NUMBER-THEORY EXERCISES, VI

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The variables n, k, and d range over the positive integers.

Exercise 1. Assuming p is an *odd* prime:

(a) $(p-1)! \equiv p-1 \pmod{1+2+\dots+(p-1)}$;
(b) $1 \cdot 3 \cdots (p-2) \equiv (-1)^{(p-1)/2} \cdot (p-1) \cdot (p-3) \cdots 2 \pmod{p}$;
(c) $1 \cdot 3 \cdots (p-2) \equiv (-1)^{(p-1)/2} \cdot 2 \cdot 4 \cdots (p-1) \pmod{p}$;
(d) $1^2 \cdot 3^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \pmod{p}$.

Exercise 2. $\tau(n) \leqslant 2\sqrt{n}$.

Exercise 3. $\tau(n)$ is odd if and only if n is square.

Exercise 4. Assuming n is odd: $\sigma(n)$ is odd if and only if n is square.

Exercise 5. $\displaystyle\sum_{d \mid n} \frac{1}{d} = \frac{\sigma(n)}{n}$.

Exercise 6. $\{n \colon \tau(n) = k\}$ is infinite (when $k > 1$), but $\{n \colon \sigma(n) = k\}$ is finite.

Exercise 7. Let $m \in \mathbb{Z}$. The number-theoretic function $n \mapsto n^m$ is multiplicative.

Exercise 8. Let $\omega(n)$ be the number of *distinct* prime divisors of n, and let m be a non-zero integer. Then $n \mapsto m^{\omega(n)}$ is multiplicative.

Exercise 9. Let $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some positive } m; \\ 0, & \text{otherwise.} \end{cases}$

(a) $\log n = \displaystyle\sum_{d \mid n} \Lambda(d)$.
(b) $\Lambda(n) = \displaystyle\sum_{d \mid n} \mu\left(\frac{n}{d}\right) \log d$.
(c) $\Lambda(n) = -\displaystyle\sum_{d \mid n} \mu(d) \log d$.

Exercise 10. Suppose $n = {p_1}^{k(1)} \cdots {p_r}^{k(r)}$, where the p_i are distinct.

(a) If f is multiplicative and non-zero, then $\displaystyle\sum_{d \mid n} \mu(d) \cdot f(d) = \prod_{i=1}^{r} (1 - f(p_i))$;
(b) $\displaystyle\sum_{d \mid n} \mu(d) \cdot \tau(d) = (-1)^r$.

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