Homework III, Math 736, Model-Theory.

These notes present an alternative approach to compactness and completeness. The proofs of the lemmas are exercises (not necessarily to be turned in).

Let $\mathrm{Fm}^n(\mathcal{L})$ comprise the n-ary formulas of $\mathcal{L}$, defined as in the notes ('Homework II') as strings of symbols, but with formulas $(\phi \to \psi)$ allowed as well, with $(\phi \to \psi)^{\mathcal{M}} = (\neg\phi \vee \psi)^{\mathcal{M}}$. We may leave off outer brackets when writing formulas, and write $\phi_0 \wedge \phi_1 * \phi_2$ for $(\phi_0 \wedge \phi_1) * \phi_2$, where $*$ is $\wedge$ or $\to$. Let $\mathrm{Fm}(\mathcal{L})$ be the union of the chain $\mathrm{Fm}^0(\mathcal{L}) \subseteq \mathrm{Fm}^1(\mathcal{L}) \subseteq \dots$. We shall assume that all tuples, terms and formulas have the arities they must, for what we say to make sense.

Let Γ be a subset of $\mathrm{Fm}(\mathcal{L})$, and let ϕ be in $\mathrm{Fm}(\mathcal{L})$. We can define

$$\Gamma \models \phi$$

to mean that, for every $\mathcal{M}$ in $\mathfrak{Mod}(\mathcal{L})$, and for every sequence $(a_i : i \in \omega)$ of elements of M, if $\mathcal{M} \models \psi(\mathbf{a})$ for each ψ in Γ, then $\mathcal{M} \models \phi(\mathbf{a})$. This agrees with the definition given in class when all formulas are sentences. We can also say that $\mathcal{M}$ is a **model** of Γ, writing

$$\mathcal{M} \models \Gamma,$$

if there is *some* sequence $(a_i : i \in \omega)$ of elements of M such that $\mathcal{M} \models \psi(\mathbf{a})$ for each ψ in Γ.

We now define

$$\Gamma \vdash \phi$$

to mean that ϕ is **derivable** from Γ, in a sense to be specified presently. The point of these notes will be to prove that this definition agrees with the one given in class. That ϕ is derivable from Γ will mean that there is a **proof** of ϕ from Γ, namely a finite sequence of formulas, ending with ϕ, of which each formula:

- is in Γ,
- is an **axiom**, or
- follows from previous formulas in the sequence by a **rule of inference**.

Before naming the axioms and rule(s) of inference, we can already check the following.

Lemma 1. *If $\Gamma \vdash \phi$, then ϕ is derivable from a finite subset of Γ.*

Lemma 2. *Derivability is* **transitive**, *in the sense that, if each formula in a set Θ is derivable from Γ, and ϕ is derivable from Θ, then $\Gamma \vdash \phi$.*

We shall use a single rule of inference, namely ***Modus Ponens***:

$$\{\phi, (\phi \to \psi)\} \vdash \psi.$$

Our axioms will be the following (where, throughout, t_i, u_i, t, u and v are terms, and ϕ and ψ are formulas of appropriate arities):

- the **tautologies**, namely formulas $F(\phi_0, \dots, \phi_{n-1})$, where F is a *tautologous* n-ary propositional formula (an n-ary term of the language of Boolean algebras whose interpretation in $\mathbf{F}_2$ is the constant-function 1);
- the **axioms of equality**, namely:
  - $(t_0 = u_0) \wedge \dots \wedge (t_{n-1} = u_{n-1}) \to (\phi(\mathbf{t}) \to \phi(\mathbf{u}))$;
  - $t = t$;
  - $(t = u) \to (u = t)$;
  - $(t = u) \wedge (u = v) \to (t = v)$;
- the **axioms of quantification**, namely:
  - $\phi(\mathbf{x}, t) \to \exists x_n\, \phi$;
  - $\forall x_n\, \neg\phi \to \neg\exists x_n\, \phi$;
  - $\phi \to \forall x_n\, \phi$, where ϕ is n-ary;
  - $\forall x_n\, (\phi \to \psi) \to (\forall x_n\, \phi \to \forall x_n\, \psi)$;
  - $\forall x_n\, \phi$, where ϕ is an axiom.

Lemma 3. $\Gamma \vdash \phi \implies \Gamma \models \phi$.

Changing a notation used in class, let us write

$$\phi \approx \psi$$

if $\phi^{\mathcal{M}} = \phi^{\mathcal{M}}$ for all $\mathcal{M}$, and let us now use

$$\phi \sim \psi$$

to mean that $\{\phi\} \vdash \psi$ and $\{\psi\} \vdash \phi$. Both $\approx$ and $\sim$ are equivalence-relations (and will turn out to be the same).

Lemma 4. $\phi \sim \psi \implies \phi \approx \psi$.

Lemma 5. *If $\Gamma \vdash \phi$, then $\Gamma' \vdash \phi'$, where Γ' is got from Γ by replacing each an element with a $\sim$-equivalent one, and $\phi' \sim \phi$.*

So, as far as derivability and interpretations are concerned, we can identify formulas with their $\sim$-classes. One point of doing this is the following.

Lemma 6. $\mathrm{Fm}(\mathcal{L})/{\sim}$ *is naturally the universe of a Boolean algebra.*

Now define

$$\langle\Gamma\rangle = \{\phi : \Gamma \vdash \phi\}.$$

Say that Γ is **consistent** if $\bot \notin \langle\Gamma\rangle$.

Lemma 7. *If Γ is consistent, then the image of $\langle\Gamma\rangle$ in $\mathrm{Fm}(\mathcal{L})/{\sim}$ is the smallest filter containing the images of the formulas in Γ.*

Now we can prove compactness—that every consistent set of formulas has a model—just as in class; but we have to do more work at some points.

Derivability depends *a priori* on signature. We must rule out the possibility that there is a proof of ϕ from Γ in a signature larger than $\mathcal{L}$, but not in $\mathcal{L}$ itself.

Lemma 8. *Suppose* $\Gamma \subseteq \mathrm{Fm}^n(\mathcal{L})$, *and* $\phi \subseteq \mathrm{Fm}^{n+k+1}(\mathcal{L})$, *and* c *is a* $k+1$-*tuple of constant-symbols not in* $\mathcal{L}$.

- *If* $\Gamma \vdash \phi$, *then* $\Gamma \vdash \forall x_n \dots \forall x_{n+k}\, \phi$.
- *If* $\Gamma \vdash \phi(\mathbf{x}, \mathbf{c})$ *in* $\mathcal{L} \cup \{c_0, \dots, c_k\}$, *then* $\Gamma \vdash \forall x_n \dots \forall x_{n+k}\, \phi$ *in* $\mathcal{L}$.

Suppose $\mathcal{L} \subseteq \mathcal{L}'$, and $\mathcal{L}' - \mathcal{L}$ contains only constant-symbols.

Lemma 9. *The inclusion of* $\mathrm{Fm}(\mathcal{L})$ *in* $\mathrm{Fm}(\mathcal{L}')$ *induces an embedding of* $\mathrm{Fm}(\mathcal{L})/{\sim}$ *in* $\mathrm{Fm}(\mathcal{L}')/{\sim}$.

For any consistent set T of formulas of $\mathcal{L}'$, there is a relation on constant-symbols given by

$$c \sim d \iff T \vdash c = d.$$

(This is distinct from the relation $\sim$ on *formulas*.)

Lemma 10. *The relation* $\sim$ *on constant-symbols is an equivalence-relation.*

Suppose in particular that $\mathcal{L}$ has a constant-symbol c_ϕ for each unary formula ϕ, and suppose Γ is a consistent set of formulas of $\mathcal{L}$.

Lemma 11. *There is a consistent set* T *of formulas of* $\mathcal{L}'$ *such that:*

- $\Gamma \subseteq T$;
- $T \vdash \exists x_0\, \phi \to \phi(c_\phi)$ *for each unary formula* ϕ;
- T *is* maximally *consistent: the image of* $\langle T \rangle$ *in* $\mathrm{Fm}(\mathcal{L}')/{\sim}$ *is an ultrafilter.*

Lemma 12. *Suppose* T *is a maximally consistent set of formulas of* $\mathcal{L}'$ *such that* $T \vdash \exists x_0\, \phi \to \phi(c_\phi)$ *for each unary formula* ϕ. *Then:*

- *There is a unique model* $\mathcal{M}$ *of* T *whose universe* M *comprises the* $\sim$-*classes of the constant-symbols* c_ϕ, *and such that*
  - $c^{\mathcal{M}} = c/{\sim}$ *for each constant-symbol* c,
  - $f^{\mathcal{M}}(\mathbf{c}/{\sim}) = d/{\sim}$ *if* $T \vdash fc_0 \dots c_{n-1} = d$, *for all function-symbols* f, *and*
  - $(\mathbf{c}/{\sim}) \in R^{\mathcal{M}}$ *if* $T \vdash Rc_0 \dots c_{n-1}$, *for all relation-symbols* R.
- *If* $\mathcal{M}$ *is this model, then* $\phi^{\mathcal{M}} = \{(\mathbf{c}/{\sim}) : T \vdash \phi(\mathbf{c})\}$ *for all formulas* ϕ.