \Rightarrow \tau_0 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \bot.$$

By the tautology $(\sigma \Rightarrow \rho) \Rightarrow (\neg\sigma \Rightarrow \rho) \Rightarrow \rho$, we conclude

$$\vdash \tau_0 \Rightarrow \cdots \Rightarrow \tau_n \Rightarrow \bot,$$

and therefore Γ is inconsistent. $\square$

Lemma 34. *If* $\Gamma\{\exists x\, \varphi(x)\}$ *is consistent, and* a *does not occur in any of its formulas, then* $\Gamma \cup \{\exists x\, \varphi(x), \varphi(a)\}$ *is consistent.*

Proof. Suppose $\Gamma \cup \{\exists x\, \varphi(x), \varphi(a)\}$ is inconsistent. Then for some sentences τ_0, …, τ_n in $\Gamma \cup \{\exists x\, \varphi(x)\}$ we have

$$\vdash \varphi(a) \Rightarrow \tau_0 \cdots \Rightarrow \tau_n \Rightarrow \bot,$$
$$\vdash \exists x\, \varphi(x) \Rightarrow \tau_0 \cdots \Rightarrow \tau_n \Rightarrow \bot.$$

Therefore $\Gamma \cup \{\exists x\, \varphi(x)\}$ is inconsistent. $\square$

Lemma 35. $\{\sigma, \neg\sigma\}$ *is inconsistent.*

Lemma 36. *Assume* Γ *is consistent. Then* $\Gamma \cup \{\sigma \Rightarrow \tau\}$ *is consistent if and only if one of* $\Gamma \cup \{\neg\sigma\}$ *or* $\Gamma \cup \{\tau\}$ *is consistent.*

A form of the following theorem was proved by Gödel [17]; but our proof is due to Henkin.

Theorem 167 (Completeness)**.** *Every logically true sentence is a logical theorem: if* $\models \sigma$*, then* $\vdash \sigma$*.*

Proof. Suppose σ is not a logical theorem. Then $\neg\sigma$ is consistent, by Lemma 31. We shall find an interpretation of $\in$ in which $\neg\sigma$ is true. To do this, we assume we have an infinite list a_0, a_1, a_2, ..., of constants, and the constants that occur in σ are on this list. We take the sentences in which constants on this list occur, and we arrange *them* in an infinite list τ_0, τ_1, τ_2, ... We recursively define a list Γ_0, Γ_1, Γ_2, ..., of collections of sentences as follows.

1. Γ_0 consists of $\neg\sigma$ alone.
2. Γ_{n+1} contains every sentence in Γ_n. Also, if $\Gamma_n \cup \{\tau_n\}$ is consistent, then Γ_{n+1} contains τ_n; otherwise, Γ_{n+1} contains $\neg\tau_n$. If Γ_{n+1} contains τ_n, and this sentence is $\exists x\,\varphi(x)$ for some variable x and formula φ, then Γ_n contains $\varphi(a_k)$, where (assuming it exists) k is the *least* number ℓ such that the constant a_ℓ that does not occur in τ_n or any formula in Γ_n.

By induction, the collections Γ_n are finite, so the desired a_k in the definition does always exist. By Lemmas 33 and 34, each Γ_n is consistent. Let Γ be the collection of all formulas belonging to some Γ_n. By finitariness, Γ is consistent. Also, for each τ_k, either it or its negation is in Γ, but not both, by Lemma 35. In short, $\Gamma \cup \{\tau_k\}$ is consistent if and only if τ_k is in Γ. Define an atomic sentence $a_j \in a_k$ to be true if it is in Γ. By induction, every sentence τ_k is true under this interpretation if and only if it belongs to Γ:

1. The claim is true by definition when τ_k is atomic.
2. If the claim is true for some τ_k, then it is true for $\neg\tau_k$, since τ_k belongs to Γ if and only if $\neg\tau_k$ does not.
3. If the claim is true when τ_k is ρ and when τ_k is π, then it is true when τ_k is $\rho \Rightarrow \pi$, since $\rho \Rightarrow \pi$ belongs to Γ if and only if either ρ does not or π does, by Lemma 36.
4. If, for some singulary formula $\varphi(x)$, the claim is true whenever τ_k is $\varphi(a_j)$ for some j, then the claim is true when τ_k is $\exists x\,\varphi(x)$, by Lemma 34.

In particular, since $\neg\sigma$ is in Γ, it is true under the given interpretation of $\in$. □

Porism. *Every logical consequence of a collection of sentences is derivable from that collection: if* $\Gamma \models \sigma$*, then* $\Gamma \vdash \sigma$*.*

Proof. Suppose σ is not derivable from Γ. Then $\Gamma \cup \{\neg\sigma\}$ is consistent. The proof of the theorem can be adapted to show that, in some interpretation of $\in$, all sentences of $\Gamma \cup \{\neg\sigma\}$ are true. (This will need transfinite recursion, if Γ is uncountable.) Then Γ does not entail σ. $\square$

An interpretation of $\in$ in which every sentence of a collection Γ is true can be called a **model.**

Corollary (Compactness)**.** *If every finite subcollection of some collection of sentences has a model, then the whole collection has a model.*

Proof. If every finite subcollection has a model, then every finite subcollection is consistent, and therefore the whole collection is consistent, so it has a model. $\square$

C. Incompleteness

In our logic for set theory, we can assign to each symbol s a different set, to be denoted by

$$\ulcorner s \urcorner.$$

A natural choice is numbers: we could start making assignments as follows.

s	$\in$	$\neg$	$($	$\Rightarrow$	$)$	$\exists$
$\ulcorner s \urcorner$	0	1	2	3	4	5

We may assume that our variables are x_k, where $k \in \omega$; then we can let $\ulcorner x_k \urcorner$ be $k+6$. We can consider *every* set a as a constant in our language; then, when we turn around and choose a set $\ulcorner a \urcorner$, it might be the set a itself. However, possibly a was already chosen as $\ulcorner s \urcorner$ for some symbol s that is not a constant. We can avoid this problem by letting $\ulcorner a \urcorner$ be the ordered pair $(0, a)$, and letting $\ulcorner s \urcorner$ have the form $(1, b)$, if s is not a constant.

Each formula φ of our logic is a string $s_0 \cdots s_{n-1}$ for some n in ω. We can form the n-tuple

$$(\ulcorner s_0 \urcorner, \dots, \ulcorner s_{n-1} \urcorner),$$

which we can denote by

$$\ulcorner \varphi \urcorner.$$

Suppose Δ is a collection of axioms. Let us call Δ **recursive** if there is a class, which we might as well call $\ulcorner \Delta \urcorner$, such that a sentence σ is in Δ if and only if $\ulcorner \sigma \urcorner \in \ulcorner \Delta \urcorner$. In particular, all collections of axioms that we consider in the text are recursive.

Lemma 37. *For every recursive collection Δ of axioms, there are classes $\boldsymbol{C}$ and $\boldsymbol{D}$ such that, for each sentence σ and each singulary formula ψ,*

1. *σ is derivable from Δ if and only if $\ulcorner \sigma \urcorner \in \boldsymbol{C}$;*
2. *$\psi(\ulcorner \psi \urcorner)$ is derivable from Δ if and only if $\ulcorner \psi \urcorner \notin \boldsymbol{D}$.*

In symbols,

$$\Delta \vdash \sigma \iff \ulcorner \sigma \urcorner \in \boldsymbol{C},$$
$$\Delta \vdash \psi(\ulcorner \psi \urcorner) \iff \ulcorner \psi \urcorner \notin \boldsymbol{D}.$$

Gödel [18] first proved the following, not for sets, but for natural numbers, equipped with the operations of addition and multiplication.

Theorem 168 (Incompleteness). *For every recursive collection of axioms, there is a true sentence that is not derivable from those axioms.*

Proof. Let Δ be a recursive collection of axioms, and let φ define the class $\boldsymbol{D}$ given by the lemma. The following are equivalent:
1) $\varphi(\ulcorner\varphi\urcorner)$ is true;
2) $\ulcorner\varphi\urcorner$ belongs to the class $\{x \colon \varphi(x)\}$, namely $\boldsymbol{D}$;
3) $\varphi(\ulcorner\varphi\urcorner)$ is not derivable from Δ.

Since every sentence derivable from Δ is true, it follows that $\varphi(\ulcorner\varphi\urcorner)$ is true, but not derivable from Δ. □

In the notation of the lemma, the consistency of Δ can be expressed by

$$\ulcorner\bot\urcorner \notin \boldsymbol{C}.$$

Theorem 169 (Second Incompleteness). *If Δ is a recursive set of axioms that includes those of* GST*, then consistency of Δ is not derivable from Δ.*

Proof. The lemma assumes only GST. Therefore, in the notation of the proof of the theorem, by Completeness (Theorem 167), we can derive from Δ that $\varphi(\ulcorner\varphi\urcorner)$ is true if and only if it is not derivable from Δ. Since we know Δ is consistent, we conclude $\varphi(\ulcorner\varphi\urcorner)$ is not derivable from Δ. Therefore, from Δ we can derive that, if Δ is consistent, then $\varphi(\ulcorner\varphi\urcorner)$ is not derivable from Δ, and therefore $\varphi(\ulcorner\varphi\urcorner)$ is true. In short, if $\ulcorner\bot\urcorner \notin \boldsymbol{C}$ is derivable from Δ, then so is $\varphi(\ulcorner\varphi\urcorner)$. But $\varphi(\ulcorner\varphi\urcorner)$ is not derivable; so consistency of Δ is not derivable. □

One might think that there was a class $\boldsymbol{E}$ such that, from GST or all of ZFC, we could prove that, for all sentences σ, $\ulcorner\sigma\urcorner \in \boldsymbol{E}$ if and only if σ is true. Then $\boldsymbol{E}$ would not contain $\ulcorner\bot\urcorner$, and this would constitute a proof that ZFC (hence GST) was consistent. Considering the definition of truth (Definition 3 in §2.4), we might proceed as follows. Let $\boldsymbol{C}$ be the class of all ordered pairs (t, f) such that the following conditions hold; these conditions ensure that, if $\ulcorner\tau\urcorner \in t$, then τ is true, but if $\ulcorner\tau\urcorner \in f$, then τ is false:

1. a) If $\ulcorner a \in b\urcorner \in t$, then $a \in b$.

b) If $\ulcorner a \in b \urcorner \in f$, then $a \notin b$.

2. a) If $\ulcorner \neg\tau \urcorner \in t$, then $\ulcorner \tau \urcorner \in f$.

   b) If $\ulcorner \neg\tau \urcorner \in f$, then $\ulcorner \tau \urcorner \in t$.

3. a) If $\ulcorner (\tau \Rightarrow \rho) \urcorner \in t$, then $\ulcorner \rho \urcorner \in t$ or $\ulcorner \tau \urcorner \in f$.

   b) If $\ulcorner (\tau \Rightarrow \rho) \urcorner \in f$, then $\ulcorner \rho \urcorner \in f$ and $\ulcorner \tau \urcorner \in t$.

4. a) If $\ulcorner \exists x\, \varphi \urcorner \in t$, then for some a, $\ulcorner \varphi(a) \urcorner \in t$.

   b) If $\ulcorner \exists x\, \varphi \urcorner \in f$, then for all a, $\ulcorner \varphi(a) \urcorner \in f$.

We then let $\boldsymbol{E}$ be the union of the class of all sets t such that, for some set f, the ordered pair (t, f) belongs to $\boldsymbol{C}$. The problem is that such a set f cannot have an element of the form $\ulcorner \exists x\, \varphi \urcorner$, because the last condition then would make f a proper class. Then t cannot have an element of the form $\ulcorner \neg\exists x\, \varphi \urcorner$.

There is a *collection* of all sets $\ulcorner \sigma \urcorner$ such that σ is true: it is the collection of all sets $\ulcorner \sigma \urcorner$ that belong to some class $\boldsymbol{T}$ such that, for some class $\boldsymbol{F}$, the following conditions are met:

1. a) If $\ulcorner a \in b \urcorner \in \boldsymbol{T}$, then $a \in b$.

   b) If $\ulcorner a \in b \urcorner \in \boldsymbol{F}$, then $a \notin b$.

2. a) If $\ulcorner \neg\tau \urcorner \in \boldsymbol{T}$, then $\ulcorner \tau \urcorner \in \boldsymbol{F}$.

   b) If $\ulcorner \neg\tau \urcorner \in \boldsymbol{F}$, then $\ulcorner \tau \urcorner \in \boldsymbol{T}$.

3. a) If $\ulcorner (\tau \Rightarrow \rho) \urcorner \in \boldsymbol{T}$, then $\ulcorner \rho \urcorner \in \boldsymbol{T}$ or $\ulcorner \tau \urcorner \in \boldsymbol{F}$.

   b) If $\ulcorner (\tau \Rightarrow \rho) \urcorner \in \boldsymbol{F}$, then $\ulcorner \rho \urcorner \in \boldsymbol{F}$ and $\ulcorner \tau \urcorner \in \boldsymbol{T}$.

4. a) If $\ulcorner \exists x\, \varphi \urcorner \in \boldsymbol{T}$, then for some a, $\ulcorner \varphi(a) \urcorner \in \boldsymbol{T}$.

   b) If $\ulcorner \exists x\, \varphi \urcorner \in \boldsymbol{F}$, then for all a, $\ulcorner \varphi(a) \urcorner \in \boldsymbol{F}$.

By the Second Incompleteness Theorem, this collection must not be a class. The collection *is* a class in Morse–Kelley set theory (see Appendix F), so this theory does entail the consistency of ZFC.

D. The German script

Writing in 1993, Wilfrid Hodges [21, Ch. 1, p. 21] observes

> Until about a dozen years ago, most model theorists named structures in horrible Fraktur lettering. Recent writers sometimes adopt a notation according to which all structures are named M, M', M^*, $\bar{M}$, M_0, M_i or occasionally N.

For Hodges, structures are A, B, C, and so forth; he refers to their universes as **domains** and denotes these by $\mathrm{dom}(A)$ and so forth. This practice is convenient if one is using a typewriter (as in the preparation of another of Hodges's books [22], from 1985). In 2002, David Marker [27] uses 'calligraphic' letters for structures, so that M is the universe of $\mathcal{M}$. I still prefer the Fraktur letters:

$$
\begin{array}{ccccccccc}
\mathfrak{A} & \mathfrak{B} & \mathfrak{C} & \mathfrak{D} & \mathfrak{E} & \mathfrak{F} & \mathfrak{G} & \mathfrak{H} & \mathfrak{I} \\
\mathfrak{J} & \mathfrak{K} & \mathfrak{L} & \mathfrak{M} & \mathfrak{N} & \mathfrak{O} & \mathfrak{P} & \mathfrak{Q} & \mathfrak{R} \\
\mathfrak{S} & \mathfrak{T} & \mathfrak{U} & \mathfrak{V} & \mathfrak{W} & \mathfrak{X} & \mathfrak{Y} & \mathfrak{Z} & \\
\\
\mathfrak{a} & \mathfrak{b} & \mathfrak{c} & \mathfrak{d} & \mathfrak{e} & \mathfrak{f} & \mathfrak{g} & \mathfrak{h} & \mathfrak{i} \\
\mathfrak{j} & \mathfrak{k} & \mathfrak{l} & \mathfrak{m} & \mathfrak{n} & \mathfrak{o} & \mathfrak{p} & \mathfrak{q} & \mathfrak{r} \\
\mathfrak{s} & \mathfrak{t} & \mathfrak{u} & \mathfrak{v} & \mathfrak{w} & \mathfrak{x} & \mathfrak{y} & \mathfrak{z} &
\end{array}
$$

A way to write these by hand is shown in Figure D.1, which is taken from a 1931 textbook of German for English-speakers [19].

Figure D.1. The German alphabet by hand

E. The Axioms

We work in a logic whose only predicate is $\in$. We expand this so that, by definition (p. 48),

$$a = b \Leftrightarrow \forall x\,(x \in a \Leftrightarrow x \in b).$$

Then our axioms are:

1. Equality (p. 49):

$$a = b \Rightarrow \forall x\,(a \in x \Leftrightarrow b \in x).$$

2. Null set (p. 52): 0 is a set.
3. Adjunction (p. 52): $a \cup \{b\}$ is a set.
4. Separation (p. 61): For every class $\boldsymbol{C}$, the class $\boldsymbol{C} \cap a$ is a set.
5. Replacement (p. 82): For every function $\boldsymbol{F}$, the class $\boldsymbol{F}[a]$ is a set (assuming $a \subseteq \operatorname{dom}(\boldsymbol{F})$).
6. Union (p. 87): The union of every set is a set.
7. Infinity (p. 94): The class ω of natural numbers is a set.
8. Power Set (p. 111): The power class of a set is a set.
9. Choice (p. 119): Every set has a choice-function.
10. Foundation (p. 138): All sets are well-founded.

F. Other set theories and approaches

As I note in the Preface, I aim in this book to introduce axioms are only when further progress is otherwise hindered. This approach is taken also by Lemmon, whose *Introduction to axiomatic set theory* [24] can be analyzed as in Table F.1. However, Lemmon explicitly (on his p. 4) follows

chapter	title	page	axiom		page
I	The General Theory of Classes	14	A1	[Extension]	14
			A2	[Classification]	15
II	Sets, Relations, and Functions	47	A3	[Null Set]	51
			A4	[Pairs]	51
			A5	[Union]	51
			A6	[Power Set]	51
			A7	[Separation]	51
			A8	[Replacement]	89
III	Numbers	92	A9	[Infinity]	105
			A10	[Foundation]	105
			A11	[Choice]	118
Appendix on some variant set theories		121			

Table F.1. Lemmon's *Introduction to axiomatic set theory*

von Neumann, Bernays, Gödel, and Morse in giving formal existence to classes. He uses only one kind of variable, which ranges over the classes. So really Lemmon's theory is a theory of classes. His 'classification axiom scheme', A2, is that, for every formula in one free variable, there is a class of all *sets* that satisfy the formula. He notes in the appendix that he thus gives up the finite axiomatizability of Gödel's theory (the theory called below NBG, after von **N**eumann, **B**ernays, and **G**ödel).

Meanwhile, Lemmon's A2 allows the next six axioms, A3–A8, to be that certain classes are sets. But then Lemmon states A9, the Axiom of Infinity, in the traditional way: there exists a set, called ω, with certain closure properties. He writes:

A9, the axiom of infinity, is an outright declaration that a certain set exists; in this respect it is like A3 [the Null Set Axiom]. In fact A3 is a consequence of A9... [24, p. 105]

Since he has already defined ω informally on page 99, it is not clear why he does not take the extra step of observing that ω does exist as a class, regardless of the Axiom of Infinity. I prefer to take this step, then see what can be said *before* going on to declare that the class ω is a set.

I prefer in addition to remain agnostic about infinity as long as possible: somewhat longer than Lemmon. Kunen [23, IV.3.13, p. 123] shows that $\mathbf{R}(\omega)$ is a model of ZFC with the Axiom of Infinity replaced with its negation; but he assumes ZF without Foundation; in particular, he assumes Infinity. This is not necessary. All one needs is GST (see p. 61).

Lemmon's classification axiom scheme is non-minimalistic in a strong sense. His theory is a version of so-called Morse–Kelley set theory; in particular, it assumes more than our ZFC. I quote here some criticism of this theory by Joseph Shoenfield.[1] Shoenfield apparently uses the word *collection* as I do in the text, as the most general collective noun (see p. 14):

In response to some rather unfavorable remarks I made about MK (Morse–Kelly set-class theory), Friedman has defended MK as natural and important. Let me try to describe briefly what (in my opinion) is the origin and purpose of NBG and MK.

For the moment, let us take a class to be a collection of sets definable from set parameters in the language of ZFC. Classes in this sense play an important role, even if one is working in ZFC. For example, the Separation Scheme is most easily stated as: every class which is included in a set is a set. In the language of ZFC, we can only state this as a scheme. If we introduce variables for classes, we can state it as a single axiom. Of course, we then need axioms to insure us that all classes are values of the class variables. This can be done by a simple scheme. As Friedman and others have pointed out, this scheme can be

[1]This Shoenfield is apparently the author of the text *Mathematical Logic* [30], who was born in 1927, and who died on November 15, 2000. The quotation is from an email, dated February 14, 2000, sent to the FOM (Foundations of Mathematics) email list and archived at `http://www.cs.nyu.edu/pipermail/fom/2000-February/003740.html` (accessed March 5, 2011). A reference to this email is given at `http://en.wikipedia.org/wiki/Morse-Kelley_set_theory` (accessed same date). I have imposed the special mathematical typography on the original plain-text email, and I have corrected one or two typographical errors.

derived from a finite number of its instances; but the proof of this is a bit tedious, and is quite useless if one wishes to develop set theory in NBG.

The nice thing about NBG it that every model M of ZFC has a least extension to a model of NBG; the classes in the extension are just the classes in M. From this it follows that NBG is a conservative extension of ZFC. Thus whether we do set theory in ZFC or NBG is a matter of taste.

Now all of this naturally suggests an extended notion of class, in which a class is an arbitrary collection of sets. We then extend our class existence scheme to make every collection of sets definable in our extended language a class. Of course not every class (in the extended sense) is so definable; but these are the only ones we can assert are classes in our extended language.

Unfortunately, it is no longer true that any model of ZFC can be extended to a model of MK. We can prove Con(ZFC)[2] in MK by proving that the class $\mathbf{V}$ is a model of ZFC; and Con(ZFC) is a statement in the language of ZFC not provable in ZFC. If the model of ZFC has strong enough closure properties, we can extend it. For example, if the model is closed under forming subsets, it is clear that the Separation Scheme will hold independent of the choice of the classes in the model. In this way we can show (as Friedman observes) that a model V_κ where κ is an inacessible cardinal can be extended to a model of MK. The trouble with such models is that they have strong absoluteness properties; most interesting set theoretic statements are true in V_κ iff they are true in $\mathbf{V}$. This makes the models useless for most independence proofs.

Friedmann has given a sketch of an independence proof in MK by forcing; but many of the details are unclear to me. He takes a model M of MK, lets M' be the included model of ZFC and N' a generic extension of M'. He then says N' canonically generates a model N of MK. I do not understand how one selects the classes of N, nor how one can prove the axioms of MK hold in N. I would be surprised if the details wouls lead me to agree with Friedman that the question he was considering is 'not very much easier to solve for NBG than it is for MK'. In any case, there seems to be little reason to solve it for MK.

Friedman concludes with some predictions about the future of MK and similar systems, He says:

[2] That is, consistency of ZFC, formalized as for example on page 154.

> We have only the bare beginnings of where the axioms of large cardinals come from or why they are canonical or why they should be accepted or why they are consistent.

I agree whole-heartedly with this, and with the implied statement that these are important questions. He then says:

> I have no doubt that further substantial progess on these crucial issues will at least partly depend on deep philosophical introspection, and I have no doubt concepts of both class and set and their 'interaction' will play a crucial role in the future.

Here I strongly disagree. I think that if there is one thing we can learn from the development of mathematical logic in the last century, it is that all the major accomplishments of this subject consist of mathematical theorems, which, in the most interesing cases, have evident foundational consequences. I do not know of any major result in the field which was largely achieved by means of philosophical introspection, as I understand the term. I do not see the the study of the interaction of sets and classes has led to any very interesting results.

If the problems about large cardinals cannot be solved by philo- sophical introspection, how can they be solved? Fortunately, I have available an example of how to proceed, furnished by the recent communication of John Steel. I think it says more about the problems of large cardinals then all the previous FOM communications combined. The idea is to examine all the results which have been proved about large cardinals and related concepts, and see if they give some hint of which large cardinals we should accept and what further results we might prove to further justify these axioms. We are still a long way from accomplishing the goal, but, as Steel shows, we have advanced a great deal since large cardinals first appeared on the scene forty years ago.

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