Some homework for Math 736, Model-Theory, given September 28, 2001.

Let $\mathcal{L}$ be a signature, and let $\mathcal{M}$ be a structure in $\mathfrak{Mod}(\mathcal{L})$.

Problem 1. We didn't actually *define* terms of $\mathcal{L}$; we simply asserted that terms t exist whose interpretations $t^{\mathcal{M}}$ have certain properties. Prove this assertion.

Problem 2. Prove the lemma that, if t is an n-ary term of $\mathcal{L}$, and u_0, …, u_{n-1} are m-ary terms of $\mathcal{L}$, then there is an m-ary term of $\mathcal{L}$ whose interpretation in $\mathcal{M}$ is the map

$$\mathbf{a} \mapsto t^{\mathcal{M}}(u_0^{\mathcal{M}}(\mathbf{a}), \dots, u_{n-1}^{\mathcal{M}}(\mathbf{a})) : M^m \to M.$$

Problem 3. Suppose that a structure in the signature $\{\wedge, \vee, \neg, 0, 1\}$ can be expanded to a signature containing $+$ in such a way that the identities $x \vee y = x + y + (x \wedge y)$ and $\neg x = x + 1$ are satisfied; suppose further that this expansion, reduced to the signature $\{+, \wedge, 0, 1\}$, is a Boolean ring. Then the original structure is, by definition, a Boolean algebra.

1. Prove that the Boolean algebras are precisely those structures $(B, \wedge, \vee, \neg, 0, 1)$ whose reducts $(B, \wedge, 1)$ and $(B, \vee, 0)$ are monoids, and that satisfy the equations $\neg\neg x = x$, and $\neg(x \wedge y) = \neg x \wedge \neg y$, and also

$$\begin{aligned} x \wedge y &= y \wedge x, & x \vee y &= y \vee x, \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z), & x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z), \\ x \wedge \neg x &= 0 & x \vee \neg x &= 1 \\ x \wedge 0 &= 0 & x \vee 1 &= 1. \end{aligned}$$

2. Find a structure $(C, +, \wedge, \curlyvee, \neg, 0, 1)$ that satisfies

$$x + y = (x \wedge \neg y) \curlyvee (y \wedge \neg x),$$

and whose reduct $(C, +, \wedge, 0, 1)$ is a Boolean ring, but whose reduct $(C, \wedge, \curlyvee, \neg, 0, 1)$ is *not* a Boolean algebra.

Problem 4. Prove that the map $x \mapsto [x]$ from a Boolean algebra to the power-set of its Stone-space is an embedding of Boolean algebras.