Math 736, Model-Theory, 2001, fall. Here are some additional notes on terms and formulas; Problems 5 and 6 constitute Homework II. Revised, 25 October 2001.

Fix a signature $\mathcal{L}$. Let c, f and R range respectively over the constant-, function- and relation-symbols of $\mathcal{L}$; let $\mathcal{M}$ range over $\mathfrak{Mod}(\mathcal{L})$; let i, k, m, and n range over ω.

For each n, we want to define a set $\mathrm{Tm}^n(\mathcal{L})$, of **$n$-ary terms of $\mathcal{L}$**. Each t in $\mathrm{Tm}^n(\mathcal{L})$ should have, for each $\mathcal{M}$, an interpretation $t^{\mathcal{M}}$, which is an n-ary function on M. We want the terms and their interpretations to satisfy the following requirements.

0. For each c, there is t in $\mathrm{Tm}^0(\mathcal{L})$ such that $t^{\mathcal{M}}$ is $c^{\mathcal{M}}$ for each $\mathcal{M}$.
1. If f is n-ary, then there is t in $\mathrm{Tm}^n(\mathcal{L})$ such that $t^{\mathcal{M}}$ is $f^{\mathcal{M}}$ for each $\mathcal{M}$.
2. There is t in $\mathrm{Tm}^1(\mathcal{L})$ such that $t^{\mathcal{M}}$ is id_M for each $\mathcal{M}$.
3. For every function $\sigma : m \to n$, and for every u in $\mathrm{Tm}^m(\mathcal{L})$, there is t in $\mathrm{Tm}^n(\mathcal{L})$ such that $t^{\mathcal{M}}$ is
$$\mathbf{a} \mapsto u^{\mathcal{M}}(a_{\sigma(0)}, \dots, a_{\sigma(m-1)}) : M^n \to M$$
for each $\mathcal{M}$.
4. For each u in $\mathrm{Tm}^m(\mathcal{L})$, and for any $t_0, \dots, t_{m-1}$ in $\mathrm{Tm}^n(\mathcal{L})$, there is t in $\mathrm{Tm}^n(\mathcal{L})$ such that $t^{\mathcal{M}}$ is $u^{\mathcal{M}} \circ (t_0^{\mathcal{M}}, \dots, t_{m-1}^{\mathcal{M}})$ for each $\mathcal{M}$.
5. No terms t exist whose interpretations $t^{\mathcal{M}}$ are not required by the preceding clauses.

Then the sets $\mathrm{Tm}^n(\mathcal{L})$ of n-ary terms t, and their interpretations $t^{\mathcal{M}}$, can be defined as follows.

(a) $\mathrm{Tm}^0(\mathcal{L})$ contains the symbols c (each a string of length 1).

(b) $\mathrm{Tm}^{i+1}(\mathcal{L})$ contains the symbol x_i (a string of length 1).

(c) $\mathrm{Tm}^{n+1}(\mathcal{L})$ includes $\mathrm{Tm}^n(\mathcal{L})$.

(d) If f is m-ary, and $u_0, \dots, u_{m-1}$ are in $\mathrm{Tm}^n(\mathcal{L})$, then $\mathrm{Tm}^n(\mathcal{L})$ contains $fu_0 \cdots u_{m-1}$ (the concatenation of the strings f, u_0, $\dots$, u_{m-1}).

(e) $\mathrm{Tm}^n(\mathcal{L})$ contains no other strings than those required by the preceding clauses, and if $t \in \mathrm{Tm}^n(\mathcal{L})$, then for every $\mathcal{M}$, the interpretation $t^{\mathcal{M}}$ is:

- $\mathbf{a} \mapsto c^{\mathcal{M}}$, if t is c;
- $\mathbf{a} \mapsto a_i$, if t is x_i;
- $f^{\mathcal{M}} \circ (u_0^{\mathcal{M}}, \dots, u_{m-1}^{\mathcal{M}})$, if t is $fu_0 \cdots u_{m-1}$ (where f is m-ary and the u_i are in $\mathrm{Tm}^n(\mathcal{L})$).

The definition of the interpretations of terms depends on how terms can be analyzed; so the validity of the definition must be checked. To do this, one can use the following.

Lemma. *A proper initial segment of a term is not a term; that is, if a string $\alpha_0\alpha_1\cdots\alpha_n$ of symbols α_i is a term, and $m < n$, then $\alpha_0\alpha_1\ldots\alpha_m$ is not a term.*

Proof. The claim is trivially true for terms of length 1. Suppose it is false for a term t of length $k+1$. Then t is $ft_0\cdots t_{n-1}$ for some terms t_i, but t has a proper initial segment of the form $fu_0\cdots u_{m-1}$, where the u_i are terms. Then there is some least i such that t_i is not u_i; but then also one of these is an initial segment of the other. Thus the claim fails for a term of length k or less—if it fails for a term of length $k+1$. By induction, the claim holds for terms of all lengths. □

Lemma (unique readability of terms). *Every term is uniquely of the form c, x_i or $ft_0\cdots t_{n-1}$, where the t_i are terms.*

Proof. If the analysis of a term as $ft_0\cdots t_{n-1}$ is not unique, then (as in the proof of the previous lemma) one of the t_i can be assumed to be a proper initial segment of another term. □

Finally, by induction on the length of terms, every n-ary term is also $n+1$-ary and has an interpretation as such. So terms and their interpretations are well-defined. Now we can check that the several numbered requirements of terms are met:

0. Let t be c.
1. Let t be $fx_0\cdots x_{n-1}$.
2. Let t be x_0.
3. The required term t can be denoted $u(x_{\sigma(0)},\ldots,x_{\sigma(m-1)})$, and can be defined inductively:
   - If u is c, then t is c.
   - If u is x_i, then t is $x_{\sigma(i)}$.
   - If u is $fu_0\cdots u_{k-1}$, then t is $ft_0\cdots t_{k-1}$, where t_i is $u_i(x_{\sigma(0)},\ldots,x_{\sigma(m-1)})$.
4. The required term t can be denoted $u(t_0,\ldots,t_{m-1})$, and can be defined inductively:
   - If u is c, then t is c.
   - If u is x_i, then t is t_i.
   - If u is $fu_0\cdots u_{k-1}$, then t is $fv_0\cdots v_{k-1}$, where v_i is $u_i(t_0,\ldots,t_{m-1})$.
5. Every interpretation $t^{\mathcal{M}}$ satisfies one of the requirements:

(a) The nullary term c is a term t such that $t^{\mathcal{M}} = c^{\mathcal{M}}$.

(b) Let u be the unary term x_0 (whose interpretation in $\mathcal{M}$, or id_M, is required); let σ be the map from 1 to $i+1$ such that $\sigma(0) = i$; then x_i is an $i+1$-ary term t such that $t^{\mathcal{M}}$ is $\mathbf{a} \mapsto u^{\mathcal{M}}(a_{\sigma(0)})$.

(c) if an n-ary term t has a required interpretation, then the interpretation of t as an $n+1$-ary term is also required, since this interpretation is $\mathbf{a} \mapsto t^{\mathcal{M}}(a_{\sigma(0)}, \dots, a_{\sigma(n-1)})$, where σ is the inclusion of n in $n+1$.

(d) Let u be $fx_0 \cdots x_{m-1}$; then its interpretation in $\mathcal{M}$, namely $f^{\mathcal{M}}$, is required. Suppose the interpretations of the terms t_i are required; then so is the interpretation of $ft_0 \cdots t_{m-1}$, since this interpretation is $u^{\mathcal{M}} \circ (t_0^{\mathcal{M}}, \dots, t_{m-1}^{\mathcal{M}})$.

Now we can move on to *formulas*. For each n, we want to define a set $\mathrm{Fm}^n(\mathcal{L})$, comprising the **$n$-ary formulas of $\mathcal{L}$**. Each ϕ in $\mathrm{Fm}^n(\mathcal{L})$ should have, for each $\mathcal{M}$, an interpretation $\phi^{\mathcal{M}}$, which is an n-ary relation on M. We want the formulas and their interpretations to satisfy the following requirements.

0. There is ϕ in $\mathrm{Fm}^2(\mathcal{L})$ such that $\phi^{\mathcal{M}}$ is $\{(a,b) \in M^2 : a = b\}$ for each $\mathcal{M}$.

1. If R is n-ary, then there is ϕ in $\mathrm{Fm}^n(\mathcal{L})$ such that $\phi^{\mathcal{M}}$ is $R^{\mathcal{M}}$ for each $\mathcal{M}$.

2. For any m-ary term F of the signature of Boolean algebras, and for any $\psi_0, \dots, \psi_{m-1}$ in $\mathrm{Fm}^n(\mathcal{L})$, there is ϕ in $\mathrm{Fm}^n(\mathcal{L})$ such that $\phi^{\mathcal{M}}$ is $F^{\mathcal{P}(M^n)}(\psi_0^{\mathcal{M}}, \dots, \psi_{m-1}^{\mathcal{M}})$ for each $\mathcal{M}$.

3. For any $t_0, \dots, t_{m-1}$ in $\mathrm{Tm}^n(\mathcal{L})$, and for any ψ in $\mathrm{Fm}^m(\mathcal{L})$, there is ϕ in $\mathrm{Fm}^n(\mathcal{L})$ such that $\phi^{\mathcal{M}}$ is
$$\{\mathbf{a} \in M^n : (t_0^{\mathcal{M}}(\mathbf{a}), \dots, t_{m-1}^{\mathcal{M}}(\mathbf{a})) \in \psi^{\mathcal{M}}\}$$
for each $\mathcal{M}$.

4. For any $u_0, \dots, u_{n-1}$ in $\mathrm{Tm}^m(\mathcal{L})$, and for any ψ in $\mathrm{Fm}^m(\mathcal{L})$, there is ϕ in $\mathrm{Fm}^n(\mathcal{L})$ such that $\phi^{\mathcal{M}}$ is
$$\{(u_0^{\mathcal{M}}(\mathbf{a}), \dots, u_{n-1}^{\mathcal{M}}(\mathbf{a})) \in M^n : \mathbf{a} \in \psi^{\mathcal{M}}\}$$
for each $\mathcal{M}$.

5. No formulas ϕ exist whose interpretations $\phi^{\mathcal{M}}$ are not required by the preceding clauses.

To meet these requirements, we propose to define the sets $\mathrm{Fm}^n(\mathcal{L})$ of n-ary formulas ϕ, and their interpretations $\phi^{\mathcal{M}}$, as follows.

(a) $\mathrm{Fm}^n(\mathcal{L})$ contains $(t = u)$ whenever t and u are in $\mathrm{Tm}^n(\mathcal{L})$.

(b) $\mathrm{Fm}^n(\mathcal{L})$ contains $Rt_0 \cdots t_{m-1}$ whenever R is m-ary and $t_0, \ldots, t_{m-1}$ are in $\mathrm{Tm}^n(\mathcal{L})$.

(c) $\mathrm{Fm}^0(\mathcal{L})$ contains $\bot$ and $\top$; and $\mathrm{Fm}^n(\mathcal{L})$ contains $\neg\psi$ and $(\psi \land \chi)$ and $(\psi \lor \chi)$ whenever $\psi, \chi \in \mathrm{Fm}^n(\mathcal{L})$. (The symbols $\bot$ and $\top$ and $\neg$ and $\land$ and $\lor$ can be supposed distinct from any symbols in $\mathcal{L}$.)

(d) $\mathrm{Fm}^n(\mathcal{L})$ contains $\exists x_n\, \psi$ and $\forall x_n\, \psi$ whenever $\psi \in \mathrm{Fm}^{n+1}(\mathcal{L})$.

(e) $\mathrm{Fm}^n(\mathcal{L})$ contains no other strings of symbols than those required by the preceding clauses, and if $\phi \in \mathrm{Fm}^n(\mathcal{L})$, then for every $\mathcal{M}$ the interpretation $\phi^{\mathcal{M}}$ is:

- $\{\mathbf{a} \in M^n : t^{\mathcal{M}}(\mathbf{a}) = u^{\mathcal{M}}(\mathbf{a})\}$, if ϕ is $(t = u)$;
- $\{\mathbf{a} \in M^n : (t_0^{\mathcal{M}}(\mathbf{a}), \ldots, t_{m-1}^{\mathcal{M}}(\mathbf{a})) \in R^{\mathcal{M}}\}$, if ϕ is $Rt_0 \cdots t_{m-1}$;
- $\emptyset$, if ϕ is $\bot$;
- $\emptyset^{\mathrm{c}}$, if ϕ is $\top$;
- $(\psi^{\mathcal{M}})^{\mathrm{c}}$, if ϕ is $\neg\psi$;
- $\psi^{\mathcal{M}} \cap \chi^{\mathcal{M}}$, if ϕ is $(\psi \land \chi)$;
- $(\neg(\neg\psi \land \neg\chi))^{\mathcal{M}}$, if ϕ is $(\psi \lor \chi)$;
- $\{\mathbf{a} \in M^n : (\mathbf{a}, b) \in \psi^{\mathcal{M}},\ \text{some } b \text{ in } M\}$, if ϕ is $\exists x_n\, \psi$;
- $(\neg\exists x_n\, \neg\psi)^{\mathcal{M}}$, if ϕ is $\forall x_n\, \psi$.

Problem 5. *Show that the proposed definition of $Fm^n(\mathcal{L})$ is valid and meets the requirements.*

Now let $\mathrm{Fm}_0^n(\mathcal{L})$ be the smallest subset of $\mathrm{Fm}^n(\mathcal{L})$ that contains the formulas $Rt_0 \cdots t_{m-1}$ and $(t = u)$ and that contains $\neg\psi$ and $(\psi \land \chi)$ and $(\psi \lor \chi)$ when it contains ψ and χ. Let $\mathrm{Fm}_{\mathrm{p}}^n(\mathcal{L})$ be the smallest subset of $\mathrm{Fm}^n(\mathcal{L})$ such that:

- $\mathrm{Fm}_0^n(\mathcal{L}) \subseteq \mathrm{Fm}_{\mathrm{p}}^n(\mathcal{L})$;
- $\mathrm{Fm}_{\mathrm{p}}^n(\mathcal{L})$ contains $\bot$ and $\top$;
- $\mathrm{Fm}_{\mathrm{p}}^n(\mathcal{L})$ contains $\exists x_n\, \psi$ and $\forall x_n\, \psi$ when $\psi \in \mathrm{Fm}_{\mathrm{p}}^{n+1}(\mathcal{L})$.

(The subscript p stands for *prenex*, which describes the elements of $\mathrm{Fm}_{\mathrm{p}}^n(\mathcal{L})$.) Say that n-ary formulas ϕ and ψ are **equivalent** if their interpretations in $\mathcal{M}$ are the same, for every $\mathcal{M}$.

Problem 6. *Show that for every formula in $Fm^n(\mathcal{L})$ there is an equivalent formula in $Fm_p^n(\mathcal{L})$.*