ELEM. N° THY II, EXAMINATION II SOLUTIONS

Solution 1. By the algorithm for finding continued fractions, when $x = \sqrt{a^2+1}$:

$$a_0 = a, \quad \xi_0 = \sqrt{a^2+1} - a, \quad \frac{1}{\xi_0} = \frac{\sqrt{a^2+1}+a}{a^2+1+a^2} = \sqrt{a^2+1}+a,$$

so $a_1 = 2a$ and $\xi_1 = \sqrt{a^2+1} - a = \xi_0$. Therefore $x = [a; \overline{2a}]$.

Remark. Alternatively, one may let

$$x = [a;\overline{2a}] = [a; a+a, \overline{2a}] = [a; a + [a, \overline{2a}]] = [a; a+x] = a + \frac{1}{a+x} = \frac{a^2+ax+1}{a+x},$$

so that $ax + x^2 = a^2 + ax + 1$ and therefore $x^2 = a^2 + 1$. Since $a > 0$, we have $[a; \overline{2a}] > 0$ and therefore $[a; \overline{2a}] = x = \sqrt{a^2+1}$.

Solution 2. (a) Since $\sqrt{5}$ is a root of $x^2 - 5$, whose leading coefficient is 1, we can conclude $\mathfrak{O}_\Lambda = \langle 1, \sqrt{5} \rangle = \Lambda$.

(b) We know that the units of $\mathfrak{O}_K$ (when $K = \mathbb{Q}(\sqrt{5})$) are $\pm\phi^n$. Of these, those that are greater than 1 form the list

$$\phi, 1+\phi, 1+2\phi, 2+3\phi, 3+5\phi, 5+8\phi, \dots$$

(and in general $\phi^n = \mathrm{F}_{n-1} + \mathrm{F}_n\,\phi$). But since $2\phi = 1 + \sqrt{5}$, we have $\mathfrak{O}_\Lambda = \langle 1, 2\phi \rangle$; also, $\mathrm{N}(\phi) = -1$. The first power of ϕ greater than 1 that belongs to $\mathfrak{O}_\Lambda$ and has norm 1 is therefore ϕ^6. Hence the elements of $\mathfrak{O}_\Lambda$ of norm 1 are $\pm\phi^{6n}$, where $n \in \mathbb{Z}$.

Solution 3. We want to solve

$$19 = x^2 + 2xy + 4y^2 = (x+y)^2 + 3y^2.$$

Hence $y^2 \leqslant 19/3 < 9$, so $|y| < 3$. When $y = \pm 2$, the equation becomes $(x \pm 2)^2 = 7$, which has no solution. When $y = \pm 1$, we get $(x \pm 1)^2 = 16$, so $(x \pm 1) \in \{4, -4\}$. When $y = 0$, there is no solution. So the solutions of the original equation are $(3, 1)$, $(-5, 1)$, $(5, -1)$, $(-3, -1)$.

Remark. Solving an equation means not only finding solutions, but showing that there are no other solutions. This is done here by noting that there are only 5 possibilities for y. Alternatively, one may rewrite the equation as

$$19 = (x + y + y\sqrt{-3})(x + y - y\sqrt{-3}) = \mathrm{N}(x + 2\omega y),$$

where we work in $\mathbb{Q}(\sqrt{-3})$. Since (x, y) is a solution if and only if $(|x|, |y|)$ is a solution, we can obtain all solutions from Figure 1.

Solution 4. (a) In $\mathbb{Q}(\sqrt{21})$, we have $(5 + \sqrt{21})/2 = 2 + \omega$, and $\mathrm{N}((5 + \sqrt{21})/2) = 1$. Hence $(5+\sqrt{21})/2$ is a unit of $\mathfrak{O}_K$, so its powers are also units of $\mathfrak{O}_K$. Let $\alpha \in \mathfrak{O}_K$. Since

$$\mathfrak{O}_K = \langle 1, \omega \rangle = \left\langle 1, \frac{1+\sqrt{21}}{2} \right\rangle,$$

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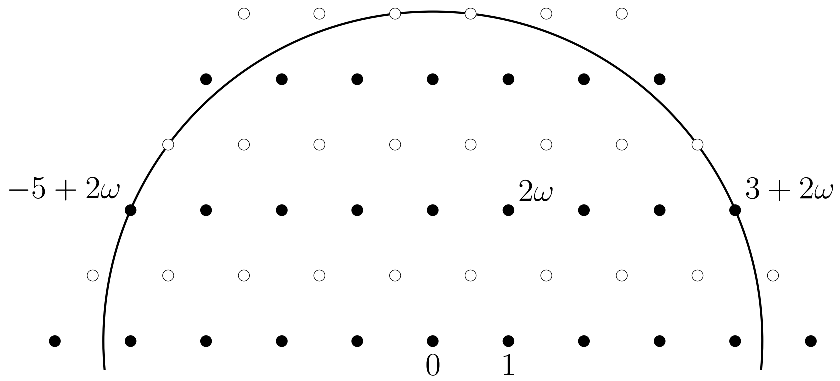

FIGURE 1. Solutions of $\mathrm{N}(\xi) = 7$ from $\langle 1, 2\omega\rangle$ in $\mathbb{Q}(\sqrt{-3})$

we have $2\alpha \in \langle 2, 1+\sqrt{21}\rangle \subseteq \langle 1, \sqrt{21}\rangle$. This proves

$$a_n + b_n\sqrt{21} = 2\Big(\frac{5+\sqrt{21}}{2}\Big)^n \in \langle 1, \sqrt{21}\rangle,$$

so $a_n, b_n \in \mathbb{Z}$.

(b) Since $\mathrm{N}(a_n + b_n\sqrt{21}) = 4$, the pairs $(\pm a_n, \pm b_n)$ are solutions of $x^2 - 21y^2 = 4$.

(c) Suppose (a, b) is an arbitrary solution of $x^2 - 21y^2 = 4$. Then $a \equiv b \pmod 2$, so $2 \mid a - b$. Hence

$$\frac{a+b\sqrt{21}}{2} = \frac{a-b+b+b\sqrt{21}}{2} = \frac{a-b}{2} + b\frac{1+\sqrt{21}}{2} = \frac{a-b}{2} + b\omega \in \langle 1, \omega\rangle,$$

so $(a + b\sqrt{21})/2$ is an element of $\mathfrak{O}_K$ of norm 1. But there is ε or $r + s\omega$ in $\mathfrak{O}_K$ of norm 1 such that $r, s > 0$ and every element of $\mathfrak{O}_K$ of norm 1 is $\pm\varepsilon^n$ for some n. But $(5 + \sqrt{21})/2 = 2 + \omega$ and has norm 1, so it must be ε. Hence $(a, b) = (\pm a_n, \pm b_n)$ for some n.

Remark. The pair $(a_n/2, b_n/2)$ solves the Pell equation $x^2 - 21y^2 = 1$, but its entries need not belong to $\mathbb{Z}$. For example, $(a_1/2, b_1/2) = (5/2, 1/2)$.

Solution 5. (a) We have $2 = 2x^2 - 3y^2 \iff 4 = 4x^2 - 6y^2 = \mathrm{N}(2x + y\sqrt{6})$ in $\mathbb{Q}(\sqrt{6})$. So let

$$K = \mathbb{Q}(\sqrt{6}), \quad \alpha = 2, \quad \beta = \sqrt{6}, \quad m = 4.$$

(b) We want the elements of $\mathfrak{O}_\Lambda$ of norm 1. But $(1/2)\Lambda = \langle 1, \sqrt{6}/2\rangle$, and $\sqrt{6}/2$ is a root of $2x^2 - 3$. Hence $\mathfrak{O}_\Lambda = \langle 1, 2\sqrt{6}/2\rangle = \langle 1, \sqrt{6}\rangle = \langle 1, \omega\rangle = \mathfrak{O}_K$. We obtain the units of $\mathfrak{O}_K$ from the continued-fraction expansion of $\sqrt{6}$:

$$\begin{aligned}
x &= \sqrt{6}, & a_0 &= 2, & \xi_0 &= \sqrt{6} - 2;\\
\frac{1}{\xi_0} &= \frac{\sqrt{6}+2}{2}, & a_1 &= 2, & \xi_1 &= \frac{\sqrt{6}-2}{2};\\
\frac{1}{\xi_1} &= \sqrt{6} + 2, & a_2 &= 4, & \xi_2 &= \sqrt{6} - 2 = \xi_0;
\end{aligned}$$

so $\sqrt{6} = [2; \overline{2, 4}]$. Since $[2; 2] = 5/2$, and $\mathrm{N}(5 + 2\omega) = 1$, we can conclude that the elements of $\mathfrak{O}_K$ of norm 1 are $\pm(5 + 2\omega)^n$. Therefore the desired parallelogram Π

can be bounded by the straight lines given by

$$2x + y\sqrt{6} = 1; \qquad 2x + y\sqrt{6} = 5 + 2\sqrt{6}.$$

Also, we are looking for points on the hyperbola

$$4 = 4x^2 - 6y^2 = (2x + y\sqrt{6})(2x - y\sqrt{6}),$$

one of whose asymptotes, given by

$$2x - y\sqrt{6} = 0,$$

forms a third side of Π; the fourth side is given by

$$2x - y\sqrt{6} = 4,$$

since this line meets the hyperbola where $2x + y\sqrt{6} = 1$ does.

(c) Same as (b).

Remark. The point in (b) is that, if γ is from Λ and solves $\mathrm{N}(\xi) = 4$, then the same is true of δ or $2a + b\omega$, where $\delta = \pm(5 + 2\omega)^n\gamma$; and we should be able to pick the sign and n so that $(a, b) \in \Pi$. But we can pick n so that

$$1 \leqslant |\delta| < |5 + 2\omega| < 10.$$

We also have $4 = \delta\delta'$, so

$$|\delta'| = \frac{4}{\delta} \leqslant 4.$$

Since

$$\delta = \frac{\delta + \delta'}{2} + \frac{\delta - \delta'}{2\omega}\omega.$$

we conclude

$$|a| = \left|\frac{\delta + \delta'}{4}\right| < 4, \quad |b| = \left|\frac{\delta - \delta'}{2\omega}\right| < 4.$$

Thus, in (b), Π can be the square with vertices $(\pm 4, 4)$ and $(\pm 4, -4)$. But this isn't good enough for (c).

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