CONIC SECTIONS AS SUCH (*SECOND DRAFT*)

DAVID PIERCE

Contents

Part 1. Preface

As an undergraduate, I attended a college[1] where Euclid and Apollonius were used as textbooks. They were so used, not because they were considered to be the *best* textbooks, but simply because they *had been* textbooks for countless generations of mathematicians: therefore (the idea was), one might gain some understanding of humanity and oneself by reading these books. (The same is true for Homer, Aeschylus, Plato, and the other great books read at the college.)

Now, having become a professional mathematician, I ask what Euclid and Apollonius have to offer the mathematician of today. It is in pursuit of an answer to this question that I prepare these notes—which therefore are part of an ongoing project.

I prepare these notes also for the sake of honesty about what students are asked to learn. The conic sections are a standard part of an elementary course of mathematics. But the sections are usually given as the curves defined by certain equations, such as $ay = x^2$ or $(x/a)^2 \pm (y/b)^2 = 1$. Or perhaps the curves are given in terms of foci and directrices. A textbook may *assert* that the curves so defined can indeed by obtained as sections of cones; but it is rare that this assertion is justified.

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[1]St. John's College, with campuses in Annapolis, Maryland, and Santa Fe, New Mexico, USA.

One calculus textbook[2] writes at the beginning of a section:

> In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

I omit the book's Figure 1. The text continues under the heading *Parabolas*:

> A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**)... In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges... We obtain a particularly simple equation for a parabola if we place its vertex at the origin O...

Is the parabola obtained from a cone, or is it obtained from a focus and a directrix? It can be obtained in either way, but the author does not say so. The author need not *prove* the equivalence of the two constructions; but at least he ought to state that he is not going to prove it.

Perhaps the author expects the reader to *infer* the equivalence of the two definitions of a parabola. But this is not like the author, who is otherwise eager to give the reader every assistance that he can think of. He apparently does not think the reader can be left to infer that parabolas are worth studying. Before concluding anything from the definition, the author feels the need to tell the reader how *useful* parabolas are.

Another textbook[3] follows a similar procedure, first defining the conic sections as such, then defining them in terms of foci and directrices. Between the two definitions, the writer observes that the intersection of a cone and a plane will be given by a second-degree equation. This suggests that the quadratic equations to be derived presently in the book may indeed define conic sections. However, no attempt is made to prove that every curve defined by a quadratic equation can be obtained as the section of a cone. The author observes:

> After straight lines the conic sections are the simplest of plane curves. They have many properties that make them useful in applications of mathematics; that is why we include a discussion of them here. Much of this material is optional from the point of view of a calculus course, but familiarity with the properties of conics can be very important in some applications. Most of the properties of conics were discovered by the Greek geometer, Apollonius of Perga, about 200 BC. It is remarkable that he was able to obtain these properties using only the techniques of classical Euclidean geometry; today most of these properties are expressed more conveniently using analytic geometry and specific coordinate systems.

[2] James Stewart, *Calculus*, fifth edition, p. 720. This text is currently in use at Middle East Technical University.

[3] Robert A. Adams, *Calculus: a complete course,* fourth edition, p. 476. This text was formerly used at METU.

Again, the justification offered for the study of the conic sections is their usefulness. But as for 'expressing' the properties of conic sections, which of the following expresses better what a conic section *is?*

(i) It is the intersection of a cone and a plane.
(ii) It is the intersection of two surfaces defined by equations $ax + by + cz + d = 0$ and $(x - ez)^2 + y^2 = fz^2$ respectively.

What the author means, I think, is that it is convenient to define certain curves analytically and obtain their properties by further analysis. But showing that those curves are conic sections is a whole other problem, not addressed in the book.

By the way, despite what the last quotation suggests, I am not sure that obtaining nice results with limited mathematical tools is remarkable in itself. The tools of an artisan depend on what is available in the physical environment; but the tools of a mathematician depend only on imagination. A mathematician without the imagination to come up with the best tool for the job would seem to be an unremarkable mathematician.

The first chapter of Hilbert and Cohn-Vossen's *Geometry and the Imagination* [9] contains a beautiful account of how various properties of the conic sections arise from consideration of the cones from which the sections are obtained. However, the cones considered by the authors are all *right* cones. Apollonius does not make this restriction. Hilbert and Cohn-Vossen give an etymology for the names of the ellipse, the hyperbola, and the parabola: it involves eccentricity. The etymology is plausible, but it appears to be literally incorrect, as a reading of Book I of Apollonius would show.

Mathematics reveals underlying correspondences between seemingly dissimilar things. Sometimes we treat these correspondences as identities. This can be a mistake. There is a correspondence between conic sections and quadratic equations. But are the sections *really* the equations? One cannot answer the question without considering *conic sections as such.*

Part 2. Introduction

Some time in the 3rd century B.C.E., Apollonius of Perga wrote eight books on conic sections. We have the first four books [2, 3] in the original Greek; the next three books survive in Arabic translation [1]; the eighth book is lost.

As Apollonius tells us in an introductory letter, his first four books are part of an elementary course on the conic sections. The present notes are for use in such a course.

Before Apollonius, around 300 B.C.E., Euclid published the thirteen books of the *Elements,* a work of mathematics of which some of the parts could well be used as a textbook today. The *Elements* provide a good example of mathematical exposition and of what it means to prove something.

Apollonius assumes some of the concepts and theorems found in Euclid. Before getting into Apollonius then, I shall in § 5 have a look at the *Elements,* and specifically at the theory of *proportion* developed there.

Before that, in §§ 3 and 4, I look at some general features of ancient mathematics as I understand it. Meanwhile, in § 2, I jump forward in history to Descartes, in order to have in mind the sorts of improvements that he thought he was making to mathematical practice.

Because I shall occasionally refer to some Greek words, I review the Greek alphabet in Table 1. (I have heard a rumor that students can improve their mathematics simply

$A\,\alpha$	**a**lpha	$H\,\eta$	**ē**ta	$N\,\nu$	**n**u	$T\,\tau$	**t**au
$B\,\beta$	**b**eta	$\Theta\,\theta$	**th**eta	$\Xi\,\xi$	**x**i	$\Upsilon\,\upsilon$	**u**psilon
$\Gamma\,\gamma$	**g**amma	$I\,\iota$	**i**ota	$O\,o$	**o**micron	$\Phi\,\phi$	**ph**i
$\Delta\,\delta$	**d**elta	$K\,\kappa$	**k**appa	$\Pi\,\pi$	**p**i	$X\,\chi$	**ch**i
$E\,\epsilon$	**e**psilon	$\Lambda\,\lambda$	**l**ambda	$P\,\rho$	**r**ho	$\Psi\,\psi$	**ps**i
$Z\,\zeta$	**z**eta	$M\,\mu$	**m**u	$\Sigma\,\sigma,\varsigma$	**s**igma	$\Omega\,\omega$	**ō**mega

TABLE 1. The first letter or two of the (Latin) name for a Greek letter provides a transliteration for that letter. However, upsilon is also transliterated by *y*. The diphthong αι often comes into English (*via* Latin) as *ae,* while οι may come as *oe.* The second form of the small sigma is used at the ends of words. In texts, the rough-breathing mark (ʽ) over an initial vowel (or ρ) is transcribed as a preceeding (or following) h (as in ὁ ῥόμβος *ho rhombos* 'the rhombus'). The smooth-breathing mark (ʼ) and the three tonal accents (ά, ᾶ, ὰ) can be ignored. Especially in the dative case (the Turkish -e hali), some long vowels may be given the iota subscript (ᾳ, ῃ, ῳ), representing what was once a following iota (αι, ηι, ωι).

by learning this alphabet, assuming they didn't grow up knowing it.)

1. Theorems and problems

The text of Apollonius as we have it consists almost entirely of theorems and problems. There are some introductory remarks, some definitions, but nothing else. The theorems and problems can be analyzed in a way described by Proclus,[4] in the fifth century C.E., in his commentaries on Euclid [11, p. 159]:

> Every problem and every theorem that is furnished with all its parts should contain the following elements: an *enunciation* (πρότασις), an *exposition* (ἔκθεσις), a *specification* (διορισμός), a *construction* (κατασκευή), a *proof* (ἀπόδειξις), and a *conclusion* (συμπέρασμα). Of these, the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.

[4]Proclus was born in Byzantium (Constantinople, İstanbul), but his parents were from Lycia (Likya), and he was educated first in Xanthus. He moved to Alexandria, then Athens, to study philosophy [11, p. xxxix].

> So many are the parts of a problem or a theorem. The most essential ones, and those which are always present, are enunciation, proof, and conclusion.

More is said in § 2 about the distinction between a theorem and a problem. In § 5, I label the parts of a theorem with the terminology of Proclus. But what I present in § 7 are merely *enunciations* in the sense of Proclus. They do not use any special mathematical notation, although they may involve technical terms. Sometimes I incorporate the *definitions* of technical terms in these enunciations. While each of Apollonius's own enunciations tends to consist of a single long sentence, I use several sentences. So these notes should not be considered as an English translation of Apollonius. The point here is to do *mathematics,* not history.

2. Synthesis and analysis

We may however be aided in our own mathematics by trying to understand how others have done it. It may be said that we are going to do **pre-Cartesian** mathematics: mathematics as done before (well before) the time of René Descartes (1596–1650).

The geometry pioneered by René Descartes is called **analytic geometry;** by contrast, the geometry of ancient mathematicians like Euclid and Apollonius is sometimes called **synthetic geometry.** But what does this *mean?* The word *synthetic* comes from the Greek **συνθετικός,** meaning *skilled in putting together* or *constructive.* This Greek adjective derives from the verb *συντίθημι* *put together, construct* (from *συν* *together* and *τίθημι* *put*). The word *analytic* is the English form of *ἀναλυτικός*, which derives from the verb *ἀναλύω* *undo, set free, dissolve* (from *ἀνα* *up,* *λύω* *loose*). Although we refer to ancient geometry as synthetic, the Ancients evidently recognize both analytic and synthetic methods. Around 320 C.E., Pappus of Alexandria writes [12, p. 597]:

> Now **analysis** (*ἀνάλυσις*) is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a *reverse solution* (*ἀνάπαλιν λύσις*).
>
> But in **synthesis** (*συνθέσις*), proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.
>
> Now analysis is of two kinds, one, whose object is to seek the truth, being called **theoretical** (*θεωρητικός*), and the other, whose object is to find something set for finding, being called **problematical** (*προβληματικός*).

This passage is not very useful without examples: I shall propose one presently. Meanwhile, I note by the way that Pappus elsewhere [12, pp. 564–567] says more about the distinction between theorems and problems:

> Those who favor a more technical terminology in geometrical research use **problem** (πρόβλημα) to mean a [proposition[5]] in which it is proposed to do or construct [something]; and **theorem** (θεώρημα), a [proposition] in which the consequences and necessary implications of certain hypotheses are investigated; but among the ancients some described them all as problems, some as theorems.

What really distinguishes Cartesian geometry from what came before is perhaps suggested by the first sentence of Descartes's *Geometry* [4, p. 2]:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.

From a straight line, Descartes abstracts something called *length.* A length is something that we might today call a positive real number.

Descartes takes the edifice of geometry that has been built up or 'synthesized' over the centuries, and reduces or 'analyzes' its study into the manipulation of numbers. To be more precise, he 'takes that which is sought as though it were admitted' in the following way. In Figure 1, straight lines BE, DR, and FS are given in position (meaning their

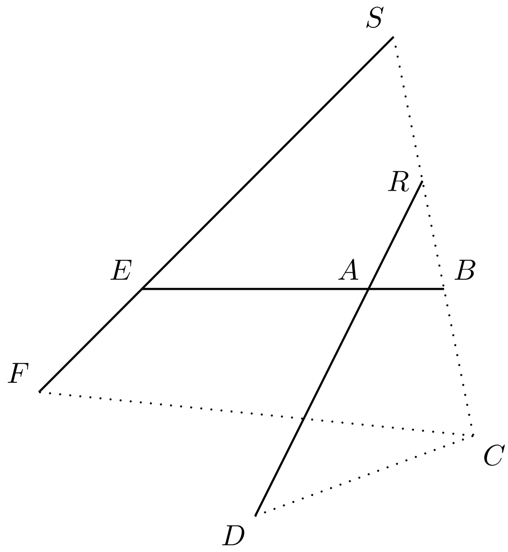


FIGURE 1

endpoints themselves are not fixed); and the sizes of angles ABC, ADC, and CFE are given. It is required to find the point C so that the rectangle with sides BC and BD

[5]Ivor Thomas [12, p. 567] uses *inquiry* here in his translation; but there is *no* word in the Greek original corresponding to this or to *proposition.*

has a given ratio to the square on CF. (This is a simplified version of the problem that Descartes takes up in the *Geometry.*)

In his analytic approach, Descartes assumes that C has already been found, as in the figure. We denote AB by x, and BC by y. The ratio $AB : BR$ is given; call it $z : b$. Then

$$RB = \frac{bx}{z}, \qquad CR = y + \frac{bx}{z} = \frac{zy + bx}{z}.$$

But $CR : CD$ is given; call it $z : c$. Then

$$CD = \frac{czy + bcx}{z^2}.$$

Also AE is given; call it k. And let $BE : BS = z : d$. Then

$$BE = k + x, \quad BS = \frac{dk + dx}{z}, \quad CS = \frac{zy + dk + dx}{z}.$$

Finally, if $CS : CF = z : e$, then

$$CF = \frac{ezy + dek + dex}{z^2}.$$

So it is given that the ratio

$$y \cdot \frac{czy + bcx}{z^2} : \left(\frac{ezy + dek + dex}{z^2}\right)^2$$

is constant. This gives us a quadratic equation in the unknowns x and y.

Descartes's method does not use explicitly drawn *axes* with respect to which x and y are measured. Also, the straight lines called x and y are not required to be perpendicular: they are merely not parallel.

Through analysis, we have found an equation that determines the point C. Since the equation is quadratic, the point C lies on a curve known as a **conic section.** When there are more straight lines in the problem, then the resulting equation may have a higher degree.

We do not get any sense here for what the curve of C *looks like.* We might get some sense by analyzing the equation for C. Apollonius will give us a sense for what conic sections look like by showing *how they are related to the cones that they come from.*

3. Conversational implicature

One apparent difference between the ancient and modern approaches to mathematics may result from a modern habit that is exemplified in a Russian textbook of the Soviet period [5, pp. 9 f.]:

> The student of mathematics must at all times have a clear-cut understanding of all fundamental mathematical concepts. . . The student will also recall the signs of weak inequalities: $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). The student usually finds no difficulty when using them in formal transformations, but examinations have shown that many students do not fully comprehend their meaning.
>
> To illustrate, a frequent answer to: "*Is the inequality* $2 \leqslant 3$ *true?*" is "No, since the number 2 is less than 3." Or, say, "*Is the inequality* $3 \leqslant 3$ *true?*" the answer is often "No, since 3 is equal to 3." Nevertheless,

> students who answer in this fashion are often found to write the result of a problem as $x \leqslant 3$. Yet their understanding of the sign $\leqslant$ between concrete numbers signifies that not a single specific number can be substituted in place of x in the inequality $x \leqslant 3$, which is to say that the sign $\leqslant$ cannot be used to relate any numbers whatsoever.

The students referred to, who will not allow that $2 \leqslant 3$, are following a habit of ordinary language, whereby the *whole* truth must be told. According to this habit, one does not say $2 \leqslant 3$, because one can make a stronger, more informative statement, namely $2 < 3$. This habit would appear to be an instance of *conversational implicature:* this is the ability of people to convey or *implicate* statements that are not logically *implied* by their words [10, ch. 1, §5, pp. 36–40]. In saying *A or B [is true],* one usually 'implicates' that one does not know *which* is true.

This habit of implicature may be reflected in the ancient understanding, according to which *one* (*ἕν*) is not a *number* (*ἀριθμός*). In Book VII of the *Elements,* Euclid somewhat obscurely defines a **unit** (*μονάς*) as that by virtue of which each being is called 'one'. (This English version of the definition is based on the Greek text supplied in [6, Vol. 2, p.279].) Then a **number** is defined as a *multitude* (*πλῆθος*) composed of units. In particular, a unit is not a number, because it is not a multitude: it is one. Euclid does not bother to state explicitly this distinction between units and numbers, but it can be inferred, for example, from his presentation of what we now call the Euclidean algorithm. Proposition VII.1 of the *Elements* involves a pair of numbers such that the algorithm, when applied to them, yields a unit (*μονάς*). Then this unit is *not* considered as a greatest common divisor of the numbers; the numbers do not *have* a greatest common divisor; the numbers are simply relatively prime. If the numbers are *not* relatively prime, then the *same* algorithm yields their greatest common divisor. This observation appears to be the contrapositive of the first, but Euclid distinguishes it as Proposition VII.2 of the *Elements.*

Conversational implicature may be seen in Apollonius's treating of the circle as different from an ellipse.

4. Lines

I follow the old understanding whereby a **line** need not be straight. A line may have endpoints, or it may be, for example, the circumference of a *circle.* Indeed, according to the definition in Euclid's *Elements* [6, 7, 8],

> A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.

A *straight* line does have endpoints; but the straight line may be *produced* (extended) beyond these endpoints, as far as desired.

5. Magnitudes and proportions

The things with size are **magnitudes.** These may be straight lines; they may be plane figures like rectangles; they may perhaps be other things (solids, angles, ...). With Descartes, we freely turned straight lines into numbers, with which we could do arithmetic. Henceforth, I want to be more careful. Here I shall attempt to suggest a theory of magnitudes based what is found in Books V and VI of Euclid. However, I shall use modern terminology, with abbreviative notation that is apparently foreign to Euclid's style.

The class of magnitudes is equipped with a binary relation, which I shall denote by $<$, so that $A < B$ is to be read as 'A is **less than** B'.

Axiom 1. *The relation $<$ is a strict partial ordering: we never have $A < A$; we never have both $A < B$ and $B < A$; but if $A < B$, and $B < C$, then $A < C$.*

The reflexive partial ordering corresponding to $<$ is $\leqslant$. That is, $A \leqslant B$ if and only if $A < B$ or $A = B$. I shall refer to this relation as **inclusion.**

Let us say that two magnitudes are **comparable** if they are equal, or if one is less than the other. Let us denote the relation of comparability by $\sim$.

Theorem 1. *Comparability of magnitudes is an equivalence: always $A \sim A$; if $A \sim B$, then $B \sim A$; if $A \sim B$ and $B \sim C$, then $A \sim C$.*

On every class of comparable magnitudes, there is an operation of **addition,** which we may denote by $+$. If $A \sim B$, then the two magnitudes have a **sum,** $A + B$.

Axiom 2. *On every class of comparable magnitudes, addition is associative and commutative: $A + (B + C) = (A + B) + C$, and $A + B = B + A$.*

Axiom 3. *If $A \sim B$, then $A < A + B$.*

Axiom 4. *If $A < B$ and $A \sim C$, then ($B \sim C$ and) $A + C < B + C$.*

Theorem 2. *If $A + B = A + C$, then $B = C$.*

Axiom 5. *If $A < B$, then there is C such that $A + C = B$.*

By Theorem 2 then, if $A < B$, then there is exactly one magnitude C such that $A + C = B$. We may write C as $B - A$, calling it the **excess** of B over A.

If two magnitudes are comparable, then it may happen that the lesser **measures** the greater. If A measures B, let us denote this by $A \mid B$. In this case, we may say that B is a **multiple** of A.

Axiom 6. *Measurement is a reflexive partial order.*

Axiom 7. *Measurement refines inclusion: if $A \mid B$, then $A \leqslant B$.*

It is not clear (to me) whether Euclid allows a magnitude to be a multiple of itself; but he does allow a magnitude to measure itself: see his Proposition X.3.

Axiom 8. *$A \mid A$.*

Axiom 9. *If $A \mid B$, then $A \mid A + B$.*

Axiom 10. *If $A \sim B$, then there is C such that $B \mid C$ and $A < C$.*

We need not talk about *how many times* a magnitude measures another; but we need to be able to say whether A measures B *as many times* as C measures D. If it does, I propose to write

$$A \mid B :: C \mid D;$$

we may say also, with Euclid, that B and D are **equimultiples** of A and C. Implicit in this notation is that, if $A \mid B :: C \mid D$, then $A \mid B$.

Axiom 11.

(i) *If* $A \mid B$, *then* $A \mid B :: A \mid B$.
(ii) $A \mid B :: C \mid D$ *if and only if* $C \mid D :: A \mid B$.
(iii) *If* $A \mid B :: C \mid D$ *and* $C \mid D :: E \mid F$, *then* $A \mid B :: E \mid F$.

Axiom 12. *Suppose* $A \mid B :: C \mid D$. *then* $A < C$ *if and only if* $B < D$.

Theorem 3. *Suppose* $A \mid B :: C \mid D$. *Then* $A = C$ *if and only if* $B = D$.

Axiom 13. *If* $A \mid B$, *and* D *is arbitrary, then there are magnitudes* C *and* E *such that* $A \mid B :: C \mid D$ *and* $A \mid B :: D \mid E$.

In Propositions V.1, 2, and 3, Euclid offers proofs of the following three axioms.

Axiom 14. *If* $A \mid B :: C \mid D$, *then* $A \mid B :: A + C \mid B + D$.

Axiom 15. *If* $A \mid B :: D \mid E$ *and* $A \mid C :: D \mid F$, *then* $A \mid B + C :: D \mid E + F$.

Axiom 16. *If* $A \mid B :: D \mid E$ *and* $B \mid C :: E \mid F$, *then* $A \mid C :: D \mid F$.

The quaternary relation of equimultiplicity is a special case of **proportion,** which we can now define. This relation holds amongst the magnitudes A, B, C, and D, and we write

$$A : B :: C : D,$$

saying 'A is to B as C is to D,' or 'A has the **same ratio** to B that C has to D,' provided $A \sim B$ and $C \sim D$, and moreover, whenever $A \mid E :: C \mid F$ and $B \mid G :: D \mid H$, we have

$$E < G \quad \text{if and only if} \quad F < H.$$

Theorem 4. *Having the same ratio is an equivalence, that is:*

(i) *If* $A \sim B$, *then* $A : B :: A : B$.
(ii) $A : B :: C : D$ *if and only if* $C : D :: A : B$.
(iii) *If* $A : B :: C : D$ *and* $C : D :: E : F$, *then* $A : B :: E : F$.

Here are Euclid's next three propositions (V.4, 5, and 6):

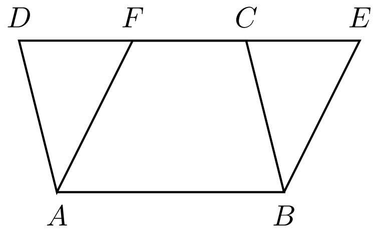
Theorem 5. *If* $A : B :: C : D$ *and* $A \mid E :: C \mid F$ *and* $B \mid G :: D \mid H$, *then* $E : G :: F : H$.

Proof. Suppose $A : B :: C : D$ and $A \mid E :: C \mid F$ and $B \mid G :: D \mid H$. Let $E \mid K :: F \mid L$ and $G \mid M :: H \mid N$. By Axiom 16, we have $A \mid K :: C \mid L$ and $B \mid M :: D \mid N$. By definition of proportionality, we have $K < M$ if and only if $L < N$. Therefore, by this definition again, $E : G :: F : H$. □

How can this theory of proportion be used to prove the proportionality of corresponding sides of similar triangles? The following theorems are found in Euclid. I state them in my own words, and analyze the first theorem according to the terminology of Proclus given in § 1. That part which is labelled 'theorem' is just the enunciation.

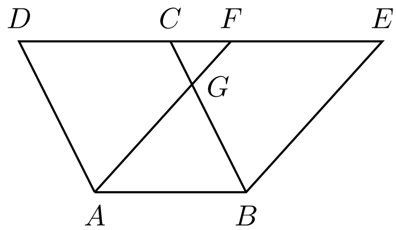
Theorem 6 (*Elements* I.35). *If two parallelograms have a common base, and the sides opposite this base are part of a common straight line, then the parallelograms are equal.*

Exposition. Suppose $ABCD$ and $ABEF$ are parallelograms, with the common base AB, such that the sides CD and EF lie in a common straight line.



Specification. Then $ABCD$ is equal to $ABEF$.

Construction. We may assume that C and F are on the same side of D. If C is between D and F, then let G be the intersection of AF and BC.

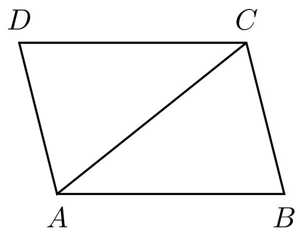


Proof. The triangle ADF is equal to BCE. If C is *not* between D and F, then $ABEF$ is obtained from $ABCD$ by removing triangle ADF and adding the equal triangle BCE. On the other hand, if C *is* between D and F, then the quadrilaterals $ADCG$ and $BEFG$ are equal, being obtained from the equal triangles ADF and BCE respectively be subtraction of the common triangle CFG. Then $ABEF$ is obtained from $ABCD$ by removing the quadrilateral $ADCG$ and adding the equal quadrilateral $BEFG$. □

Conclusion. Therefore, if two parallelograms have a common base, and the sides opposite this base are part of a common line, then the parallelograms are equal.

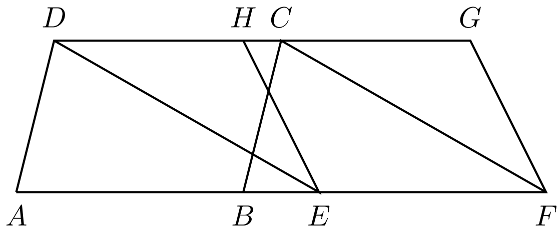
The conclusion repeats the enunciation. The construction is not always needed, even in principle. Henceforth I shall follow the usual custom of treating what is not part of the enunciation as part of the proof. I omit some proofs, as being clear from the picture.

Theorem 7 (I.34)**.** *A diagonal divides a parallelogram into two equal triangles.*



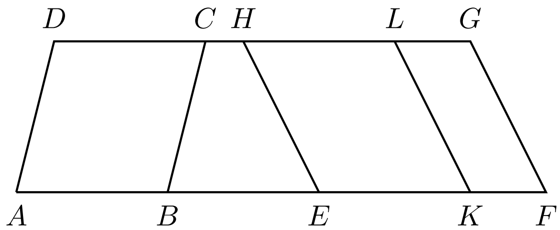
Proof. Let $ABCD$ be a parallelogram, with diagonal AC. Then angle BAC equals angle ACD, and angle BCA equals angle CAD, and the side AC is common: so triangle ABC equals triangle CDA. □

Theorem 8 (I.36)**.** *Parallelograms between the same parallel straight lines and with equal bases are equal.*



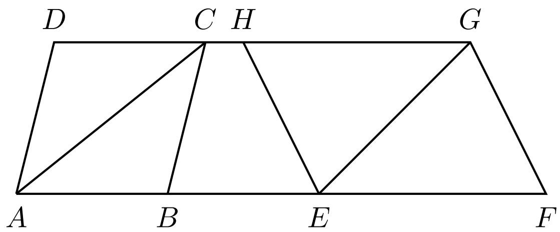
Proof. Let $ABCD$ and $EFGH$ be parallelograms such that the bases AB and EF are equal and are part of the same straight line; and suppose CD and GH are in the same straight line. Draw the straight lines CF and DE. Then $CDEF$ is a parallelogram equal to both $ABCD$ and $EFGH$, by Theorem 6. □

Theorem 9 (VI.1)**.** *Parallelograms drawn between the same parallel straight lines are to one another as their bases.*



Proof. Suppose $ABCD$ and $EFGH$ are parallelograms such that the bases AB and EF are part of the same straight line, and also CD and GH are in the same straight line. We may assume that EF is greater than AB. Between E and F, let K be chosen so that EF is equal to AB. Through K, a straight line drawn parallel to EH meets GH at a point L. Then parallelograms $ABCD$ and $EKLH$ are equal. And $EFGH$ is greater than $EKLH$; therefore it is greater than $ABCD$. In short, for parallelograms between the same parallels, the parallelogram with the greater base is the greater. A multiple of $ABCD$ is equal to another parallelogram between the same parallels; likewise for a multiple of $EFGH$; and the greater multiple corresponds to the greater base. Therefore $ABCD$ is to $EFGH$ as AB is to EF. □

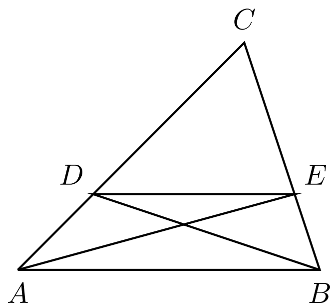
Theorem 10 (VI.1)**.** *Triangles drawn between the same parallel straight lines are to one another as their bases.*



Proof. Let ABC and EFG be triangles whose bases AB and EF are in a straight line, and whose apices C and G are in a straight line DG parallel to the bases. Through A draw straight line AD parallel to BC and meeting DG at D; likewise, through E, draw straight line EH parallel to FG and meeting DG at H. Then parallelograms $ABCD$ and

$EFGH$ are to one another as their bases AB and EF, which are the bases of ABC and EFG; and these triangles are half of the corresponding parallelograms, by Theorem 7. Therefore the triangles are to one another as their bases. □

Theorem 11 (VI.2). *A straight line drawn parallel to the base of a triangle cuts the two sides in proportion.*



Proof. Let ABC be a triangle, and on sides AC and BC respectively let D and E be chosen so that DE is parallel to BC. Let straight lines AE and BD be drawn. Then triangles ADE and BDE are equal, by Theorem 10. Add to these the common part CDE: then the triangles ACE and BCD are equal. Therefore CD is to CA as CDE is to ACE, that is, as CDE is to BCD. But also CDE is to BCD as CE is to CB. Therefore CD is to CA as CE is to CB. Hence also CD is to DA as CE is to EB. □

Theorem 12 (VI.4). *If two triangles have corresponding angles equal, then the triangles are to one another as their corresponding sides.*

Part 3. **Conic Sections**

6. Definitions

A **cone** (κῶνος) has, and is determined by:

(i) a **base** (βάσις), which is a circle;
(ii) an **apex** (κορυφής), which is a point not in the plane of the base.

Indeed, the base and the apex determine a **conic surface** (κωνικὴ ἐπιφάνεια): this comprises every point of every straight line that passes through both the apex and the circumference of the base. One conceives of the cone as obtained by taking a straight line from the apex to the circumference of the base, then moving the line around this circumference. The cone itself is the solid figure bounded by the base and the conic surface.

The straight line through the apex and the center of the base is the **axis** (ἄξων) of the cone. If this is perpendicular to the base, then the cone is **right** (ορθός); otherwise, the cone is **oblique** (σκαληνός). We shall work with arbitrary cones: they may be right or oblique.

A conic surface really comprises two surfaces, on opposite sides of the apex. We shall just call these **opposite surfaces.** Each of these has an inside and an outside.

7. Propositions

Proposition 1. Pick a point on a conic surface. Draw a straight line from the apex to this point. Then this line lies on the conic surface.

Proposition 2. Pick one of two opposite surfaces. Pick two points on it. Join them with a straight line. Either this line falls within the surface, and outside when produced; or else, when produced, the straight line meets the apex.

Proposition 3. Let a cone be cut by a plane that contains the apex. The resulting section of the cone is a triangle. If the cutting plane contains the axis of the cone, then the section is called an **axial triangle.**

Proposition 4. Let either of two opposite surfaces be cut by a plane parallel to the base of the cone. The resulting section is the circumference of a circle. This circle is the base of a new cone with the same apex and axis as the original cone.

Proposition 5. In an *oblique* cone, suppose an axial triangle is perpendicular to the base. (How is this possible?) Cut the cone by a plane that:

- (i) is not parallel to the base, but
- (ii) is perpendicular to the axial triangle, and
- (iii) cuts off from the axial triangle, on the side of the apex, another triangle.

This new triangle may still be similar to the axial triangle, while lying 'subcontrariwise' to it (about the apex). In this case, the section of the cone is a circle. This section may be called a **subcontrary** section.

In a cone, suppose a particular axial triangle is chosen. In the base of the cone, suppose a straight line is drawn perpendicular to the base of the axial triangle. Straight lines drawn parallel to this straight line will be said to be drawn **ordinate-wise.**

Henceforth, cones will implicitly have distinguished axial triangles.

Proposition 6. On the surface of a cone, pick a point that is not on a side of the axial triangle. From this point, draw a straight line ordinate-wise towards the axial triangle. This straight line will meet the conic surface again, on the other side of the axial triangle; and the axial triangle will bisect the straight line between the two points on the conic surface.

In the plane of the base of a cone, let a straight line be drawn perpendicular to the base of the axial triangle, or to this base produced. Let a plane containing this straight line cut the cone. This plane cuts the conic surface in a line called a **conic section.**

Proposition 7. Choose a point on a conic section that is not on the axial triangle. A straight line drawn ordinate-wise from this point meets the conic section again. The resulting chord of the conic section is bisected by the axial triangle. The intersection of the axial triangle and the plane of the conic section is therefore a **diameter** of the conic section. A point where the diameter meets the conic section is a **vertex** of the section. A straight line drawn ordinatewise from the conic section to the diameter is an called an **ordinate.** So an ordinate meets the diameter at some point. The part of the diameter between this point and a vertex is the corresponding **abscissa** (Latin: *[line] cut off*). In a right cone, ordinates are perpendicular to abscissas. In an oblique cone, they are not, unless the axial triangle is perpendicular to the base of the cone.

Proposition 8. Suppose the diameter of a conic section, when extended beyond a vertex, meets the extension of a side of the axial triangle beyond the apex. Then, for every straight line, there is an abscissa equal to that straight line. The same is true if the diameter is parallel to a side of the axial triangle.

Proposition 9. Suppose a conic section has two vertices. If the plane of the conic section is not parallel to the base of the cone, and if the conic section is not subcontrary, then the conic section is not a circle.

Proposition 10. A chord of a conic section falls inside the section (that is, inside the conic surface of which the section is a section); produced, the chord falls outside.

Proposition 11. Suppose the diameter of a conic section is parallel to a side of the axial triangle. From the vertex of the conic section, we shall draw a certain straight line perpendicular to the plane of the conic section. This straight line will be called the **upright side** (ὀρθία πλευρά; in Latin, *latus rectum*). It is drawn so as to have the same ratio to the part of the side of the axial triangle between the vertex of the conic section and the apex of the triangle that the square on the base of the axial triangle has to the rectangle whose sides are the sides of the axial triangle. The square on any ordinate is then equal to the rectangle whose sides are the abscissa and the upright side. The conic section is therefore called a **parabola** (παραβολή *juxtaposition, comparison, application,* from the verb παραβάλλω *throw beside,* from παρά *beside,* βάλλω *throw.*)

Proposition 12. Suppose the diameter of a conic section, produced, meets a side of the axial triangle produced beyond the apex. The segment of the diameter produced, between the two points where it meets the sides of the axial triangle, is the **transverse side** of the conic section. An upright side is constructed, as for the parabola; but now the upright side is to the *transverse side* as a certain square is to a certain rectangle. Namely, the square is on the straight line drawn from the apex of the cone to the base, parallel to the diameter of the conic section; this straight line cuts the base into two segments, and these are the sides of the rectangle. Then the square on an ordinate *exceeds* the rectangle whose sides are the abscissa and the upright side: the excess is the rectangle whose base is the abscissa, and whose height is to its base as the upright side of the conic section is to its transverse side. The conic section is therefore called a **hyperbola** (ὑπερβολή *excess, pre-eminence, throwing beyond,* from ὑπερβάλλω *throw beyond*).

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Mathematics Department, Middle East Technical University, Ankara 06531, Turkey
E-mail address: dpierce@metu.edu.tr
URL: http://metu.edu.tr/~dpierce/