ELEMENTARY NUMBER THEORY II, EXAMINATION I

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Instructions. Take at most 90 minutes to write reasonably legible solutions on the blank sheets provided. You may want to do scratch-work first, on sheets that you will keep. But the sheets that you turn in should show sufficient work to justify your answers. You may keep this problem-sheet for future study. *Kolay gelsin.*

Problem 1. This problem involves the Gaussian integers. Let $\alpha = 40 + 5\mathrm{i}$ and $\beta = 39\mathrm{i}$.

(i) Find a greatest common divisor of α and β.
(ii) If γ is your answer to (i), solve

$$(40 + 5\mathrm{i}) \cdot \xi + 39\mathrm{i} \cdot \eta = \gamma.$$

Problem 2. This problem involves the Diophantine equation

$$2x^2 - 3y^2 = 2. \tag{$*$}$$

(i) Express $\sqrt{3/2}$ as a continued fraction.
(ii) Find a positive solution to $(*)$.
(iii) Find a solution (a, b) to $(*)$ in which each of a and b has two digits (in the usual decimal notation).
(iv) Find a solution (a, b) to $(*)$ in which each of a and b has three digits.

Problem 3. In class we found the bijection

$$t \longmapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

between $\mathbb{Q}$ and the set of rational solutions (other than $(-1, 0)$) to the equation

$$x^2 + y^2 = 1.$$

(i) Find all rational solutions to the equation

$$x^2 + 3y^2 = 1.$$

(ii) Find α in $\mathbb{Q}(\mathrm{i})$ such that $\mathrm{N}(\alpha) = 1$, but α is not a Gaussian integer.
(iii) Find β in $\mathbb{Q}(\sqrt{-3})$ such that $\mathrm{N}(\beta) = 1$, but β is not an integer (that is, not an Eisenstein integer).

Problem 4.

(i) Find all distinct solutions (from $\mathbb{Z}$) of the Diophantine equation

$$x^2 + y^2 = 221.$$

(ii) Find a factorization of $27 - 57\mathrm{i}$ as a product of Gaussian primes.

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