Model-Theory to Compactness

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0 Introduction

These notes are an attempt to develop model theory, as economically as possible, on a foundation of some familiarity with algebraic structures. References include [1], [2] and [3].

1 The natural numbers

By one standard definition, the set ω of **natural numbers** is the smallest set that contains the empty set and that contains $x \cup \{x\}$ whenever it contains x. The empty set will be denoted 0 here, and $x \cup \{x\}$, the **successor** of x, can be denoted x'. The triple $(\omega, ', 0)$ will turn out to be an example of a *structure*.

Throughout these notes, n will be a natural number, understood as the set $\{0, 1, 2, \dots, n-1\}$, possibly empty; and i will range over the elements of this set. Also m will be a natural number.

2 Cartesian powers

Let M be a set. The **Cartesian power** M^n is the set of functions from n to M. Such a function will be denoted by a boldface letter, as $\mathbf{a}$, but then its value $\mathbf{a}(i)$ at i will be denoted a_i. The function $\mathbf{a}$ can be identified with the n**-tuple** $(a_0, \dots, a_{n-1})$ of its values.

In particular, the power M^0 has but a single member, 0; hence $M^0 = 1$. This is so, even if $M = 0$; however, $0^n = 0$ when n is *positive* (different from 0). The set M itself can be identified with the power M^1.

Any function $f : m \to n$ determines the map

$$\mathbf{a} \mapsto (a_{f(0)}, \dots, a_{f(m-1)}) : M^n \to M^m,$$

no matter what set M is. In case $m = 1$, we have the **coordinate projections** $\mathbf{a} \mapsto a_i$.

The **Cartesian product** $A \times B$ of sets A and B is identified with the set of (ordered) pairs (a, b) such that $a \in A$ and $b \in B$. There is a map

$$\begin{aligned} M^n \times M^m &\longrightarrow M^{n+m} \\ (\mathbf{a}, \mathbf{b}) &\longmapsto (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1}), \end{aligned}$$

often considered an identification.

3 Structures and signatures

A **function on** the set M is a map $M^n \to M$; the function then is n**-ary**—its **arity** is n. A nullary (that is, 0-ary) function is a **constant** and can be identified with an element of M.

An n**-ary relation on** M is a subset of M^n. There are two nullary relations, namely 0 and 1. The relation of *equality* is binary (2-ary).

A **structure** is a set equipped with some distinguished constants and with some functions and relations of various positive arities. The set then is the **universe** of the structure. If the universe is M, then the structure might be denoted $\mathcal{M}$ or just M again. However, the structure $(\omega, ', 0)$ is denoted $\mathbf{N}$. (This structure is often considered to contain the binary functions of addition and multiplication as well, but these are uniquely determined by the successor-function.)

Examples. A set with no distinguished relations, functions or constants is trivially a structure. Groups, rings and partially ordered sets are structures. A vector space is a structure whose unary functions are the multiplications by the scalars. A valued field can be understood as a structure when the valuation ring is distinguished as a unary relation.

The **signature** of a structure contains a **symbol** for each function, relation and constant in the structure; the function, relation or constant is then the **interpretation** of the symbol. Notationally, the symbols are primary; their interpretations can be distinguished, if need be, by superscripts indicating the structure.

Examples. The complete ordered field $\mathbf{R}$ has the signature $\{+, -, \cdot, \leqslant, 0, 1\}$. The ordered field $\mathbf{Q}$ of rational numbers has the same signature. The binary function-symbol $+$ is interpreted in $\mathbf{R}$ by addition of real numbers; the interpretation is also denoted by $+$, or by $+^{\mathbf{R}}$ if it should be distinguished from addition $+^{\mathbf{Q}}$ of rational numbers. To make its signature explicit, we can write $\mathbf{R}$ as the tuple $(\mathbf{R}, +, -, \cdot, \leqslant, 0, 1)$.

Throughout these notes, $\mathcal{L}$ will be a signature, and f, R and c will range respectively over the function-, relation- and constant-symbols in $\mathcal{L}$. The structures with signature $\mathcal{L}$ compose the class $\mathfrak{Mod}(\mathcal{L})$.

4 Homomorphisms and embeddings

Suppose M and N are in $\mathfrak{Mod}(\mathcal{L})$, and h is a map $M \to N$. (So, N must be nonempty, unless M is empty.) Then h induces maps $M^n \to N^n$ in the obvious way, even when $n = 0$; so, $h(\mathbf{a})(i) = h(a_i)$, and $h(0) = 0$. The map

h is a **homomorphism** if it *preserves* the functions, relations and constants symbolized in $\mathcal{L}$, that is,

- $h(f^M(\mathbf{a})) = f^N(h(\mathbf{a}))$;
- $h(\mathbf{a}) \in R^N$ when $\mathbf{a} \in R^M$;
- $h(c^M) = c^N$.

Any map preserves *equality.* A homomorphism is an **embedding** if it preserves both inequality and the complements of the relations symbolized in $\mathcal{L}$.

Examples. A group-homomorphism is a homomorphism of groups; a group-monomorphism is an embedding of groups. The zero-map on the ordered field $\mathbf{R}$ can be seen as a homomorphism, but not an embedding. (It would not even be a homomorphism if the signature of an ordered ring contained $<$ instead of $\leqslant$.)

5 Boolean algebras

An essential and notationally exceptional example is the *Boolean algebra* of subsets of a nonempty set Ω; this structure is the tuple

$$(\mathcal{P}(\Omega), \cap, \cup, {}^c, \emptyset, \Omega),$$

but we shall consider the signature of Boolean algebras to be the set

$$\{\wedge, \vee, \neg, 0, 1\}.$$

A **Boolean ring** is a (unital) ring in which every element is idempotent, that is, satisfies

$$x^2 = x.$$

In particular, in such a ring we have $(x + y)^2 = x + y$, whence

$$xy + yx = 0;$$

replacing y with x, we get $2x = 0$, so every element is its additive inverse; hence also $xy = yx$, so the ring is commutative. We have $x(1 + x) = 0$, so if x is a unit, then $1 + x = 0$, so $x = 1$. Thus also every nonzero nonunit of a Boolean ring is a zero-divisor. Hence the only Boolean integral domain is the two-element ring $\{0, 1\}$ or $\mathbf{F}_2$, and this is a field. Therefore prime ideals of Boolean rings are maximal, since the quotient of a Boolean ring by an ideal is Boolean.

A structure $(M, \wedge, \vee, \neg, 0, 1)$—call it $\mathcal{M}^{\mathrm{a}}$—in the signature of Boolean algebras determines a structure $\mathcal{M}^{\mathrm{r}}$ with the same universe in the signature of rings: This structure $\mathcal{M}^{\mathrm{r}}$—that is, $(M, +, \cdot, 0, 1)$—is given by the rules

$$\begin{aligned} x + y &= (x \wedge \neg y) \vee (y \wedge \neg x), \\ xy &= x \wedge y \end{aligned}$$

and the rule that 0 and 1 have the same interpretation in each structure. The structure $\mathcal{M}^{\mathrm{a}}$ is a **Boolean algebra** just in case $\mathcal{M}^{\mathrm{r}}$ is a Boolean ring. Any Boolean ring $(M, +, \cdot, 0, 1)$ is determined in this way by the Boolean algebra $(M, \wedge, \vee, \neg, 0, 1)$ such that

$$\begin{aligned} x \wedge y &= xy, \\ x \vee y &= x + y + xy, \\ \neg x &= 1 + x. \end{aligned}$$

A Boolean algebra has a partial order $\leqslant$ such that

$$x \leqslant y \iff x \wedge y = x.$$

An **ideal** of a Boolean algebra is just an ideal of the corresponding ring. A **filter** of a Boolean algebra is *dual* to an ideal, so F is a filter just in case $\{\neg x : x \in F\}$ is an ideal. An **ultrafilter** is dual to a maximal ideal. So, F is a filter just in case

$$\begin{aligned} &1 \in F, \\ &x, y \in F \implies x \wedge y \in F, \\ &x \in F \text{ and } x \leqslant y \implies y \in F, \\ &0 \notin F; \end{aligned}$$

also, a filter F is an ultrafilter just in case

$$x \vee y \in F \implies x \in F \text{ or } y \in F,$$

equivalently, $x \notin F \implies \neg x \in F$.

The set of ultrafilters of a Boolean algebra is the **Stone-space** of the algebra. For every element x of a Boolean algebra, the corresponding Stone-space has a subset $[x]$ comprising the ultrafilters containing x. Then

$$[x] \cap [y] = [x \wedge y]$$

since the elements of these sets are filters; since they are ultrafilters, we have also

$$[x] \cup [y] = [x \vee y],$$
$$[x]^c = [\neg x].$$

Finally, [1] is the whole Stone-space, and [0] is empty. Therefore the map

$$x \longmapsto [x]$$

is a homomorphism of Boolean algebras; it is an embedding, since $[x]$ is nonempty when $x \neq 0$.

Since the collection of sets $[x]$ contains the whole Stone-space and the empty set and is closed under finite unions, it is a **basis** for the **closed** sets of a **topology** for the Stone-space. By definition then, every closed subset is an intersection of some closed sets $[x]$. These basic closed sets are also open—they are **clopen**. The topology is **Hausdorff**, since distinct points are respectively contained in some disjoint sets $[x]$ and $[\neg x]$.

Suppose B is a subset of a Boolean algebra. Then the following are equivalent:

- the collection $\{[x] : x \in B\}$ has the **finite-intersection property**, meaning any finite sub-collection has nonempty intersection;
- the set B generates a filter of the algebra;
- B included in an ultrafilter of this algebra;
- $\{[x] : x \in B\}$ has nonempty intersection.

That the first condition implies the last means that the topology of the Stone-space is **compact**. Consequently, every clopen set is one of the sets $[x]$.

Of the nonempty set Ω, we can see the Boolean ring $\mathcal{P}(\Omega)$ of its subsets as a compact **topological ring**. For, we can identify any subset A of Ω with its **characteristic function**, the map from Ω to $\mathbf{F}_2$ taking x to 1 just in case $x \in A$. The set of such maps can be denoted $\mathbf{F}_2^{\Omega}$. With the **discrete** topology, in which every subset is closed, $\mathbf{F}_2$ is a compact topological ring. Therefore on $\mathbf{F}_2^{\Omega}$ is induced a ring-structure and a compatible topology—the **product**-topology or topology of **pointwise convergence**, compact in this case since $\mathbf{F}_2$ is compact. The induced ring-structure makes the bijection from $\mathcal{P}(\Omega)$ to $\mathbf{F}_2^{\Omega}$ a homomorphism. In the induced topology, every finite subset of Ω determines for the zero-map on Ω an open neighborhood, comprising those maps into $\mathbf{F}_2$ that are zero on that finite subset. Translating such a neighborhood by an element of $\mathbf{F}_2^\Omega$ gives an open neighborhood of that element, and every open subset of $\mathbf{F}_2^\Omega$ is a union of such neighborhoods; the finite unions are precisely the clopen subsets.

6 Functions and terms

Suppose M is in $\mathfrak{Mod}(\mathcal{L})$. Various functions on M can be derived, by composition, from:

- the functions f^M,
- the constants c^M, and
- the coordinate projections.

These compositions can be described without reference to M; the result is the **terms** of $\mathcal{L}$.

The **interpretation** in M, or t^M, of an n-ary term t of $\mathcal{L}$ will be an n-ary function on M. Terms can be defined precisely as follows:

- Each constant-symbol c is also an n-ary term whose interpretation is the constant map $\mathbf{a} \mapsto c^M$ on M^n.
- There is an n-ary term x_i whose interpretation is the coordinate projection $\mathbf{a} \mapsto a_i$ on M^n.
- If t_0, …, t_{n-1} are m-ary terms, and f is n-ary, then there is an m-ary term $f(t_0, \dots, t_{n-1})$ whose interpretation is the map
$$\mathbf{a} \mapsto f^M(t_0^M(\mathbf{a}), \dots, t_{n-1}^M(\mathbf{a})).$$

By this account, an n-ary term is also $n+1$-ary. The nullary terms are the **constant** terms.

Lemma. *If t is an n-ary term, and u_0, …, u_{n-1} are m-ary terms, then there is an m-ary term whose interpretation in M is the map*
$$\mathbf{a} \mapsto t^M(u_0^M(\mathbf{a}), \dots, u_{n-1}^M(\mathbf{a})).$$

The new term in the lemma can of course be denoted $t(u_0, \dots, u_{n-1})$.

We can identify terms whose interpretations are indistinguishable in every structure. In particular, if t is n-ary, but not $(n-1)$-ary, then t is precisely $t(x_0, \dots, x_{n-1})$, which we may abbreviate as $t(\mathbf{x})$.

If A is a subset of M, we let $\mathcal{L}(A)$ be the signature $\mathcal{L}$ augmented with a constant-symbol for each element of A. The symbols and the elements are generally not distinguished notationally, and an $\mathcal{L}$-structure M naturally determines an $\mathcal{L}(A)$-structure, denoted M_A if there is a need to distinguish.

Lemma. *Every term of $\mathcal{L}(A)$ is $t(\mathbf{a},\mathbf{x})$ for some term t of $\mathcal{L}$ and tuple $\mathbf{a}$ from A.*

7 Propositional logic

The terms in the signature of Boolean algebras—the **Boolean terms**—can be considered as strings of symbols generated by the following rules:

- each constant-symbol 0 or 1 is a term;
- each symbol x_i for a coordinate projection is a term;
- if t and u are terms, then so are $(t \wedge u)$ and $(t \vee u)$ and $\neg t$.

A term here is n-ary just in case $i < n$ whenever x_i appears in the term. Instead of $(\dots(t_0 * t_1) * \dots * t_{n-1})$ we can write

$$t_0 * \dots * t_{n-1},$$

where $*$ is $\wedge$ or $\vee$.

Lemma. *Every n-ary function on $\mathbf{F}_2$ is the interpretation of an n-ary Boolean term.*

Proof. Suppose f be an n-ary function on $\mathbf{F}_2$, and let $\mathbf{a}^0$, ..., $\mathbf{a}^{m-1}$ be the elements of $\mathbf{F}_2^n$ at which f is 1. If $m = 0$, then f is the interpretation of 0. If $m > 0$, then f is the interpretation of

$$t^0 \vee \dots \vee t^{m-1},$$

where t^j is $u_0^j \wedge \dots u_{n-1}^j$, where u_i^j is x_i, if $a_i^j = 1$, and otherwise is $\neg x_i$. □

The Boolean terms can be considered as the *propositional formulas* composing a *propositional logic*—call it PL. The constant-symbols 0 and 1 can then be taken to stand for **false** and **true** statements, respectively; an element of $\mathbf{F}_2^\omega$ is a **truth-assignment** to the **propositional variables** x_i, and under such an assignment σ, a propositional formula t takes on the **truth-value**

$$t^{\mathbf{F}_2}(\sigma(0), \dots, \sigma(n-1))$$

if t is n-ary. Write $\langle\sigma, t\rangle$ for the truth-value of t under σ. A *model* for a set of propositional formulas is a truth-assignment σ sending the set to 1 under the map $t \mapsto \langle\sigma, t\rangle$.

Theorem (Compactness for sentential logic). *A set of propositional formulas has a model if each finite subset does.*

Proof. If a set of sentences t satisfies the hypothesis, then the collection of closed subsets $\{\sigma : \langle\sigma, t\rangle = 1\}$ of $\mathbf{F}_2^\omega$ has the finite-intersection property. □

The sets $\{\sigma : \langle\sigma, t\rangle = 1\}$ are precisely the clopen subsets of $\mathbf{F}_2^\omega$.

8 Relations and formulas

From the relations R^M and the interpretations t^M of terms t, new relations on M can be derived by various techniques. These relations will be the **0-definable** relations of M, and each of them will be the interpretation of a **formula** of $\mathcal{L}$. (The **definable** relations of M are the interpretations of formulas of $\mathcal{L}(M)$.) Distinctions are made according to which techniques are needed to derive the relations.

The **atomic** formulas are given thus:

- If t_0 and t_1 are n-ary terms, then there is an n-ary atomic formula $t_0 = t_1$ whose interpretation $(t_0 = t_1)^M$ is $\{\mathbf{a} \in M^n : t_0^M(\mathbf{a}) = t_1^M(\mathbf{a})\}$.
- If $t_0, \dots, t_{n-1}$ are m-ary terms, and R is n-ary, then there is an m-ary atomic formula $R(t_0, \dots, t_{n-1})$ whose interpretation $R(t_0, \dots, t_{n-1})^M$ is $\{\mathbf{a} \in M^m : (t_0^M(\mathbf{a}), \dots, t_{n-1}^M(\mathbf{a})) \in R^M\}$.

(In particular, $R(x_0, \dots, x_{n-1})^M = R^M$.)

If t is an m-ary Boolean term, and $\phi_0, \dots, \phi_{n-1}$ are n-ary atomic formulas, then there is an n-ary **basic** or **quantifier-free** formula, say $t(\phi_0, \dots, \phi_{n-1})$, whose interpretation is

$$t^{\mathcal{P}(M^n)}(\phi_0^M, \dots, \phi_{n-1}^M).$$

If we identify formulas with indistinguishable interpretations in every structure, then the set of basic formulas is a Boolean algebra generated by the atomic formulas. The set of **formulas** is the smallest Boolean algebra containing the atomic formulas and closed under the operation converting an $n+1$-ary formula ϕ into an n-ary formula $\exists x_n\, \phi$ whose interpretation is the image of ϕ^M under the map

$$(a_0, \dots, a_n) \mapsto (a_0, \dots, a_{n-1}) : M^{n+1} \to M^n.$$

The Boolean algebra of n-ary formulas of $\mathcal{L}$ can be denoted $\mathrm{Fm}^n(\mathcal{L})$.

The formula $\neg\exists x_n\,\phi$ is also denoted $\forall x_n\,\neg\phi$, and $\neg\phi\vee\psi$ is denoted $\phi\to\psi$. If ϕ is an n-ary formula, and t_0, …, t_{n-1} are m-ary terms, then there is an m-ary formula $\phi(t_0,\dots,t_{n-1})$ with the obvious interpretation; in particular, if it is not also $(n-1)$-ary, then ϕ is the same as the formula $\phi(x_0,\dots,x_{n-1})$.

The A-definable relations of M are the interpretations in M of formulas of $\mathcal{L}(A)$. In particular, they are the sets $\phi(a_0,\dots,a_{m-1},x_0,\dots,x_{n-1})^M$, where ϕ is an $m+n$-ary formula of $\mathcal{L}$, and $\mathbf{a}$ is a tuple from A.

Sentences are 0-ary formulas.

9 Substructures

Suppose M and N are members of $\mathfrak{Mod}(\mathcal{L})$. We can now say that an embedding of M in N is a map $h: M\to N$ such that

$$h^{-1}(\phi^N)=\phi^M$$

for all basic formulas ϕ of $\mathcal{L}$; if the same holds for *all* formulas ϕ of $\mathcal{L}$, then h is an **elementary embedding**. If the universe of N includes the universe of M, and the inclusion-map is an embedding, we say M is a **substructure** of N and write

$$M\subseteq N;$$

if the inclusion-map is an elementary embedding, we write

$$M\preccurlyeq N$$

and say M is an **elementary** substructure of N.

Lemma (Tarski–Vaught). *Suppose $M\subseteq N$. Then $M\preccurlyeq N$, provided that*

$$\phi(\mathbf{a},x_0)^N\cap M$$

is nonempty whenever $\phi(\mathbf{a},x_0)^N$ is, for all $\mathcal{L}$-formulas ϕ and all tuples $\mathbf{a}$ from M.

Proof. Let Σ comprise the formulas ϕ such that

$$\phi(x_0,\dots,x_{n-1})^M=\phi(x_0,\dots,x_{n-1})^N\cap M^n. \qquad (*)$$

Then Σ contains all the basic formulas and is closed under the Boolean operations. Suppose ϕ is in Σ and $\mathbf{a}$ is in M^n. Then

$$\phi(\mathbf{a},x_0)^M=\phi(\mathbf{a},x_0)^N\cap M.$$

By hypothesis then, $\phi(\mathbf{a},x_0)^M$ and $\phi(\mathbf{a},x_0)^N$ are alike empty or not. Hence $(*)$ holds, *mutatis mutandis*, with $\exists x_{n-1}\phi$ in place of ϕ. Therefore $\Sigma=\mathrm{Fm}(\mathcal{L})$. □

10 Models and theories

Suppose ϕ is an n-ary formula of $\mathcal{L}$, and $\mathbf{a}$ is an n-tuple of elements of M, so that $\phi(\mathbf{a})$ is a sentence of $\mathcal{L}(M)$. We write

$$M \models \phi(\mathbf{a})$$

if $\phi(\mathbf{a})^M = 1$, equivalently, $\mathbf{a} \in \phi^M$. The map $h : M \to N$ is an elementary embedding just in case

$$M \models \phi(\mathbf{a}) \iff N \models \phi(h(\mathbf{a}))$$

for all such ϕ and $\mathbf{a}$.

If $\mathcal{K}$ is a subclass of $\mathfrak{Mod}(\mathcal{L})$, then the **theory** $\mathrm{Th}(\mathcal{K})$ of $\mathcal{K}$ is the subset of $\mathrm{Fm}^0(\mathcal{L})$ comprising σ such that $M \models \sigma$ whenever $M \in \mathcal{K}$; this subset is a filter, if $\mathcal{K}$ is nonempty; otherwise it contains every sentence. In general, a **theory** of $\mathcal{L}$ is $\mathrm{Fm}^0(\mathcal{L})$ or a filter of it; a **consistent** theory is a proper filter; a **complete** theory is an ultrafilter. A **model** of a set Σ of sentences is a structure M such that $\Sigma \subseteq \mathrm{Th}(M)$. We write

$$\Sigma \models \sigma$$

if every model of Σ is a model of σ (that is, of $\{\sigma\}$). We write

$$\Sigma \vdash \sigma$$

if σ is in the theory generated by Σ. If $\Sigma \vdash \sigma$, then $\Sigma \models \sigma$.

11 Compactness

It is a consequence of the following that $\Sigma \vdash \sigma$ if $\Sigma \models \sigma$.

Theorem (Compactness). *Every consistent theory has a model.*

Proof. Let T be a consistent theory in the signature $\mathcal{L}$. We shall extend $\mathcal{L}$ to a signature $\mathcal{L}'$, and extend T to a complete theory T' of $\mathcal{L}'$. We shall do this in such a way that, for every unary formula ϕ of $\mathcal{L}'$, there will be a constant-symbol c_ϕ not appearing in ϕ such that

$$T' \vdash \exists x_0\, \phi \to \phi(c_\phi).$$

Then T' and the constant-symbols c_ϕ will determine a structure M in the following way. The universe of M will consist of equivalence-classes $[c_\phi]$ of the symbols c_ϕ, where

$$[c_\phi] = [c_\psi] \iff T' \vdash c_\phi = c_\psi.$$

Then we require

$$\phi^M = \{[\mathbf{c}] : T' \vdash \phi(\mathbf{c})\} \tag{$*$}$$

for all basic formulas ϕ of $\mathcal{L}'$ and all tuples $\mathbf{c}$ of symbols c_ϕ. The requirements $(*)$ do make sense. In particular, $c_\phi^M = [c_\phi]$. The requirements determine a well-defined structure, since T' is complete.

If T' is as claimed, then $(*)$ holds for all formulas ϕ; we show this by induction. If ϕ is an n-ary formula, and $[\mathbf{c}]$ is an $(n-1)$-tuple from M, let d be the constant-symbol determined by the unary formula $\phi(\mathbf{c}, x_0)$. If $(*)$ holds for ϕ, then we have:

$$\begin{aligned}
[\mathbf{c}] \in (\exists x_n\,\phi)^M &\implies M \models \phi(\mathbf{c}, [e]),\ \text{some } [e] \text{ in } M \\
&\implies T' \vdash \phi(\mathbf{c}, e) \\
&\implies T' \vdash \exists x_0\,\phi(\mathbf{c}, x_0) \\
&\implies T' \vdash \phi(\mathbf{c}, d) \\
&\implies M \models (\mathbf{c}, [d]) \\
&\implies [\mathbf{c}] \in (\exists x_n\,\phi)^M;
\end{aligned}$$

so $(*)$ holds with $\exists x_n\,\phi$ in place of ϕ.

Once $(*)$ holds for all formulas ϕ, then in particular it holds when ϕ is a sentence in T; so $M \models T$.

It remains to find T' as desired. First we construct a chain $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots$ of signatures, where $\mathcal{L}_{n+1} - \mathcal{L}_n$ consists of a constant-symbol c_ϕ for each unary formula ϕ in $\mathcal{L}_n$. Taking the union of the chain gives $\mathcal{L}'$.

Now we work in the Stone space of $\mathrm{Fm}^0(\mathcal{L}')$. We claim that the collection

$$\{[\sigma] : \sigma \in T\} \cup \{[\forall x_0\,\neg\phi \vee \phi(c_\phi)] : \phi \in \mathrm{Fm}^1(\mathcal{L}')\}$$

of closed sets has the finite-intersection property; from this, by compactness, we can take T' to be an element of the intersection.

To establish the f.i.p., suppose that $[\psi]$ is a nonempty finite intersection of sets in the collection. Then $\psi \in \mathrm{Fm}^0(\mathcal{L}_n)$ for some n. If $\phi \in \mathrm{Fm}^1(\mathcal{L}') - \mathrm{Fm}^1(\mathcal{L}_{n-1})$, then c_ϕ does not appear in ψ. If also $[\psi] \cap [\forall x_0\,\neg\phi]$ is empty, then

$$[\psi] \cap [\phi(c_\phi)]$$

is nonempty; for, if $M \models \psi \wedge \exists x_0\,\phi$, then we may assume $M \models \psi \wedge \phi(c_\phi)$. □

Theorem. *Suppose $N \in \mathfrak{Mod}(\mathcal{L})$, and κ is a cardinal such that*

$$\aleph_0 + |\mathcal{L}| \leqslant \kappa \leqslant |N|.$$

Then there is M in $\mathfrak{Mod}(\mathcal{L})$ such that $M \preccurlyeq N$ and $|M| = \kappa$.

Proof. Use the proof of Compactness, with $\mathrm{Th}(N)$ for T. We can choose T', and we can choose c_ϕ^N in N, so that $N \models T'$. Then we may assume $M \subseteq N$, and so $M \preccurlyeq N$ by the Tarski–Vaught test. By construction, $|M| \leqslant |\mathcal{L}'| = \aleph_0 + |\mathcal{L}|$.

To ensure $M = \kappa$, we first add κ-many new constant-symbols to $\mathcal{L}$ and let their interpretations in N be distinct. □

Example. In the signature $\{\in\}$ of set-theory, any infinite structure has a countably infinite elementary substructure, even though the power-set of an infinite set is uncountable.

Corollary. *Suppose A is an infinite $\mathcal{L}$-structure and $|A| + |\mathcal{L}| \leqslant \kappa$. Then there is M in $\mathfrak{Mod}(\mathcal{L})$ such that $A \preccurlyeq M$ and $|M| = \kappa$.*

Proof. Let $\{c_\mu : \mu < \kappa\}$ be a set of new constant-symbols, and let T be the theory generated by $\mathrm{Th}(A_A)$ and $\{c_\mu \neq c_\nu : \mu \neq \nu\}$. Use Compactness to get a model N of T; then use the last Theorem to get M as desired. □

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