NUMBER-THEORY EXERCISES, II

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These exercises are for Elementary Number Theory I (Math 365 at METU). In them, if a *statement* is given that is not a definition, then the exercise is to prove the statement.

A first set of exercises is the proofs not given in [2]. The exercises below are mostly inspired by exercises in [1, Ch. 2].

Recall that the **triangular numbers** compose a sequence $(t_n : n \in \mathbb{N})$, defined recursively by $t_0 = 0$ and $t_{n+1} = t_n + n + 1$.

Exercise 1. An integer n is a triangular number if and only if $8n + 1$ is a square number.

Exercise 2.

(a) If n is triangular, then so is $9n + 1$.
(b) Find infinitely many pairs (k, ℓ) such that, if n is triangular, then so is $kn + \ell$.

Exercise 3. If $a = n(n + 3)/2$, then $t_a + t_{n+1} = t_{a+1}$.

Exercise 4. The **pentagonal numbers** are 1, 5, 12, . . . : call these p_1, p_2, &c.

(a) Give a recursive definition of these numbers.
(b) Find a closed expression for p_n (that is, an expression not involving p_{n-1}, p_{n-2}, &c.).
(c) Find such an expression involving triangular numbers and square numbers.

Exercise 5.

(a) $7 \mid 2^{3n} + 6$.
(b) Given a in $\mathbb{Z}$ and k in $\mathbb{N}$, find integers b and c such that $b \mid a^{kn} + c$ for all n in $\mathbb{N}$.

Exercise 6. $\gcd(a, a + 1) = 1$.

Exercise 7. $(k!)^n \mid (kn)!$ for all k and n in $\mathbb{N}$.

Exercise 8. If a and b are co-prime, and a and c are co-prime, then a and bc are co-prime.

Exercise 9. Let $\gcd(204, 391) = n$.

(a) Compute n.
(b) Find a solution of $204x + 391y = n$.

Exercise 10. Let $\gcd(a, b) = n$.

(a) If $k \mid \ell$ and $\ell \mid 2k$, then $|\ell| \in \{|k|, |2k|\}$.
(b) Show $\gcd(a + b, a - b) \in \{n, 2n\}$.
(c) Find an example for each possibility.
(d) $\gcd(2a + 3b, 3a + 4b) = n$.
(e) Solve $\gcd(ax + by, az + bw) = n$.

Exercise 11. $\gcd(a, b) \mid \operatorname{lcm}(a, b)$.

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Exercise 12. When are $\gcd(a,b)$ and $\operatorname{lcm}(a,b)$ the same?

Exercise 13. The binary operation $(x,y) \mapsto \gcd(x,y)$ on $\mathbb{N} \setminus \{0\}$ is commutative and associative.

Exercise 14. The co-prime relation on $\mathbb{N} \setminus \{0\}$, namely

$$\{(x,y) \in \mathbb{N} \setminus \{0\} \colon \gcd(x,y) = 1\}$$

—is it reflexive? irreflexive? symmetric? anti-symmetric? transitive?

Exercise 15. Give complete solutions, or show that they do not exist, for:

(a) $14x - 56y = 34$;
(b) $10x + 11y = 12$.

Exercise 16. I have some 1-YTL pieces and some 50- and 25-YKr pieces: 16 coins in all. They make 6 YTL. How many coins of each denomination have I got?

References

[1] David M. Burton. *Elementary Number Theory*. McGraw-Hill, Boston, sixth edition, 2007.
[2] David Pierce. Foundations of number-theory. `http://www.math.metu.edu.tr/~dpierce/courses/365/`. 4 pp.

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