CONIC SECTIONS AS SUCH *(FIRST DRAFT)*

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1. Introduction

The following notes investigate the conic sections in a manner inspired by Apollonius of Perga and other ancient mathematicians. The point here is to do *mathematics,* not history; but we may be aided in our own mathematics by trying to understand how others have done it. It may be said that we are going to do *pre-Cartesian* mathematics: mathematics as done before (well before) the time of René Descartes (1596–1650).

As I shall be referring to some works in Greek; as Greek is the origin of much of our technical vocabulary; as mathematicians use Greek letters all the time: for these reasons, I review the Greek alphabet in Table 1.

$A\,\alpha$	**a**lpha	$H\,\eta$	**ē**ta	$N\,\nu$	**n**u	$T\,\tau$	**t**au
$B\,\beta$	**b**eta	$\Theta\,\theta$	**th**eta	$\Xi\,\xi$	**x**i	$\Upsilon\,\upsilon$	**u**psilon
$\Gamma\,\gamma$	**g**amma	$I\,\iota$	**i**ota	$O\,o$	**o**micron	$\Phi\,\phi$	**ph**i
$\Delta\,\delta$	**d**elta	$K\,\kappa$	**k**appa	$\Pi\,\pi$	**p**i	$X\,\chi$	**ch**i
$E\,\epsilon$	**e**psilon	$\Lambda\,\lambda$	**l**ambda	$P\,\rho$	**r**ho	$\Psi\,\psi$	**ps**i
$Z\,\zeta$	**z**eta	$M\,\mu$	**m**u	$\Sigma\,\sigma,$ς	**s**igma	$\Omega\,\omega$	**ō**mega

Table 1. In this table, the first letter or two of the (Latin) name for a Greek letter provides a transliteration for that letter. In texts, the rough-breathing mark (ʽ) over an initial vowel (or ρ) is transcribed as a preceeding (or following) `h`; the smooth-breathing mark (ʼ) and the three tonal accents (ά, ᾶ, ὰ) can be ignored. Especially in the dative case (the Turkish `-e hali`), some long vowels may be given the iota subscript (ᾳ, ῃ, ῳ), representing what was once a following iota (αι, ηι, ωι).

2. Synthesis and analysis

The geometry pioneered by René Descartes is called **analytic geometry;** by contrast, the geometry of ancient mathematicians like Euclid and Apollonius of Perge is sometimes called **synthetic geometry.** But what does this *mean?* The word *synthetic* comes from the Greek συνθετικός meaning *skilled in putting together* or *constructive.* This Greek adjective derives from the verb συντίθημι *put together, construct.* The word *analytic* is the English form of ἀναλυτικός, which derives from the verb ἀναλύω *undo, set free, dissolve.* Although we refer to ancient geometry as synthetic, the Ancients evidently recognize both analytic and synthetic methods. Around 320 C.E., Pappus of Alexandria writes [10, p. 597]:

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> Now **analysis** (ἀνάλυσις) is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a *reverse solution* (ἀνάπαλιν λύσις).
>
> But in **synthesis** (συνθέσις), proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis.
>
> Now analysis is of two kinds, one, whose object is to seek the truth, being called **theoretical** (θεωρητικός), and the other, whose object is to find something set for finding, being called **problematical** (προβληματικός).

By the way, Pappus elsewhere [10, pp. 564–567] says more about the distinction brought up here between theorems and problems:

> Those who favor a more technical terminology in geometrical research use **problem** (πρόβλημα) to mean a [proposition[1]] in which it is proposed to do or construct [something]; and **theorem** (θεώρημα), a [proposition] in which the consequences and necessary implications of certain hypotheses are investigated; but among the ancients some described them all as problems, some as theorems.

What really distinguishes Cartesian geometry from what came before is perhaps suggested by the first sentence of Descartes's *Geometry* [4, p. 2]:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.

From a line, Descartes abstracts something called length. A length is something that we might today call a positive real number. Descartes takes the edifice of geometry that has been built up or 'synthesized' over the centuries, and reduces or 'analyzes' its study into the manipulation of numbers. This is just what we are *not* going to do in these notes.

Again, my main interest will be in the conic sections. If one has time and interest, one should just read Apollonius on the subject, if not in the original Greek,[2] then in a proper translation [2, 3] as opposed to a paraphrase [1]. However, it may be that some features of the text of Apollonius are dictated by customs of mathematical exposition that have no great connexion with the mathematics itself. The style of Apollonius's text as we have it is reflected in the observations of Proclus, in the fifth century C.E., in his commentaries on Euclid:

> Every problem and every theorem that is furnished with all its parts should contain the following elements: an *enunciation* (πρότασις), an *exposition* (ἔκθεσις), a *specification* (διορισμός), a *construction* (κατασκευή), a *proof*

[1]Ivor Thomas [10, p. 567] uses *inquiry* here in his translation; but there is *no* word in the Greek original corresponding to this or to *proposition*.

[2]Of the eight books of Apollonius's treatise on the conic sections, the last has been lost, and the fifth, sixth, and seventh come down to us only in Arabic translation.

> (ἀπόδειξις), and a *conclusion* (συμπέρασμα). Of these, the enunciation states what is given and what is being sought from it, for a perfect enunciation consists of both these parts. The exposition takes separately what is given and prepares it in advance for use in the investigation. The specification takes separately the thing that is sought and makes clear precisely what it is. The construction adds what is lacking in the given for finding what is sought. The proof draws the proposed inference by reasoning scientifically from the propositions that have been admitted. The conclusion reverts to the enunciation, confirming what has been proved.
>
> So many are the parts of a problem or a theorem. The most essential ones, and those which are always present, are enunciation, proof, and conclusion. [9, p. 159].

The propositions of Apollonius can be analyzed in these terms; but such analysis is not my concern here.

3. Conversational implicature

Another difference between my approach and the ancient approach to mathematics may result from a modern habit exemplified, for example, in a Russian textbook of the Soviet period:

> The student of mathematics must at all times have a clear-cut understanding of all fundamental mathematical concepts. . . The student will also recall the signs of weak inequalities: $\leqslant$ (less than or equal to) and $\geqslant$ (greater than or equal to). The student usually finds no difficulty when using them in formal transformations, but examinations have shown that many students do not fully comprehend their meaning.
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> To illustrate, a frequent answer to: "*Is the inequality* $2 \leqslant 3$ *true?*" is "No, since the number 2 is less than 3." Or, say, "*Is the inequality* $3 \leqslant 3$ *true?*" the answer is often "No, since 3 is equal to 3." Nevertheless, students who answer in this fashion are often found to write the result of a problem as $x \leqslant 3$. Yet their understanding of the sign $\leqslant$ between concrete numbers signifies that not a single specific number can be substituted in place of x in the inequality $x \leqslant 3$, which is to say that the sign $\leqslant$ cannot be used to relate any numbers whatsoever. [5, pp. 9 f.]

The students referred to, who will not allow that $2 \leqslant 3$, are following a habit of ordinary language, whereby the *whole* truth must be told. According to this habit, one does not say $2 \leqslant 3$, because one can make a stronger, more informative statement, namely $2 < 3$. This habit would appear to be an instance of *conversational implicature:* the ability of people to convey or *implicate* statements that are not logically *implied* by their words [8, ch. 1, §5, pp. 36–40]. In saying *A or B [is true],* one usually implicates that one does not know *which* is true.

I believe this habit of implicature may be reflected in the Greek understanding, according to which *one* (ἕν) is not a *number* (ἀριθμός). In the *Elements* [7], Book VII, Euclid somewhat obscurely defines a **unit** (μονάς) as that by virtue of which each being is called 'one' [6]. Then a **number** is defined as a *multitude* (πλῆθος) composed of units. In particular, a unit is not a number, because it is not a multitude: it is one. Euclid does not bother to state explicitly this distinction between units and numbers, but it can be inferred, for example, from his presentation of what we now call the Euclidean algorithm. Proposition VII.1 of the *Elements* involves a pair of numbers such that the Euclidean algorithm, when applied to them, yields a unit ($\mu o \nu \acute{\alpha} \varsigma$). Then this unit is *not* considered as a greatest common divisor of the numbers; the numbers do not *have* a greatest common divisor; the numbers are simply relatively prime. If the numbers are *not* relatively prime, then the *same* algorithm yields their greatest common divisor. This observation appears to be the contrapositive of the first, but Euclid distinguishes it as Proposition VII.2 of the *Elements.*

In these notes, I shall treat one as a number. Finally, I shall use special symbolism more freely than the Ancients did. However, I shall *not* abstract numbers from the things numbered, or sizes distinct from the things with size.

4. Magnitude and proportion

The things with size are **magnitudes.** These may be straight lines (what we call *line segments;* they may be plane figures like rectangles; they may perhaps be other things (solids, angles, ...). The class of magnitudes is equipped with a binary relation, denoted by $<$, where $A < B$ is read as 'A is less than B'. There is also the partial binary operation of **addition,** denoted by $+$, where $A + B$ is the **sum** of A and B. Axioms governing magnitudes might be laid out as follows, starting with:

(1) The relation $<$ is a partial ordering.

Two magnitudes are **comparable** if they are equal, or if one is less than the other. Let us denote the relation of comparability by $\sim$. Then we can state some more axioms:

(2) $\sim$ is an equivalence-relation.
(3) Two magnitudes have a sum if and only if they are comparable.
(4) On every class of comparable magnitudes, addition is associative and commutative.

We can now speak simply of **multiples** of a magnitude; we may abbreviate these as follows:

$$A + A = 2A, \quad A + A + A = 3A, \quad \dots, \quad \underbrace{A + \dots + A}_{n} = nA, \quad \dots$$

(5) If $A \sim B$, then $A < A + B$.
(6) If $A < B$ and $A \sim C$, then $A + C < B + C$.

Consequently, $A + C = B + C \implies A = B$.

(7) If $A < B$, then there is C such that $A + C = B$.

This C is unique by the previous observation and can be denoted by $B - A$: it is the **difference** of B from A.

(8) (Archimedean Axiom) If $A \sim B$, then $A < nB$ for some multiple nB of B.

We define the quaternary relation of **proportionality** as follows. This relation holds amongst the magnitudes A, B, C, and D, and we write

$$A : B :: C : D,$$

saying 'A is to B as C is to D,' provided $A \sim B$ and $C \sim D$, and moreover

$$mA < nB \iff mC < nD$$

for all equimultiples mA and mC of A and C, and nB and nD of B and D.

5. Definitions

A **cone** (κῶνος) has, and is determined by:

(1) a **base** (βάσις), which is a circle;

(2) an **apex** (κορυφής), which is a point not in the plane of the base.

Indeed, the base and the apex determine a **conic surface** (κωνικὴ ἐπιφάνεια), which comprises every point of every straight line that passes through both the apex and the circumference of the base. The cone itself is the solid figure bounded by the base and the conic surface.

The straight line through the apex and the center of the base is the **axis** (ἄξων) of the cone. If this is perpendicular to the base, then the cone is **right** (ορθός); otherwise, the cone is **oblique** (σκαληνός). We shall work with an arbitrary cone.

Let the apex of the cone be A. The conic surface has two parts, connected at A. The cone itself is bounded by one of these. Suppose a plane, not containing A, and not parallel to the base, cuts the part of the conic surface that bounds the cone. We shall study the **conic section** (τομή) so made. We may assume that the plane cuts the base in some chord BC, with midpoint D. There is a diameter EF of the base that is perpendicular to BC; these two chords meet at D. We have a triangle AEF, called an **axial triangle** (διὰ του ἄξονος τριγώνος), since it contains the axis of the cone. See Figure 1.

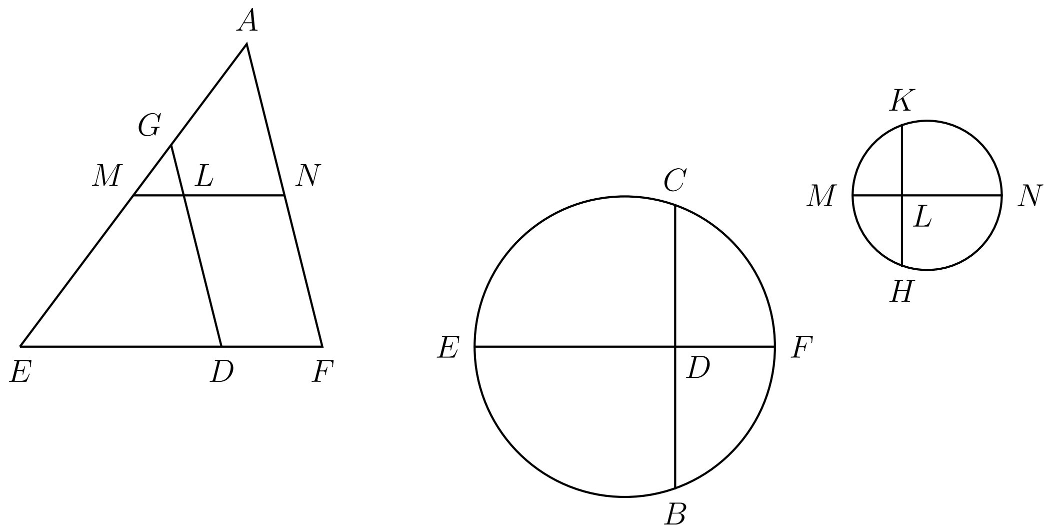

Figure 1. Axial triangle, base, and section parallel to the base

We may assume that the conic section contains a point G of AE (extended as necessary). Then G is the **vertex** (κορυφής) of the section, and GD is the **diameter** (διάμετρος) of the section. Indeed, let a point H be chosen at random on the section. If H is not G, then the section has a chord HK that is parallel to BC, and HK is bisected by GD, say at L.

Let the straight line through L parallel to EF meet AE at M and AF at N. We shall refer to HL as an **ordinate** (καταγόμενη τεταγμένως), and to GL as the corresponding **abscissa.** Our first aim is to relate ordinates and abscissas. We may note first of all

$$HL^2 = ML \cdot LN; \qquad BD^2 = ED \cdot DF.$$

6. Kinds of conic sections

There are three cases to consider. Suppose first of all that $GD \parallel AF$. Then $LN = DF$, so that

$$HL^2 : BD^2 :: ML : ED.$$

Thus abscissas are to one another as the squares on the ordinates. Draw GP perpendicular to GD so that

$$PG \cdot GD = BD^2 = ED \cdot DF$$

and therefore also $PG \cdot GL = KL^2$. The segment PG is called, in Latin, the ***latus rectum,*** and in English, the **upright side** (ὀρθία). The conic section itself is a **parabola** (παραβολή *juxtaposition, comparison, application*), from the verb παραβάλλω *throw beside,* because the square on the ordinate is equal to a rectangle on the abscissa that can be applied to the upright side. For an alternative characterization of the upright side, we have

$$PG : DF :: ED : GD :: EF : AF.$$

But also $DF : GA :: EF : EA$. Compounding ratios, we have

$$\boxed{PG : GA :: EF^2 : EA \cdot AF}.$$

In the second case, DG extended meets FA extended at a point Q.

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