

## NUMBER-THEORY EXERCISES, VIII

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**Exercise 1.** We have  $(\pm 3)^2 \equiv 2 \pmod{7}$ . Compute the orders of 2, 3, and  $-3$ , *modulo* 7.

**Exercise 2.** Suppose  $\text{ord}_n(a) = k$ , and  $b^2 \equiv a \pmod{n}$ .

- (a) Show that  $\text{ord}_n(b) \in \{k, 2k\}$ .
- (b) Find an example for each possibility of  $\text{ord}_n(b)$ .
- (c) Find a condition on  $k$  such that  $\text{ord}_n(b) = 2k$ .

**Exercise 3.** This is about 23:

- (a) Find a primitive root of least absolute value.
- (b) How many primitive roots are there?
- (c) Find these primitive roots as powers of the root found in (a).
- (d) Find these primitive roots as elements of  $[-11, 11]$ .

**Exercise 4.** Assuming  $\text{ord}_p(a) = 3$ , show:

- (a)  $a^2 + a + 1 \equiv 0 \pmod{3}$ ;
- (b)  $(a + 1)^2 \equiv a \pmod{3}$ ;
- (c)  $\text{ord}_p(a + 1) = 6$ .

**Exercise 5.** Find all elements of  $[-30, 30]$  having order 4 *modulo* 61.

**Exercise 6.**  $f(x) \equiv 0 \pmod{n}$  may have more than  $\deg(f)$  solutions:

- (a) Find four solutions to  $x^2 - 1 \equiv 0 \pmod{35}$ .
- (b) Find conditions on  $a$  such that the congruence  $x^2 - a^2 \equiv 0 \pmod{35}$  has four distinct solutions, and find these solutions.
- (c) If  $p$  and  $q$  are odd primes, find conditions on  $a$  such that the congruence  $x^2 - a^2 \equiv 0 \pmod{pq}$  has four distinct solutions, and find these solutions.

**Exercise 7.** If  $\text{ord}_n(a) = n - 1$ , then  $n$  is prime.

**Exercise 8.** If  $a > 1$ , show  $n \mid \phi(a^n - 1)$ .

**Exercise 9.** If  $2 \nmid p$  and  $p \mid n^2 + 1$ , show  $p \equiv 1 \pmod{4}$ .

**Exercise 10.**

- (a) Find conditions on  $p$  such that, if  $r$  is a primitive root of  $p$ , then so is  $-r$ .
- (b) If  $p$  does not meet these conditions, then what is  $\text{ord}_p(-r)$ ?

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