# Lobachevski's Geometry

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# Preface

This is a rough draft of my own attempt at retelling the *Geometrical Researches on the Theory of Parallels,* by Nicholas Lobachevski. Though I tried to correct it while composing, I have not read the present through from the beginning. A significant feature is the diagrams, in which the appearance of *parallelism* is preserved, at the expense of straightness. In Lobachevski's own diagrams, the emphasis is the reverse. Often my diagrams are from the Poincaré half-plane or half-space model. In future I hope to add a section on these models.

In the translation by George Bruce Halstead, Lobachevski's *Researches* are printed as a supplement to Roberto Bonola, *Non-Euclidean Geometry* (New York: Dover, 1955).

One may read the present document independently, or for the sake of elucidating Lobachevski himself. The exposition of Lobachevski is sometimes confusing, in a way perhaps not absolutely required by the mathematics itself. The order in which Lobachevski presents his propositions (or "theorems") seems good, and I preserve this order. I do skip the first fifteen propositions, which summarize what can be known, apart from any assumption about parallelism. Those fifteen propositions can be found in or inferred from Euclid's *Elements.* Lobachevski does not cite Euclid specifically for the propositions, but mentions him when opening his treatise:

> In geometry I find certain imperfections which I hold to be the reason why the science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid.

Book I of Euclid's *Elements* should not be "elucidated" for students. Students entering my own mathematics department at Mimar Sinan in Istanbul read this book (in Turkish translation), and we hope that they will figure out for themselves both *what* Euclid is saying about mathematics, and *whether* he says it in a good way. The same exercise can be repeated with any writer of mathematics, in the manner of my *alma mater,* St John's College in Annapolis and Santa Fe; but with a modern writer like Lobachevki, perhaps just understanding his mathematics is enough of a challenge, without the stumbling block of obscurities in his proofs.

Before the preparation of the present notes, I had worked through Lobachevski four times:

1) as a student in the senior mathematics tutorial of St John's College, 1986–7;
2) as the teacher of an elective upper-level course called *Geometriler* at Mimar Sinan, fall semester, 2015–6;
3) as the teacher of a course called Geometries at the Nesin Mathematics Village, Şirince, September 12–25, 2016;
4) as the teacher of Hyperbolic Geometry in the same place, August 7–13, 2017.

The course at Mimar Sinan covered projective geometry in its first half. In the style of St John's, and of the Euclid course that my own students had taken in their first semester in my department, I had these students present, at the board, some relevant propositions of Pappus, which I had translated into Turkish from the Greek. In the second half of the course, students presented the propositions of Lobachevski in the same way, but from the English translation mentioned above. I kept a detailed record of the course and of the mathematics involved; the record is on my website; I have used the record as a source for the present notes.

In the following summer, I tried to imitate my Mimar Sinan course in Şirince. Since the Mimar Sinan course had met two hours a week, and a Şirince course met two hours a day, two weeks in Şirince should be enough to cover what the Mimar Sinan course did. I kept a detailed record of the first week, on projective geometry; but not of the second week, on hyperbolic geometry. I do remember that almost none of the students from the first week stayed on for the second. Also, students in the second week were generally not able to understand the propositions of Lobachevsky that they were supposed to present—if they even tried to understand, and sometimes they did not, or else they just did not show up for class.

Our last meeting during that second week was on Friday, since I would attend the Thales Meeting in Miletus the next day, September 24. Near the beginning of the week, I had introduced the Poincaré half-plane model of Lobachevskian geometry. This may have been a mistake, since students continued to show throughout the week that they did not understand that lines appearing curved in the model were supposed to be understood as straight.

In the following summer in Şirince, I covered the same material, but in the standard lecture format. Perhaps this was better than expecting students to come to the board, since Şirince is too crowded in both space and time for students to work well on their own. In any case, a number of students in 2017 seemed more engaged than students in the previous summer. I treated the two weeks officially as different courses, and again there was little or no continuity among the students. In the first week, after the first day, I lectured in English, because a foreign student started attending; in the second week, I used Turkish. Over the course of the second week, some of the actively questioning students stopped attending the lectures without warning. Attendence figures were thus (dates in July and August on the first line, numbers attending on the second):

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 31 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 | 12 | 13 |
| ? | 12 | 12 | 9 | 6 | 3 | ? | 15 | 12 | 10 | 8 | 4 |

Numbers for the first day of each week are unknown, since class rosters had not been distributed, and I did not make my own lists. On the other days, figures are only approximate, since I may not have counted students who came late.

# Contents

# List of Figures

# Introduction

By a **line,** we shall always understand an *unbounded, straight* line. We allow ourselves to do everything that Euclid does, *without* using the postulate that, when two lines are crossed by another, and the interior angles on the same side are less than two right angles, then the two lines intersect when extended on that side.

The juxtaposition, as $AB$, of two named points can have five different meanings:

1) the unique line passing through $A$ and $B$,
2) the same line, *directed* from $A$ to $B$,
3) the ray that begins at $A$ and passes through $B$,
4) the line segment bounded by $A$ and $B$,
5) the same segment, directed from $A$ to $B$.

The **measure** of an angle is its congruence class. The measure of a right angle is denoted by $\pi/2$ or

$$\frac{\pi}{2}.$$

There is no need to understand $\pi$ as a real number. However, the measure of every acute angle is $t\pi/2$ for some real number $t$ in the interval $(0,1)$. Indeed, since we can bisect angles, for every acute angle with a measure $\alpha$, we can define a sequence $(e_k\colon k \in \mathbb{N})$ such that, for each $k$,

$$\sum_{i=1}^{k} e_i \frac{\pi}{2^{i+1}} \leqslant \alpha < \sum_{i=1}^{k} e_i \frac{\pi}{2^{i+1}} + \frac{\pi}{2^{k+1}}.$$

We make the **Archimedean** assumption that the sequence $(e_k : k \in \mathbb{N})$ uniquely determines $\alpha$. We shall assume **completeness** also, in the sense that every such binary sequence determines the measure of some angle.

Similarly, the **length** of a line segment is its congruence class, and any two lengths have a ratio, which is a positive real number. We shall also want that, for every length and every positive real number, the latter is the ratio of some other length to the given length.

# The propositions of Lobachevski

## Proposition 16

*For every length $x$, there is a measure, denoted by*

$$\Pi(x),$$

*such that, in a plane, under the hypothesis that a segment $AB$ has length $x$ as in Figure 1, and $ABC$ is a right angle, and $AD$ is on the same side of $AB$ that $BC$ is,—under this hypothesis, $BC$ and $AD$ intersect when extended if and only if the measure of $BAD$ is less than $\Pi(x)$. There is a ray $AD$ such that the measure of $BAD$ is precisely $\Pi(x)$.*

*Proof.* By SAS, the supremum of the measures of all angles $BAX$, where $X$ is on $BC$ extended, depends only on $x$ and is $\Pi(x)$ as desired. By completeness, $\Pi(x)$ is the measure of some angle. $\square$

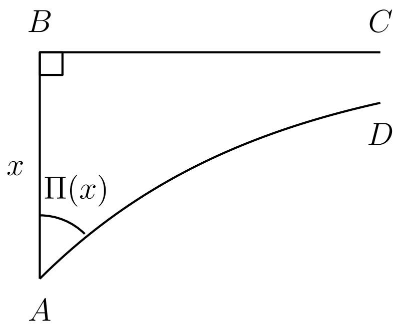



Figure 1: Proposition 16. $AD \parallel BC$.

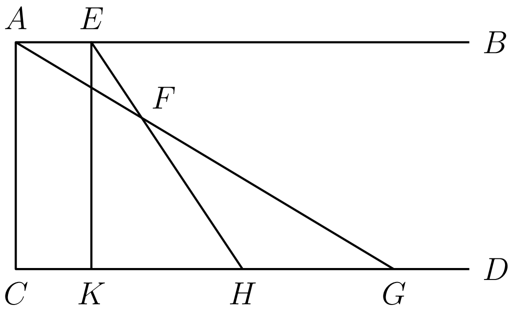


Figure 2: Proposition 17, first case

In the theorem, when the measure of angle $BAD$ is precisely $\Pi(x)$, we say $BC$ is **parallel** to $AD$. At the moment, while $AD$ is only a directed *line,* $BC$ must be understood as a *ray.* In particular, the specification of the point $B$ is important. We may refer to $\Pi(x)$ as the **angle of parallelism** of $x$.

In Figure 1, the ray $AD$ must be understood as being straight, even though it appears to us as curved. The same proviso will apply to most figures here.

## Proposition 17

*Two rays of the same directed line are parallel to the same directed lines.*

*Proof.* We suppose $AB$ is parallel to $CD$, $ACD$ being a right angle. There are two cases.

1. In Figure 2, we drop the perpendicular $EK$ to $CD$ and then draw $EF$ arbitrarily within the angle $BEK$. Then $AF$, extended, must meet $CD$ at some $G$, and so $EF$, extended, must cut $KG$.

2. In Figure 3, we drop the perpendicular $E'K'$ to $CD$, then

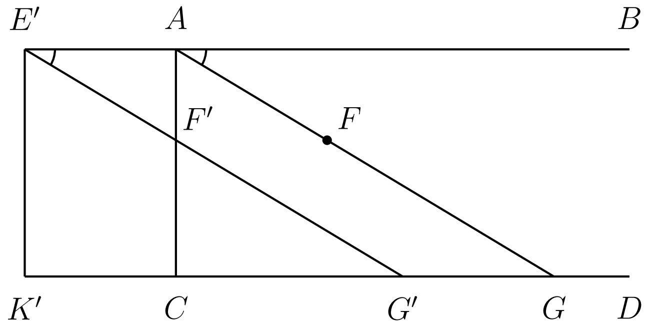


Figure 3: Proposition 17, second case

draw $E'F'$ arbitrarily in the angle $BE'K'$. Extended, $E'F'$ must cut $K'C$ or $CA$. In the latter case, draw $AF$ so that $FAB = F'E'B$; then $AF$ cuts $CD$ at some $G$, but also $E'F'$, extended, cannot cut $AF$, so it cuts $CG$ at some $G'$. □

Parallelism is now a possible relation of one directed line to another.

## Proposition 18

*The relation of parallelism is symmetric.*

*Proof.* In Figure 4, suppose $AB$ is parallel to $CD$, and $ACD$ is a right angle, within which $CE$ is drawn. Drop the perpendicular $AF$ to $AE$, let $AG = AF$, erect $GH$ perpendicular to $AC$, and let $CAK = FAB$. Then $AK$ must meet $CD$ at some point $K$, and so $AK$ cuts $GH$ at some $L$. By SAS, $CE$ will cut $AB$ at a point whose distance from $AB$ is the length of $GL$. □

We shall show in Proposition 25 that the relation of parallelism is transitive.

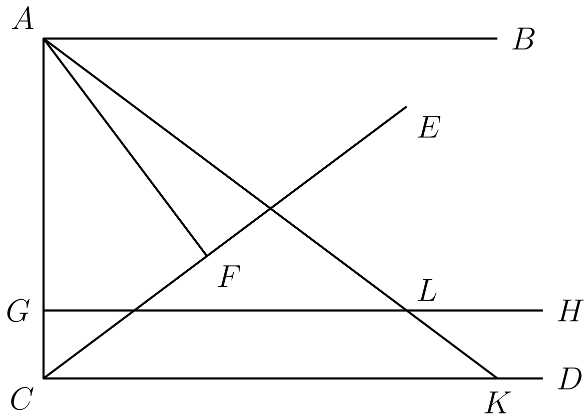



Figure 4: Proposition 18

## Proposition 19

*The sum of angles in a triangle is never greater than two right angles.*

*Proof.* In Figure 5, suppose the least angle of triangle $ABC$ is at $A$. Bisect $BC$ at $D$, extend $AD$ to $E$ so that $AD = DE$,

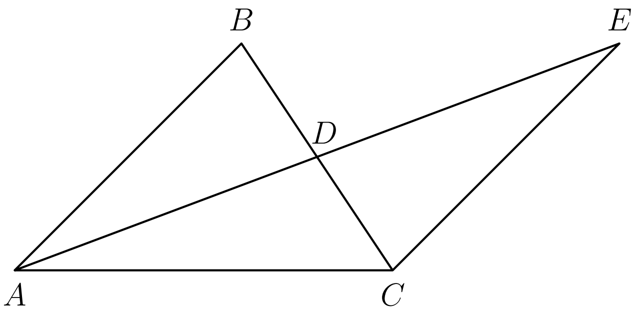



Figure 5: Proposition 19

and complete triangle $ACE$. This has the same angle sum as $ABC$, but two angles of $ACE$, namely those at $A$ and $E$, are equal in sum to angle $BAC$, so one of them is no greater than half. We can continue this construction until obtaining a triangle whose angle sum is that of $ABC$, but with two angles whose sum is less than any pre-assigned positive measure $\alpha$. (Here we use the Archimedean property of angle measure.) Thus the angle sum of $ABC$ cannot exceed $\pi$ by $\alpha$. Since $\alpha$ is arbitrary, the angle sum of $ABC$ cannot exceed $\pi$. □

Let us define the **defect** of a triangle to be what must be added to the angle sum of the triangle to reach $\pi$. By Proposition 19, the defect of a triangle is positive or zero. A triangle with positive defect is **defective.**

**Lemma.** *If one triangle is divided into two by a line through a vertex, the defect of the original triangle is the sum of the defects of the smaller two triangles.*

A **rectangle** is a quadrilateral figure, each of whose four angles is right. If such a figure does exist, a diagonal divides it into two congruent defectless right triangles.

## Proposition 20

*If one triangle has angle measure $\pi$, then all triangles do.*

*Proof.* Suppose some triangle has no defect. One of its altitudes has its foot on a side, thus dividing the triangle into two right triangles, each having no defect. One of these is half a rectangle. By the Archimedean property of lengths, we can multiply the sides of the rectangle, so as to exceed the legs of a given right triangle, as, in Figure 6, $AB$ and $BC$ exceed,

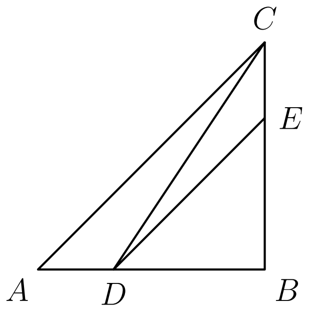



Figure 6: Proposition 20

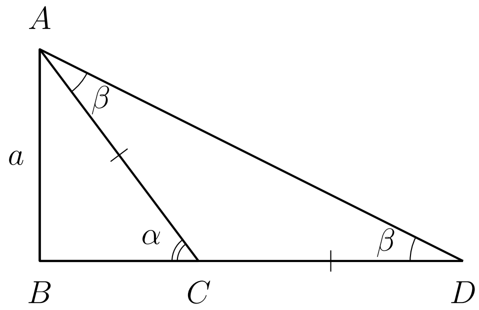



Figure 7: Proposition 21

respectively, the legs $DB$ and $BE$ of the right triangle $DBE$. By drawing $CD$, we can conclude that, as $ABC$ is defectless, so must $DBE$ be. ☐

## Proposition 21

*If there is an acute angle of parallelism, then there is a defective triangle.*

*Proof.* In Figure 7, suppose $\Pi(a) < \pi/2$. If $DE = AD$, we have $\alpha \geqslant 2\beta$. Continuing, as in Figure 8, we obtain a right

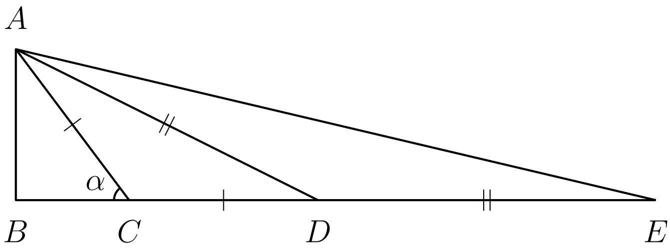



Figure 8: Proposition 21 continued

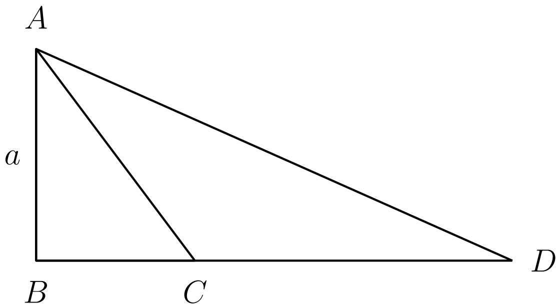



Figure 9: Proposition 22

triangle with one angle measuring less than $\pi/2 - \Pi(a)$, and another angle measuring no greater than $\Pi(a)$; thus the defect of the triangle is positive. □

## Proposition 22

*If there is a right angle of parallelism, then there is a defectless triangle.*

*Proof.* In Figure 9, by placing $D$ on $BC$ far enough away from $C$, we can make the measure angle $BAD$ as close as we like to $\Pi(a)$. If this measure is $\pi/2$, then the defect of $BAD$ is as

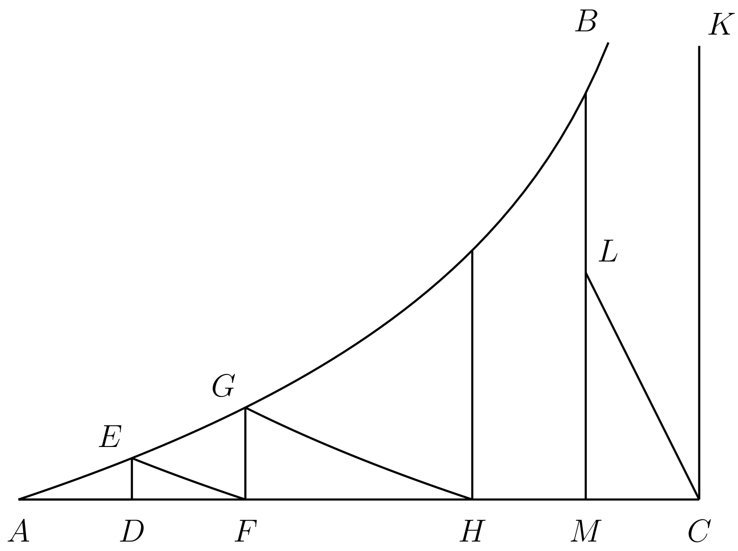


Figure 10: Proposition 23

small as we like. Since the defect of $ABC$ is no greater, this can only be 0. □

The following are now equivalent.

1. There is a defective triangle.
2. All triangles are defective.
3. There is an acute angle of parallelism.
4. All angles of parallelism are acute.

We henceforth assume these.

## Proposition 23

*Every acute angle is the angle of parallelism of some length.*

*Proof.* In Figure 10, acute angle $BAC$ being given, we erect

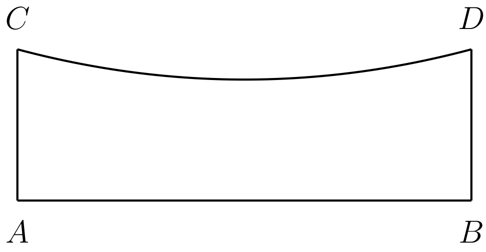



Figure 11: Proposition 24

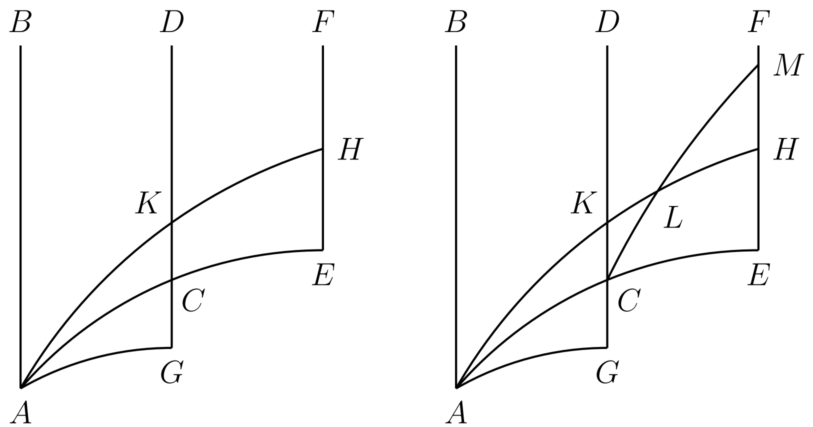
the perpendicular $DE$. If this meets $AB$ at $E$, we make $DF = AD$ and erect the perpendicular $FG$. If this meets $AB$ at $G$, we make $FH = AF$ and erect a perpendicular at $H$, and so on. The defects of $ADE$ and $FDE$ are equal (the triangles themselves being congruent), so the defect of $AFG$ is more than twice that of $ADE$. Since no defect can exceed $\pi$, we must eventually find a point on $AC$ where the perpendicular does not meet $AB$. By completeness, there is a closest such point to $A$; let it be $C$, and let the perpendicular be $CK$. If we draw $CL$ in the angle $ACK$, then, dropping the perpendicular $LM$ to $AC$, we can extend $ML$ to meet $AB$. Then $CL$, extended, must also meet $AB$. Thus $CK$ is parallel to $AB$. □

The following does not say parallel lines approach one another arbitrarily closely.

## Proposition 24

*Prolonged, parallel lines approach one another.*

*Proof.* If equal perpendiculars $AC$ and $BD$ are erected on a line $AB$, and $CD$ is connected, as in Figure 11, then, in the



(a) $AB \parallel EF$ & $CD \parallel EF$ (b) $AB \parallel CD$ & $CD \parallel EF$

Figure 12: Proposition 25, planar case

quadrilateral $ABDC$, the angles at $C$ and $D$ are equal and acute, and so $CD$ cannot be parallel to $AB$, since the exterior angle at $D$ is obtuse. □

## Proposition 25

*Parallelism is transitive, whether in the plane or in space. Indeed, in space, when each of two intersecting planes contains one of two parallel lines, the intersection of the planes is parallel to those lines.*

*Proof.* There are two cases in the plane. Suppose first $AB \parallel EF$ and $CD \parallel EF$, as in Figure 12a. Let the perpendicular $AE$ to $EF$, cutting $CD$ at $C$, be dropped. Since $ACD$ is acute, when perpendicular $AG$ to $CD$ is dropped, $AC$ lies within the angle $AGD$. Another line $AH$ drawn within $AGD$ either cuts

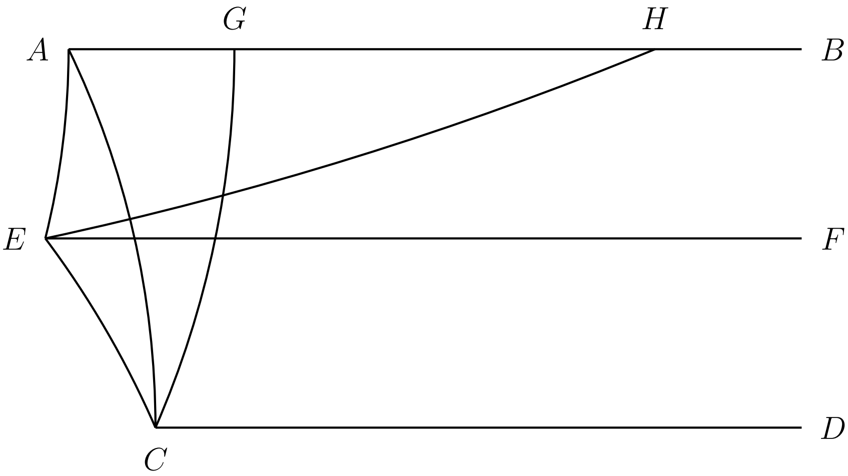


Figure 13: Proposition 25, spatial case

$GC$ or else cuts $EF$ at some $H$, and in the latter case $AH$ must cut $CD$ at some $K$. Thus $AB \parallel CD$.

Suppose next $AB \parallel CD$ and $CD \parallel EF$. Again let the perpendicular $AE$ be dropped to $EF$. A line $AK$ drawn within angle $BAG$ cuts $CD$ at some $K$. When $AK$ is extended to $L$, the line $CL$ within angle $DCE$ must cut $EF$ at some $M$, and then $AL$ must cut $EM$ at some $H$. Thus $AB \parallel EF$.

Suppose finally $AB \parallel CD$, and two planes containing $AB$ and $CD$ respectively meet along $EF$, as in Figure 13. Drop perpendiculars $EA$ to $AB$, $AC$ to $CD$, and $CG$ to $AB$. If a line $EH$ is drawn in the angle $AEF$, then the plane $CEH$ cuts the plane $ACD$ along a line that is bound to cut $AB$ at some point $H$. Thus $EF \parallel AB$. $\square$

We make use now of spherical geometry, and later of spherical trigonomety. On a sphere, we shall let the antipodal point of $X$ be $X'$. We could take the following theorem to be obvious, by "symmetry"; but like Lobachevski, we can also establish it by the kind of rearrangement of parts that Euclid employs.

## Proposition 26

*On a sphere, triangles whose vertices are respectively antipodal are equal.*

*Proof.* Given the triangle $ABC$ on the sphere, we let the perpendicular dropped from the center of the sphere to the plane of $ABC$ cut the sphere at $D$. Then $D$ is equidistant from the vertices of $ABC$. Since $ABD$ is isosceles, it is congruent to $A'B'D'$, and likewise for $BCD$ and $CAD$. Thus $ABC = A'B'C'$. $\square$

Lobachevski treats the total solid angle as $2\pi$; we convert to $4\pi$ (although Lobachevski's convention has its own convenience). With Lobachevski, we follow the convention whereby the measures of the surface angles of a spherical triangle $ABC$ are respectively $A$, $B$, and $C$.

## Proposition 27

*At the center of a sphere, the solid angle subtended by the triangle $ABC$ in the surface of the sphere measures*

$$A + B + C - \pi.$$

*Proof.* We use that twice the measure of (for example) $A$ is the sum of $ABC$ and $A'BC$. Thus

$$\begin{multline*}2(A+B+C) \\ =(ABC+A'BC)+(ABC+AB'C)+(ABC+ABC') \\ =2ABC+ABC+A'BC+AB'C+A'B'C \\ =2ABC+2\pi. \quad \square\end{multline*}$$

## Proposition 28

*If the intersections of three planes taken in pairs are parallel, the sum of the angles of the planes taken in pairs is* $\pi$*.*

*Proof.* Given parallels $AA'$, $BB'$, and $CC'$ as in Figure 14, picking a point $P$ along $BB'$, we construct, centered at $A$, $P$, and $C$ respectively, the spherical triangles $DEF$, $GHK$, and $LMN$. The angle sum of the three planes containing the parallels in pairs is $D+H+N$. Also

$$E+G=\pi, \qquad F=L, \qquad K+M=\pi.$$

Thus

$$\begin{multline*}DEF+GHK+LMN \\ =D+E+F+G+H+K+L+M+N-3\pi \\ =D+H+N-\pi+2F,\end{multline*}$$

and so

$$D+H+N-\pi=GHK-\bigl(2F-(DEF-LMN)\bigr).$$

The difference $2F-(DEF-LMN)$ is positive, and $F$ can be as small as desired. So can the angle $GPH$, by the proof of

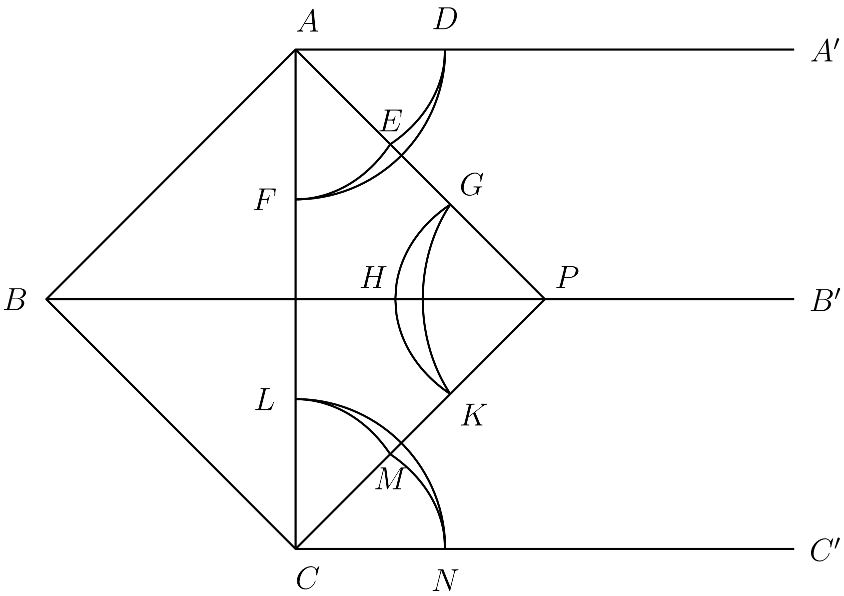



Figure 14: Proposition 28

Proposition 21, and this means the solid angle $GHK$ can be as small as desired. Thus we can only conclude

$$D + H + N = \pi. \qquad \square$$

## Proposition 29

*If the perpendicular bisectors of two sides of a (rectilineal) triangle meet at a point, then the perpendicular bisector of the third side also passes through that point.*

*Proof.* This is entirely as would be in Euclid. □

## Proposition 30

*If the perpendicular bisectors of two sides of a (rectilineal) triangle are parallel to one another, then they are parallel to the perpendicular bisector of the third side.*

*Proof.* Let the perpendicular bisectors be $DE$, $FG$, and $HK$, with $HK$ between the others, as in Figure 15. There are two cases. If $DE \parallel FG$, then, since $HK$ meets neither of these by Proposition 29, it must be parallel to them, as in the proof of Proposition 25.

Now we assume $HK \parallel FG$. Let the parallels $AA'$, $BB'$, and $CC'$ to $HK$ be drawn. As before, if $DE \nparallel HK$, then it cannot cut $HK$, so it must cut $AA'$. Letting the angles of $ABC$ at $B$ and $C$ be $\beta$ and $\gamma$ respectively, we have

$$\beta = \Pi(a) - \Pi(c), \qquad \gamma > \Pi(a) + \Pi(b).$$

If we rotate $CA$ about $C$ into a position $CQ$ as in Figure 16, so that $\angle QCB = \Pi(a) + \Pi(b)$, then $\angle QBC > \Pi(a) -$

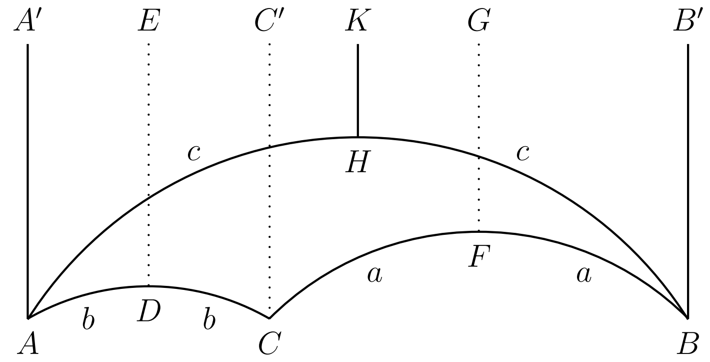


Figure 15: Proposition 30

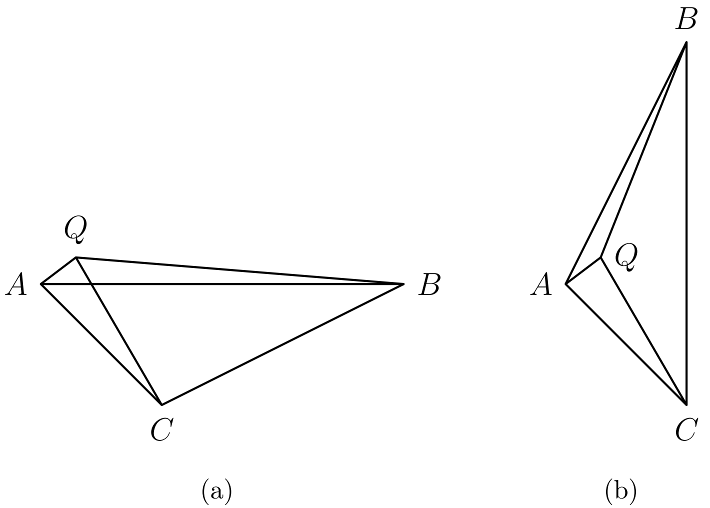


Figure 16: Proposition 30 continued

$\Pi(c)$. But we now have a triangle $QBC$ to which the earlier argument applies, namely that the perpendicular bisector of $QB$ is parallel to the other two. Thus, letting $QB = 2c'$, we have

$$\begin{aligned}\Pi(a) - \Pi(c') &> \Pi(a) - \Pi(c),\\ \Pi(c') &< \Pi(c),\\ c' &> c.\end{aligned}$$

But $AC = QC$, and so, since the greater angle is subtended by the greater side, as in Euclid's *Elements* I.24, we have $AB > QB$, that is, $c > c'$, contradicting the earlier inequality. Thus $DE \parallel HK$. □

## Proposition 31

*Given a ray $AB$, for every acute angle measure $\xi$, we can find $x$ such that $\Pi(x) = \xi$, and then, on either side of $AB$, we can find a point $X$ such that*

$$\angle BAX = \xi, \qquad |AX| = 2x,$$

*as in Figure 17. The locus of the points $X$ is a curve, the perpendicular bisector of whose every chord is parallel to $AB$.*

*Proof.* By Proposition 23, $x$ exists for every $\xi$. By Proposition 30, the perpendicular bisector of, for example, $XY$, is parallel to those of $AX$ and $AY$ and thus to $AB$ itself. □

The curve given by Proposition 31 is the **horocycle,** and $AB$, along with the lines parallel to this, is an **axis** of the horocycle. Though Lobachevski does not use the term, we may refer to the intersection of an axis with the horocycle as a **vertex.**

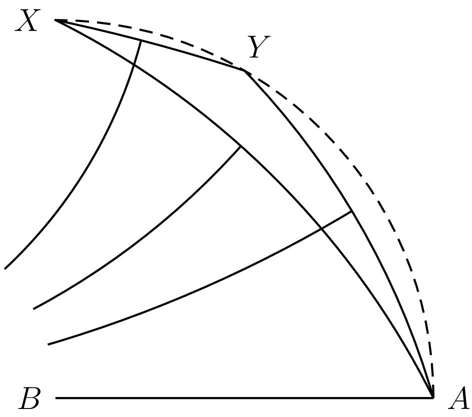



Figure 17: Proposition 31. The horocycle

## Proposition 32

*Every point of the horocycle with a given axis is approached as close as desired by a circle through the vertex with center on the axis.*

*Proof.* In Figure 18 then, a horocycle has axes $AC$ and $BD$, and the circle with center $E$ on $AC$, and passing through $A$, cuts $BD$ at $F$. The sum of the angles in triangle $ABF$ is

$$\alpha - \beta + \alpha + (\pi - \beta - \gamma),$$

which is $2\alpha - 2\beta - \gamma + \pi$, but is also less than $\pi$, and so

$$\alpha - \beta \leqslant \frac{1}{2}\gamma.$$

Since we can make $\gamma$ as small as desired, we can make $BF$ the same. $\square$

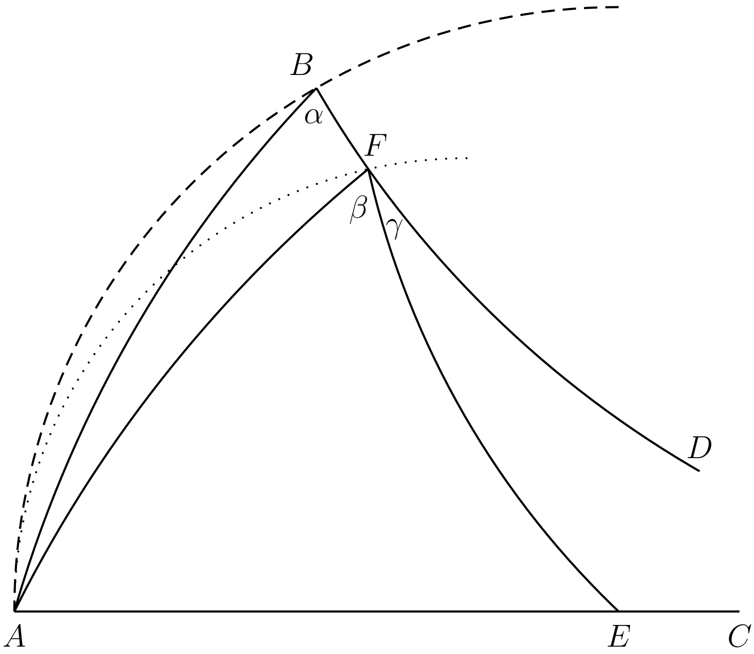


Figure 18: Proposition 32

Since any two segments of two horocycles are comparable, in the sense that one is congruent to a part or the whole of the other, the segments have a ratio.

When two parallel line segments are joined by segments of two horocycles of which they are axes, we may say that a **rectangle** is formed, whose **length** is that of either of the two line segments, and whose two **widths** are the lengths of the two segments of horocycles. Lobachevski does not use such terminology, but just assigns letters to the lengths that he wants to talk about.

## Proposition 33

*For some unit length, the ratio of the widths of any rectangle, the larger to the smaller, is the power of* e *by the ratio of the length of the rectangle to the unit length.*

*Proof.* Let $f_w(x)$ be the greater width of the rectangle of length $x$ and lesser width $w$. For all counting numbers $m$ and $n$, we have

$$f_{mw}(x) = mf_w(x), \qquad\qquad f_{(1/n)w} = \frac{1}{n}f_w(x),$$

and therefore

$$f_{(m/n)w}(x) = \frac{m}{n}f_w(x).$$

By "continuity" then, for all positive real numbers $t$,

$$f_{tw}(x) = tf_w(x),$$

and so for all other widths $w'$,

$$\frac{f_{w'}(x)}{w'} = \frac{f_w(x)}{w}.$$

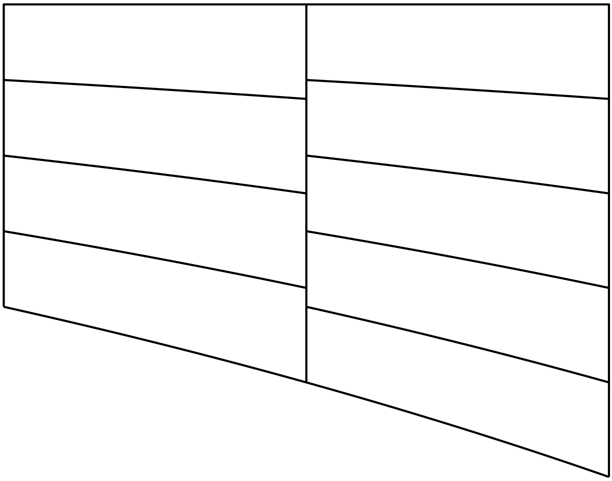


Figure 19: Proposition 33

We may now denote this common ratio by $g(x)$; it is the ratio of widths, greater to less, of any rectangle of length $x$. At least, in case this is a ratio of counting numbers, we have

$$g(mx) = g(x)^m, \qquad g\left(\frac{1}{n}x\right) = g(x)^{1/n},$$

and therefore

$$g\left(\frac{m}{n}x\right) = g(x)^{m/n}.$$

Figure 19 shows $g(2x) = g(x)^2$ where $g(x) = 5/4$. By "continuity" then, for all $x$ and for all positive real numbers $t$,

$$g(tx) = g(x)^t.$$

Since by "continuity" one more time, the equation $g(x) = \mathrm{e}$ is soluble by some length $u$, we have then

$$g(tu) = \mathrm{e}^t,$$

as claimed. $\square$

We extend the range of $\Pi$ to obtuse angles by defining

$$\Pi(x) + \Pi(-x) = \pi.$$

## Proposition 34

*When a horocycle is rotated about an axis, so as to generate a surface, if another axis of the horocycle is selected when this is in either of two positions, the two new axes are axes of a horocycle lying within the surface.*

*Proof.* When the horocycle $AB$, shown as a dashed line in Figure 20, is rotated about the axis $AA'$, let one of its positions be $AC$, and let the parallels $BB'$ and $CC'$ to $AA'$ be drawn. The chord $BC$ having midpoint $G$, we shall show $B'BG = GCC'$.

Within the plane $A'AB$, we erect a perpendicular bisector $DD'$ of $AB$; it is parallel to $AA'$ (and therefore to $BB'$). The angle between the planes $A'AB$ and $CAB$ being $\Pi(a)$ for some $a$, possibly 0 or negative, we erect, in the plane $ABC$, a perpendicular bisector $DF$ of $AB$ having directed length $a$, measured into the triangle. Then

$$AF = FB.$$

When $FF'$ is erected perpendicular to the plane $ABC$, it is parallel to $DD'$ and thus to $AA'$.

Within the plane $A'AC$, we erect a perpendicular bisector $EE'$ of $AC$; it is parallel to $AA'$ and thus to $FF'$. We erect also $EK$ perpendicular to the plane $ABC$. Then $AE$ is perpendicular to both $EE'$ and $EK$, which determine a plane in

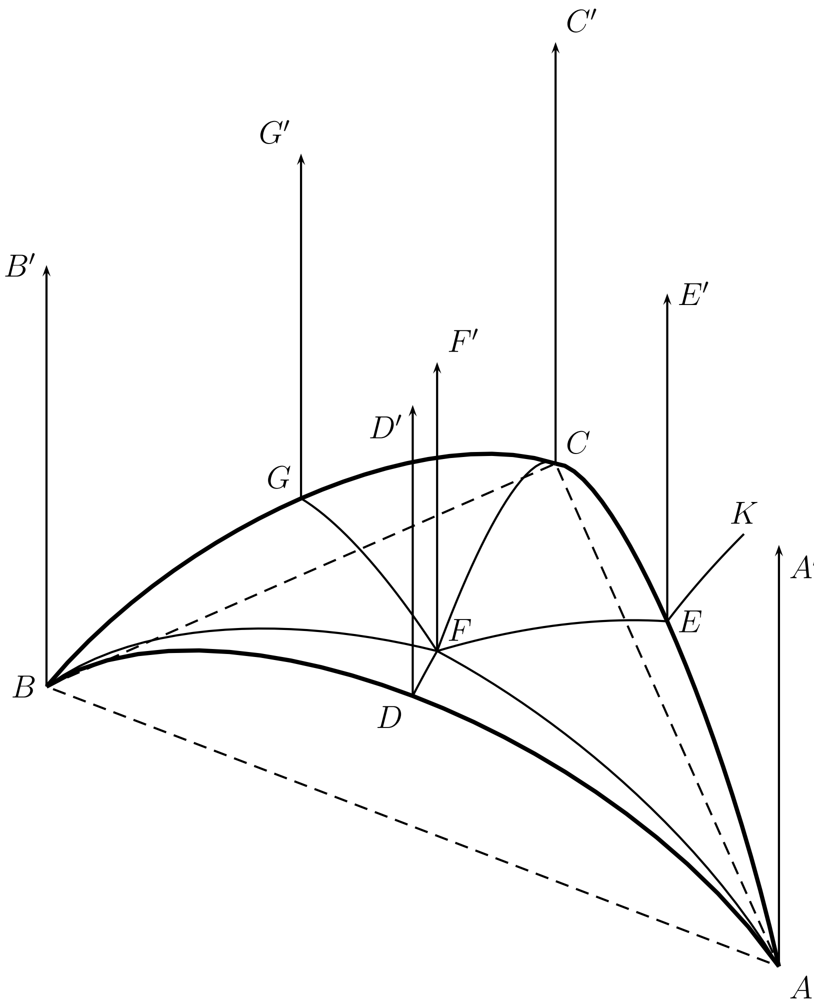


Figure 20: Proposition 34

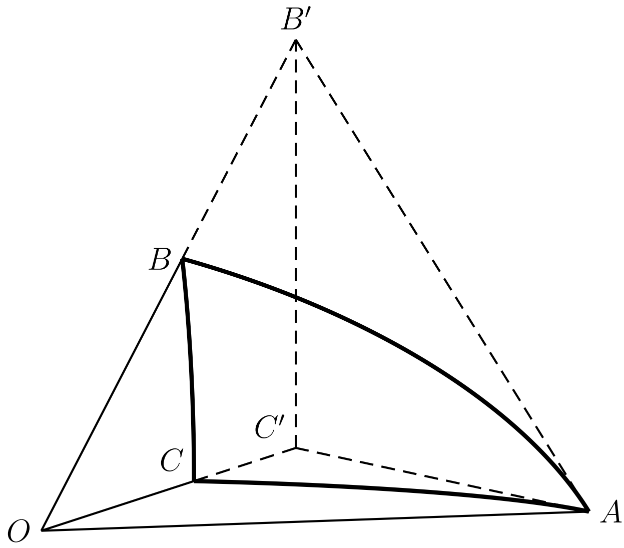



Figure 21: Euclidean spherical trigonometry

which $EF$ lies—assuming for the moment that $F$ is not the point $E$. Therefore $AE$ is perpendicular to $EF$. Thus $EF$ is a perpendicular bisector of $AC$, and so we have

$$CF = FA = BF.$$

We have this as well if $F$ should happen to be the point $E$.

Finally, letting $G$ be the midpoint of $BC$, we let the intersection of the planes $ABC$ and $B'BC$ be $GG'$. As $BB' \parallel FF'$, it follows that $GG' \parallel BB'$ and similarly $GG' \parallel CC'$. This yields the desired conclusion. $\square$

In Euclidean spherical trigonometry, one can reason as follows about the spherical right triangle $ABC$ in Figure 21, where the angle $C$ is right. (A reference is Todhunter, *Spher-*

*ical Geometry,* fifth edition [London: Macmillan, 1886], available from Project Gutenberg.) The center of the sphere being $O$, and the rectilineal triangle $AB'C'$ lying in a plane tangent to the sphere at $A$, so that the planar angles $AC'B'$, $OC'B'$, and $OAC'$ are right, we have

$$\frac{B'C'}{OB'} = \frac{B'C'}{B'N} \cdot \frac{B'N}{OB'}$$

which means

$$\sin a = \sin A \sin c,$$

and so by symmetry also

$$\sin b = \sin B \sin c.$$

Moreover,

$$\frac{OA}{OB'} = \frac{OA}{OC'} \cdot \frac{OC'}{OB'},$$

which means

$$\cos c = \cos b \cos a.$$

Therefore, since from

$$\frac{C'A}{B'A} = \frac{C'A}{OA} \cdot \frac{OA}{B'A}$$

we have

$$\cos A = \tan b \cot c = \frac{\sin b}{\sin c} \cdot \frac{\cos c}{\cos b},$$

we can conclude

$$\cos A = \sin B \cos a,$$

and by symmetry

$$\cos B = \sin A \cos b.$$

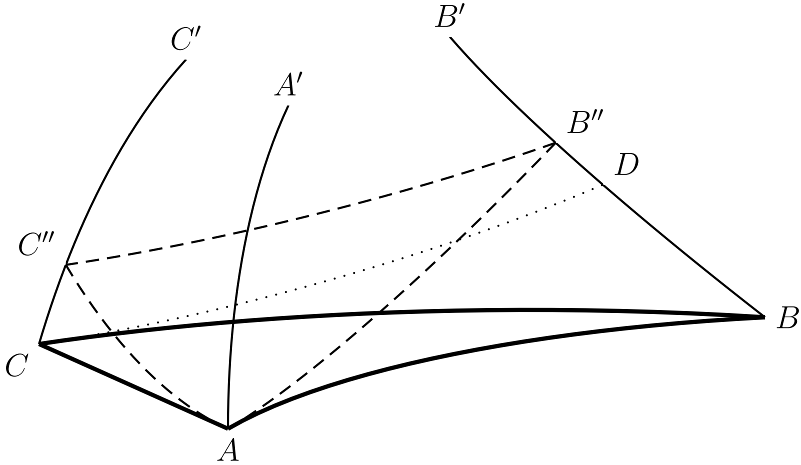



Figure 22: Proposition 35

## Proposition 35

*Spherical trigonometry is unchanged by the new postulate on parallels.*

*Proof.* Let $ABC$ be a triangle with right angle at $C$. As usual, the side opposite vertex $X$ has length $x$. For some additional lengths $\alpha$ and $\beta$, we have

$$\angle BAC = \Pi(\alpha), \qquad \angle ABC = \Pi(\beta).$$

Now we erect $AA'$ perpendicular to the plane of $ABC$, and we draw $BB'$ and $CC'$ parallel to $AA'$, as in Figure 22 (see the appendix on the actual drawing of the figure). We shall use the notation

$$\Pi(x') + \Pi(x) = \frac{\pi}{2}.$$

Thus the angles of the planes that meet, two by two, in the three parallels are $\pi/2$ at $CC'$, $\Pi(\alpha)$ at $AA'$, and therefore $\Pi(\alpha')$ at $BB'$, by Proposition 28.

Suppose now a sphere centered at $B$ cuts $BB'$, $BA$, and $BC$ respectively at $E$, $F$, and $G$. In the spherical triangle $EFG$,

$$\begin{aligned} g &= \Pi(c), & e &= \Pi(\beta), & f &= \Pi(a), \\ G &= \Pi(b), & E &= \Pi(\alpha'), & F &= \frac{\pi}{2}. \end{aligned}$$

Conversely, given a spherical triangle with these parameters, we can recover the planar triangle. In other words, for any ordered quintuple $(a, b, c, \alpha, \beta)$ of lengths, a right triangle $ABC$ exists as above, with sides and angles

$$\big(a, b, c, \Pi(\alpha), \Pi(\beta)\big),$$

if and only if a spherical right triangle $EFG$ exists as above, with sides and angles

$$\big(\Pi(c), \Pi(\beta), \Pi(a), \Pi(b), \Pi(\alpha')\big).$$

We can interchange the angles that are not right angles. Thus the existence of such a spherical right triangle as was just mentioned is equivalent to the existence of one with sides and angles

$$\big(\Pi(\beta), \Pi(c), \Pi(a), \Pi(\alpha'), \Pi(b)\big).$$

By what we already saw, the existence of this last spherical right triangle is equivalent to the existence of a right triangle with sides and angles

$$\big(a, \alpha', \beta, \Pi(b'), \Pi(c)\big),$$

simply because the correspondence

$$(a, b, c, \alpha, \beta) \leftrightarrow (c, \beta, a, b, \alpha')$$

can also be written as

$$(a, \alpha', \beta, b', c) \leftrightarrow (\beta, c, a, \alpha', b).$$

We now let $AA'$ be an axis of a horocycle through $A$, cutting $BB'$ and $B''$ and $CC'$ at $C''$. If we define

$$|B''C''| = p, \qquad |C''A| = q, \qquad |AB''| = r,$$

then

$$p = r \sin \Pi(\alpha), \qquad q = r \cos \Pi(\alpha),$$

by Proposition 34. If parallels such as $AA'$ and $CC'$ are given, and at a point $A$ on one of them a perpendicular is erected, cutting the other parallel cuts this at $C$, while the horocycle passing through $A$ (and having the parallels as axes) cuts the other parallel at $C''$, then the length of $CC''$ and the arc length of $AC''$ are determined by the length of $AC$ alone. Thus we have functions $f$ and $g$ given by

$$f(b) = |CC''|, \qquad g(b) = q.$$

Then also

$$f(c) = |BB''|, \qquad g(c) = r.$$

Now we let a new horosphere with axis $CC''$, passing through $C$, cut $BB''$ at $D$, so that

$$BB'' = BD + DB'' = BD + CC'';$$

and we define

$$|CD| = t.$$

Then

$$f(a) = |BD|, \qquad\qquad g(a) = t.$$

Consequently

$$f(c) = f(a) + f(b).$$

Also, considering the rectangle $CDB''C''$, we have

$$t = p\mathrm{e}^{f(b)} = r \sin \Pi(\alpha)\mathrm{e}^{f(b)},$$

and thus

$$g(a) = g(c) \sin \Pi(\alpha)\mathrm{e}^{f(b)}.$$

By symmetry,

$$g(b) = g(c) \sin \Pi(\beta)\mathrm{e}^{f(a)},$$

and this gives us

$$\cos \Pi(\alpha) = \sin \Pi(\beta)\mathrm{e}^{f(a)}.$$

By the transformation discussed earlier,

$$\sin \Pi(b) = \cos \Pi(b') = \sin \Pi(c)\mathrm{e}^{f(a)},$$
$$\sin \Pi(b)\mathrm{e}^{f(b)} = \sin \Pi(c)\mathrm{e}^{f(a)}\mathrm{e}^{f(b)} = \sin \Pi(c)\mathrm{e}^{f(c)},$$

and therefore also, by symmetry again,

$$\sin \Pi(a)\mathrm{e}^{f(a)} = \sin \Pi(b)\mathrm{e}^{f(b)}.$$

Since $a$ and $b$ are independent of one another, and $\sin \Pi(a)\mathrm{e}^{f(a)}$ approaches the limit of 1 at 0, we can conclude

$$\mathrm{e}^{-f(a)} = \sin \Pi(a).$$

Thus from equations found earlier we obtain

$$\sin \Pi(\beta) = \sin \Pi(a) \cos \Pi(\alpha),$$
$$\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b).$$

By symmetry, the former yields

$$\sin \Pi(\alpha) = \sin \Pi(b) \cos \Pi(\beta).$$

From this, by the more elaborate permutation,

$$\cos \Pi(b) = \cos \Pi(\alpha) \cos \Pi(c). \tag{0.1}$$

Finally, by interchanging $a$ with $b$ and $\alpha$ with $\beta$ again,

$$\cos \Pi(a) = \cos \Pi(\beta) \cos \Pi(c).$$

If we relabel the earlier spherical triangle $GEF$ as $ABC$, then the five equations that we have found become

$$\begin{aligned} \sin b &= \sin c \sin B, \\ \sin a &= \sin c \sin A, \\ \cos B &= \sin A \cos b, \\ \cos A &= \sin B \cos a, \\ \cos c &= \cos b \cos a. \end{aligned}$$

□

## Proposition 36

*For some unit length, the power of* e *by the ratio to that length of any length is the reciprocal of the tangent of half the angle of parallelism of that length:*

$$\tan \frac{\Pi(x)}{2} = \mathrm{e}^{-x}.$$

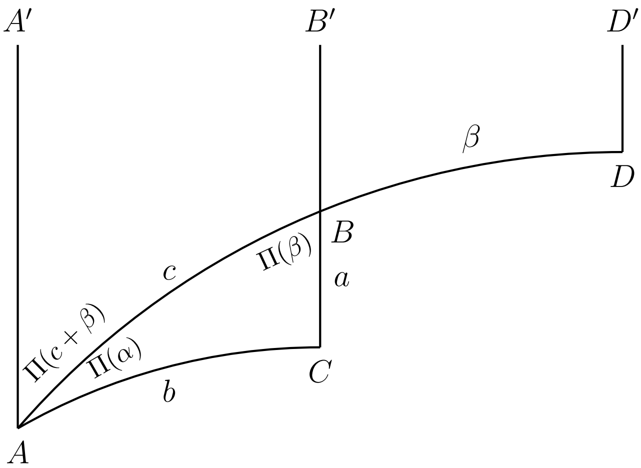



Figure 23: Theorem 36, first case

*Proof.* We are given triangle $ABC$ as in Figure 23; as in Proposition 35, the angle at $C$ is right, and for some *distances* $\alpha$ and $\beta$, the angles at $A$ and $B$ are $\Pi(\alpha)$ and $\Pi(\beta)$ respectively. The sides opposite $A$, $B$, and $C$ are $a$, $b$, and $c$ respectively. We first extend $AB$ by the distance $\beta$ to $D$. Then the perpendicular $DD'$ to $AD$ is parallel to $CB$ (which is in turn extended to $B'$). If the parallel $AA'$ to $BB'$ is also drawn, then, considering that $AA'$ is parallel to two different straight lines to which perpendiculars are dropped from $A$, we have

$$\Pi(b) = \Pi(\alpha) + \Pi(c + \beta). \tag{0.2}$$

We derive a related equation by measuring $\beta$ along $BA$ in the other direction. There are three possibilities here. If $\beta < c$, we have Figure 24, from which we can infer

$$\Pi(c - \beta) = \Pi(\alpha) + \Pi(b). \tag{0.3}$$

In case $\beta = c$, the diagram is as in Figure 25a, and then

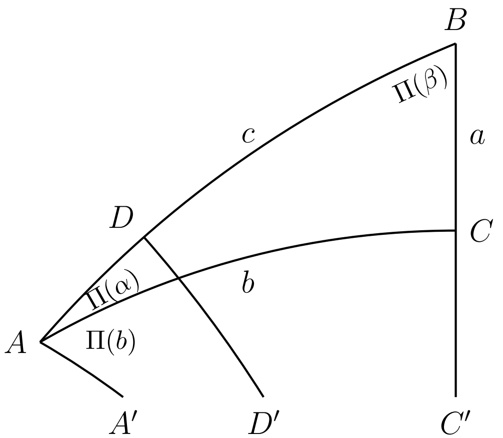



Figure 24: Theorem 36, second case, when $\beta < c$

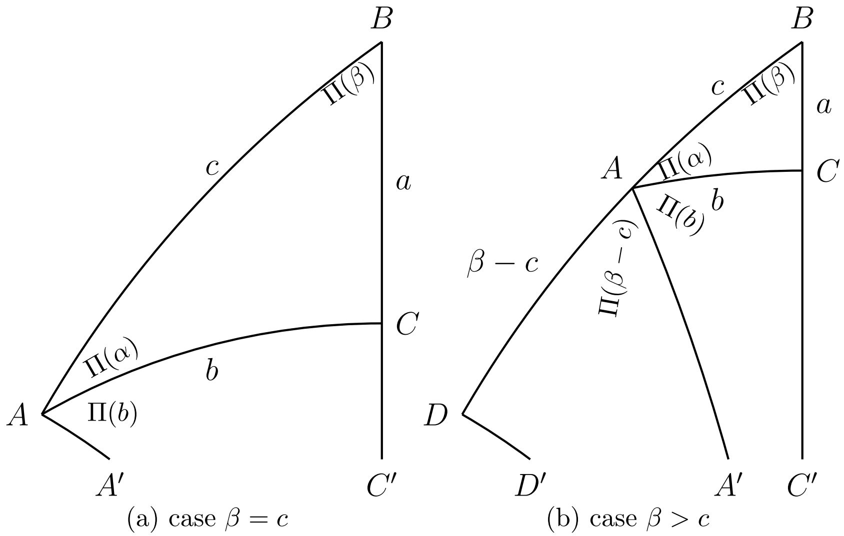



Figure 25: Theorem 36, second case, when $\beta \geqslant c$

$$\Pi(\alpha) + \Pi(b) = \frac{1}{2}\pi;$$

but now $\Pi(c-\beta) = \Pi(0) = \pi/2$ by definition; so again (0.3) holds. Finally, if $\beta > c$, then as in Figure 25b,

$$\Pi(\beta - c) + \Pi(b) + \Pi(\alpha) = \pi,$$

so (0.3) still holds since $\pi - \Pi(\beta - c) = \Pi(c-\beta)$ by definition.

From the system of (0.2) and (0.3), we obtain

$$\begin{aligned} 2\Pi(b) &= \Pi(c-\beta) + \Pi(c+\beta), \\ 2\Pi(\alpha) &= \Pi(c-\beta) - \Pi(c+\beta), \end{aligned}$$

which yield immediately

$$\frac{\cos \Pi(b)}{\cos \Pi(\alpha)} = \frac{\cos\left(\frac{1}{2}\Pi(c-\beta) + \frac{1}{2}\Pi(c+\beta)\right)}{\cos\left(\frac{1}{2}\Pi(c-\beta) - \frac{1}{2}\Pi(c+\beta)\right)}.$$

Now, from the proof of Theorem 35, we use (0.1) to obtain

$$\cos \Pi(c) = \frac{\cos\left(\frac{1}{2}\Pi(c-\beta) + \frac{1}{2}\Pi(c+\beta)\right)}{\cos\left(\frac{1}{2}\Pi(c-\beta) - \frac{1}{2}\Pi(c+\beta)\right)}. \tag{0.4}$$

From this, we shall obtain

$$\left(\tan \frac{\Pi(c)}{2}\right)^2 = \tan \frac{\Pi(c-\beta)}{2} \cdot \tan \frac{\Pi(c+\beta)}{2}. \tag{0.5}$$

Lobachevski does not give a derivation, but if we write (0.4) as

$$\cos\theta = \frac{\cos(\varphi+\psi)}{\cos(\varphi-\psi)}, \tag{0.6}$$

then, since

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta},$$

so that

$$\left(\tan\frac{\theta}{2}\right)^2 = \frac{1+\cos\theta}{1-\cos\theta},$$

we obtain from (0.6)

$$\left(\tan\frac{\theta}{2}\right)^2 = \frac{\cos(\varphi-\psi)-\cos(\varphi+\psi)}{\cos(\varphi-\psi)+\cos(\varphi+\psi)} = \frac{\sin\varphi\cdot\sin\psi}{\cos\varphi\cdot\cos\psi}$$
$$= \tan\varphi\cdot\tan\psi.$$

Now Lobachevski proposes replacing $\beta$ with $c$, $2c$, $3c$, and so forth. One can do this; that is, one can use induction to obtain

$$\left(\tan\frac{\Pi(c)}{2}\right)^n = \tan\frac{\Pi(nc)}{2}.$$

But it seems neater to me to rewrite (0.5) as

$$\frac{\tan(\Pi(c)/2)}{\tan(\Pi(c-\beta)/2)} = \frac{\tan(\Pi(c+\beta)/2)}{\tan(\Pi(c)/2)};$$

for then

$$\left(\tan\frac{\Pi(c)}{2}\right)^n = \prod_{k=1}^{n}\frac{\tan\bigl(\Pi(kc)/2\bigr)}{\tan\Bigl(\Pi\bigl((k-1)c\bigr)/2\Bigr)} = \tan\frac{\Pi(nc)}{2}$$

since $\tan\bigl(\Pi(0)/2\bigr) = 1$. By continuity,

$$\left(\tan\frac{\Pi(c)}{2}\right)^t = \tan\frac{\Pi(tc)}{2}.$$

for all positive real numbers $t$. For some unit length $u$ we have

$$\tan\frac{\Pi(u)}{2} = \mathrm{e}^{-1},$$

and then

$$\tan\frac{\Pi(tu)}{2} = \mathrm{e}^{-t}. \qquad \square$$

## Proposition 37

*[Equations for solving arbitrary planar triangles.]*

# Drawing a figure

I record here the process of reproducing, with precision, the diagrams of Lobachevski, and specifically his Figure 28, used for [Theorem] 35. Drawing such a figure tests the computational limits, both of myself when working by hand, and of the PostScript program. The figure shows a rectilineal triangle $ABC$ in which the angle at $C$ is right. The line $AA'$ is perpendicular to the plane of $ABC$, and $BB'$ and $CC'$ parallel to $AA'$. The three parallels are axes of a horosphere passing through $A$, and the horosphere cuts $BB'$ and $CC'$ at $B''$ and $C''$ respectively. The solid angle at $B$ bounded by $BA$, $BC$, and $BB'$ is also considered, as is the horocycle with axis $CC'$ passing through $C$.

## The triangle

I propose to depict $ABC$ as being flat to us, the leg $AC$ being straight to us. This leg then should be perpendicular the bounding plane of a Poincaré half-plane. I take that plane to be the $yz$ plane, in the coordinate system of Figure 26, and $ABC$ can be as shown. Thus $AC$ is along the $x$ axis, and $BC$ is an arc centered at the origin. Thus there are positive parameters $a$, $b$, and $c$ such that

$$A = (a, 0, 0), \qquad C = (c, 0, 0), \qquad B = (\sqrt{c^2 - b^2}, -b, 0).$$

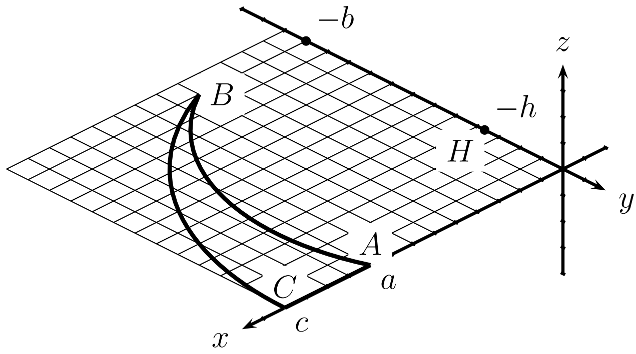



Figure 26: The triangle

Also $AB$ is an arc centered at a point $H$ on the $y$ axis. We may let

$$H = (0, -h, 0),$$

and then

$$a^2 + h^2 = c^2 - b^2 + (b - h)^2,$$
$$h = \frac{c^2 - a^2}{2b}.$$

The circle of which $AB$ is an arc is given by

$$x^2 + (y + h)^2 = a^2 + h^2,$$
$$x^2 = a^2 - 2hy - y^2,$$

so that, as an arc,

$$AB = \left\{ \left( \sqrt{a^2 + t(c^2 - a^2) - t^2 b^2}, -bt, 0 \right) : 0 \leqslant t \leqslant 1 \right\}.$$

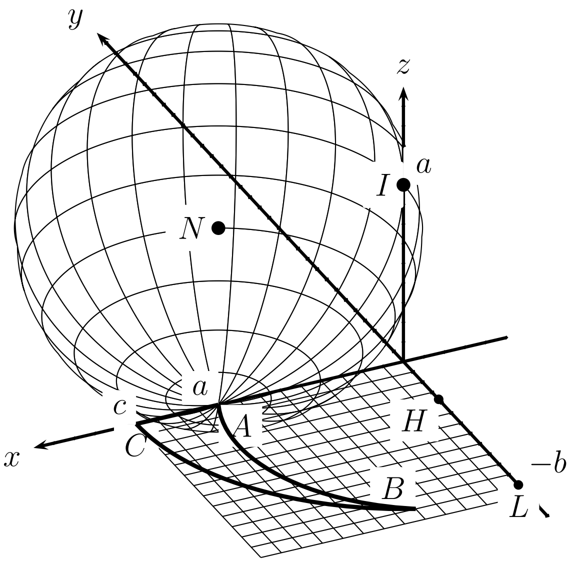



Figure 27: The horosphere

## The horosphere

The horosphere tangent at $A$ to $ABC$ is the real sphere of radius $a$ having center at $N$, where

$$N = (a, 0, a),$$

as in Figure 27. This sphere is tangent to the $yz$ plane at a point $I$, and so

$$I = (0, 0, a).$$

## The easy parallel

The name of the point $I$ can be understood to stand for "infinity." The parallels $AA'$, $BB'$, and $CC'$ at the vertices of $ABC$

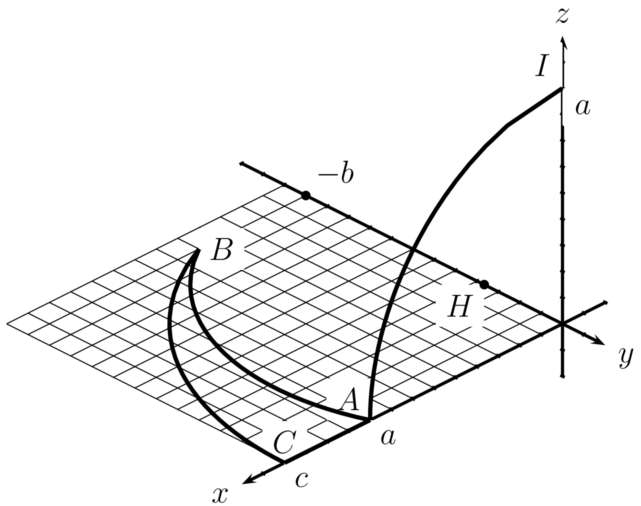



Figure 28: The easy parallel

will for us be arcs of circles passing through $I$, with centers on the $yz$ plane, in planes parallel to the $x$ axis. We shall concentrate on the arcs $AI$, $BI$, and $CI$; if they are all parametrized similarly, then they can be cut off uniformly at the points $A'$, $B'$, and $C'$.

The arcs $AI$ and $CI$ have centers on the $z$ axis. The center of $AI$ is just the origin, so the arc is as in Figure 28.

## The middle parallel

A direction vector of the line $CI$ is $(c, 0, -a)$, and the midpoint of $CI$ is $(c/2, 0, a/2)$, and so the perpendicular bisector of $CI$

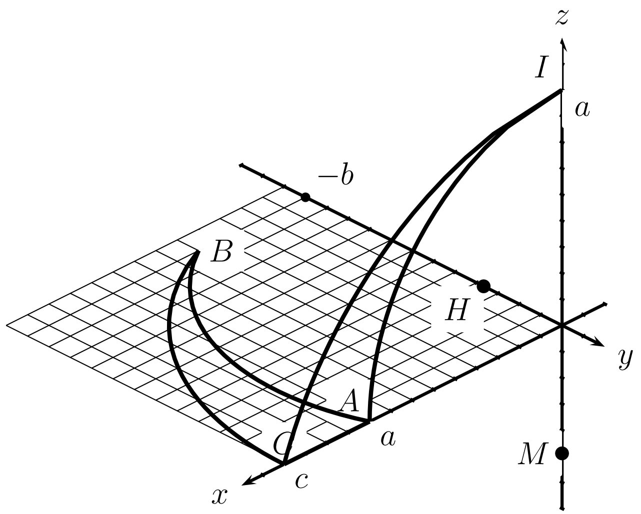


Figure 29: The middle parallel

is

$$c\left(x-\frac{c}{2}\right)-a\left(z-\frac{a}{2}\right)=0,$$
$$2cx-2az=c^2-a^2,$$

which cuts the $z$ axis at $-(c^2-a^2)/2a$. Thus the center of the arc $CI$ is $M$, given by

$$M=\left(0,0,-\frac{c^2-a^2}{2a}\right),$$

as in Figure 29. The radius $MI$ is then $a+(c^2-a^2)/2a$, so the points of the arc $CI$ are thus

$$\left(\sqrt{\left(a+\frac{c^2-a^2}{2a}\right)^2-\left(ta+\frac{c^2-a^2}{2a}\right)^2},0,ta\right),$$

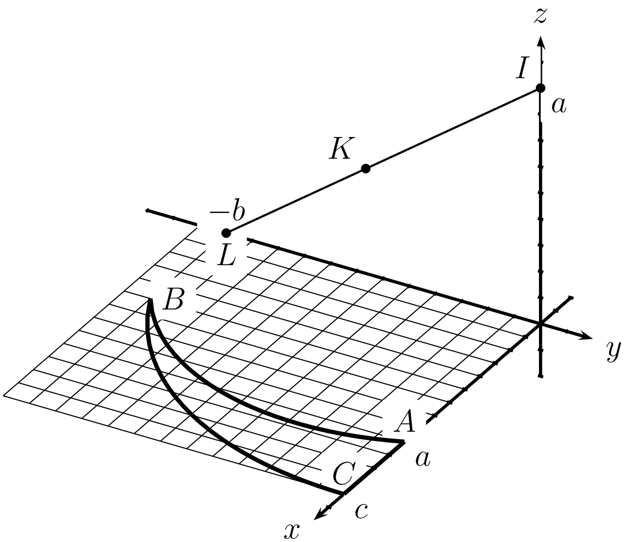



Figure 30: Setup for the hard parallel

where $0 \leqslant t \leqslant 1$. The radicand simplifies to

$$a^2 + c^2 - a^2 - t^2a^2 - t(c^2 - a^2),$$
$$c^2 - t(c^2 - a^2) - t^2a^2,$$

so that

$$CI = \left\{ \left( \sqrt{c^2 - t(c^2 - a^2) - t^2a^2}, 0, ta \right) : 0 \leqslant t \leqslant 1 \right\}.$$

## The hard parallel

The arc $BI$ has center $K$ on the line through $I$ and the foot $L$ of the perpendicular dropped from $B$ to the $yz$ plane, as in Figure 30. Thus

$$K = (0, -k_1, k_2), \qquad L = (0, -b, 0),$$

and the line $IL$ is given by

$$x = 0, \qquad\qquad ay - bz = -ab.$$

To find the parameters $k_1$ and $k_2$, we observe that a direction vector and the midpoint of $BI$ are respectively

$$(\sqrt{c^2 - b^2}, -b, -a), \qquad\qquad \left(\frac{\sqrt{c^2 - b^2}}{2}, -\frac{b}{2}, \frac{a}{2}\right),$$

so the perpendicular bisecting plane is

$$\sqrt{c^2 - b^2}\left(x - \frac{\sqrt{c^2 - b^2}}{2}\right) - b\left(y + \frac{b}{2}\right) - a\left(z - \frac{a}{2}\right) = 0,$$
$$2\sqrt{c^2 - b^2}x - 2by - 2az = c^2 - a^2.$$

Thus we obtain $K$ by solving simultaneously

$$\begin{cases} -ay & + & bz & = & ab \\ -2by & - & 2az & = & c^2 - a^2. \end{cases}$$

We may use Cramer's Rule:

$$\begin{vmatrix} ab & b \\ c^2 - a^2 & -2a \end{vmatrix} = -b\begin{vmatrix} a & -1 \\ c^2 - a^2 & 2a \end{vmatrix} = -b(a^2 + c^2),$$
$$\begin{vmatrix} -a & ab \\ -2b & c^2 - a^2 \end{vmatrix} = a\begin{vmatrix} 1 & b \\ -2b & a^2 - c^2 \end{vmatrix} = a(a^2 + 2b^2 - c^2),$$
$$\begin{vmatrix} -a & b \\ -2b & -2a \end{vmatrix} = 2\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = 2(a^2 + b^2),$$

and so

$$k_1 = \frac{b(a^2 + c^2)}{2(a^2 + b^2)}, \qquad\qquad k_2 = \frac{a(a^2 + 2b^2 - c^2)}{2(a^2 + b^2)}.$$

The point $(0,-k_1,k_2)$ or $K$ lies on the *ray* $IL$. The segment $LI$ consists of the points

$$\big(0,(t-1)b,ta\big),$$

where $0 \leqslant t \leqslant 1$. The arc from $B$ to $I$ with center at $K$ has the radius $r$, this being the distance from $B$ to $K$ and so satisfying

$$\begin{aligned} r^2 &= c^2 - b^2 + (b-k_1)^2 + {k_2}^2 \\ &= c^2 - 2bk_1 + {k_1}^2 + {k_2}^2. \end{aligned}$$

The points of the arc $IB$ are thus

$$\left(\sqrt{r^2 - \big((1-t)b-k_1\big)^2 - (ta-k_2)^2}, (t-1)b, ta\right).$$

The radicand simplifies to

$$\begin{gathered} c^2 - 2bk_1 - (1-t)^2b^2 + 2(1-t)bk_1 - t^2a^2 + 2tak_2, \\ c^2 - 2bk_1 - b^2 + 2tb^2 - t^2b^2 + 2bk_1 - 2tbk_1 - t^2a^2 + 2tak_2, \\ c^2 - b^2 + 2t(b^2 - bk_1 + ak_2) - t^2(a^2+b^2), \end{gathered}$$

and here

$$\begin{aligned} 2(bk_1 - ak_2) &= \frac{b^2(a^2+c^2)}{a^2+b^2} - \frac{a^2(a^2+2b^2-c^2)}{a^2+b^2} \\ &= \frac{a^2b^2 + b^2c^2 - a^4 - 2a^2b^2 + a^2c^2}{a^2+b^2} \\ &= \frac{a^2c^2 + b^2c^2 - a^4 - a^2b^2}{a^2+b^2} = c^2 - a^2, \end{aligned}$$

so that

$$\begin{aligned} 2(b^2 - bk_1 + ak_2) &= a^2 + 2b^2 - c^2 \\ &= a^2 + b^2 - (c^2 - b^2), \end{aligned}$$

and therefore the radicand is

$$\begin{gathered}c^2 - b^2 + t\big(a^2 + b^2 - (c^2 - b^2)\big) - t^2(a^2 + b^2),\\ (1 - t)\big((c^2 - b^2) + t(a^2 + b^2)\big).\end{gathered}$$

Thus

$$BI = \left\{ \left( \sqrt{(1-t)\big((c^2 - b^2) + t(a^2 + b^2)\big)}, (t-1)b, ta \right) : \right.$$
$$\left. 0 \leqslant t \leqslant 1 \right\},$$

as in Figure 31.

## The easy plane

We now want to know where the horosphere with center $I$ passing through $A$ cuts the three planes $ACI$, $CBI$, and $BAI$. Again, the points where the horosphere cuts the three parallels $AA'$, $BB'$, and $CC'$ are $A$, $B''$, and $C''$ respectively. The plane of $ACI$ is in the $xz$ plane, and the intersection of the horosphere with this plane is given by

$$\begin{gathered}(x - a)^2 + (z - a)^2 = a^2,\\ x^2 - 2ax = -a^2 + 2az - z^2,\end{gathered}$$

while by our earlier computations the circle of $CI$ is given by

$$x^2 = c^2 - \frac{c^2 - a^2}{a} z - z^2.$$

To find $C''$, we eliminate the squares, obtaining

$$2ax = a^2 + c^2 - \left(2a + \frac{c^2 - a^2}{a}\right) z = \frac{a^2 + c^2}{a}(a - z),$$

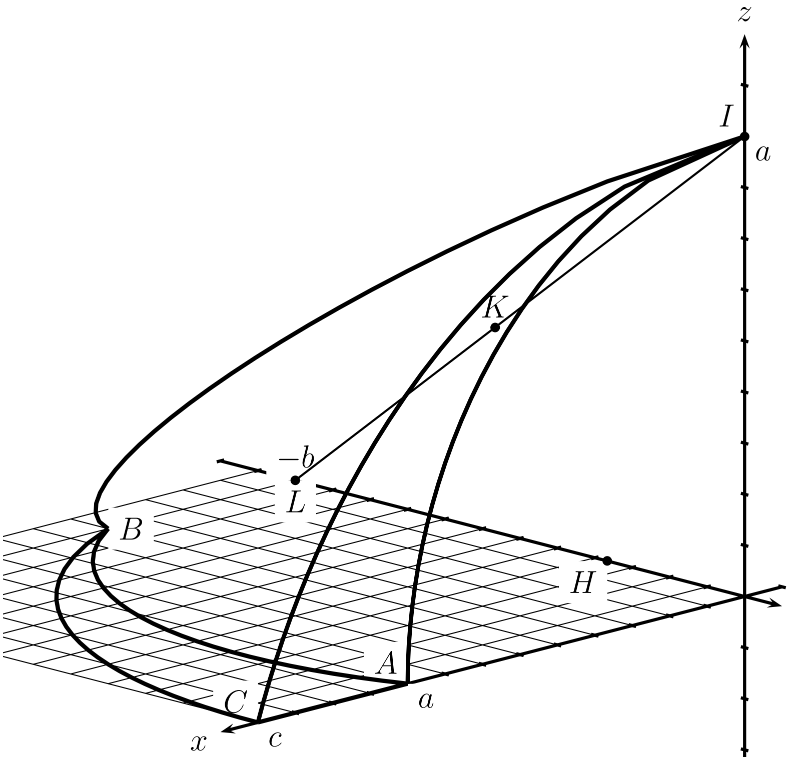



Figure 31: All of the parallels

which defines (with $y = 0$) a straight line through $(0, 0, a)$, as we already expect. Plugging into the first equation for a circle, and assuming $x \neq 0$, we have

$$\begin{aligned} a^2 &= (x-a)^2 + \frac{4a^4}{(a^2+c^2)^2} x^2 \\ &= \left(1 + \frac{4a^4}{(a^2+c^2)^2}\right) x^2 - 2ax + a^2, \\ 2a &= \left(1 + \frac{4a^4}{(a^2+c^2)^2}\right) x, \\ x &= \frac{2a(a^2+c^2)^2}{(a^2+c^2)^2 + 4a^4}. \end{aligned}$$

Thus

$$AC'' = \left\{ \left(t, 0, a - \sqrt{2at - t^2}\right) : a \leqslant t \leqslant \frac{2a(a^2+c^2)^2}{(a^2+c^2)^2 + 4a^4} \right\},$$

as in Figure 32.

## The middle plane

The arc $AB''$ lies in the intersection of two spheres:

(1) the horosphere, which is the sphere with center $N$ passing through $A$, and therefore also $I$, given by

$$(x-a)^2 + y^2 + (z-a)^2 = a^2,$$

and

(2) the sphere in which the arcs $AB$ and $AI$ lie, which is just the sphere with center $H$ passing through $A$.

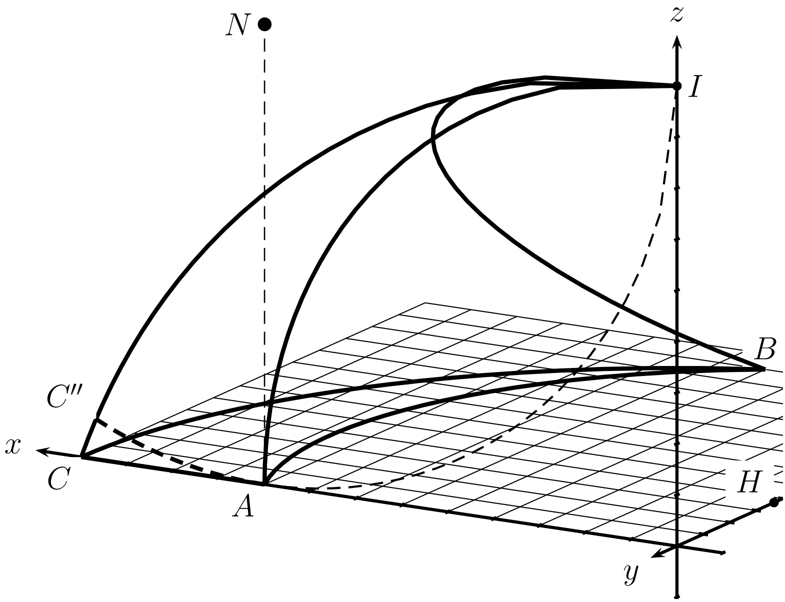



Figure 32: The easy plane

The intersection of the two spheres is then a circle lying in the plane through $A$ normal to $HN$ and thus to $(a, h, a)$. The plane of the circle is thus given by

$$a(x-a) + hy + az = 0.$$

Using this to eliminate $x - a$ from the equation of the horosphere, we obtain

$$\begin{aligned}(az + hy)^2 + a^2y^2 + a^2(z-a)^2 &= a^4,\\ 2a^2z^2 + 2a(hy - a^2)z + (a^2+h^2)y^2 &= 0,\\ 2az = a^2 - hy &\mp \sqrt{(hy-a^2)^2 - 2(a^2+h^2)y^2}\\ = a^2 - hy &\mp \sqrt{a^4 - 2a^2hy - (2a^2+h^2)y^2}.\end{aligned}$$

Rewriting the plane equation as

$$2ax = 2(a^2 - hy) - 2az,$$

from the expression for $2az$ we obtain

$$2ax = a^2 - hy \pm \sqrt{a^4 - 2a^2hy - (2a^2+h^2)y^2}.$$

Now we know the points of the circle in terms of their projections onto the $y$ axis. Along the arc $AB''$, the value of $z$ is the less of the two found. For the range of $y$, we need the $y$ coordinate of $B''$. The point $B''$ itself is at the intersection of the circle with the arc $BI$. This arc lies in a plane with $L$, and so this plane is given by

$$\begin{aligned}-ay + bz &= ab,\\ bz &= ab + ay.\end{aligned}$$

We use this in the earlier expression for $2az$, which we first rewrite as

$$2abz = a^2b - bhy \mp \sqrt{a^4b^2 - 2a^2b^2hy - (2a^2 + h^2)b^2y^2}.$$

Thus the $y$ coordinate of $B''$ satisfies

$$\begin{aligned} 2a^2(b+y) &= a^2b - bhy \mp \sqrt{a^4b^2 - 2a^2b^2hy - (2a^2 + h^2)b^2y^2}, \\ \big(a^2b + (2a^2 + bh)y\big)^2 &= a^4b^2 - 2a^2b^2hy - (2a^2 + h^2)b^2y^2, \end{aligned}$$

Performing all of the multiplications gives

$$\begin{aligned} a^4b^2 + 4a^4by + 2a^2b^2hy + 4a^4y^2 + 4a^2bhy^2 + b^2h^2y^2 & \\ = a^4b^2 - 2a^2b^2hy - 2a^2b^2y^2 - b^2h^2y^2, & \end{aligned}$$

and so

$$\begin{aligned} (4a^4 + 2a^2b^2 + 4a^2bh + 2b^2h^2)y^2 + (4a^4b + 4a^2b^2h)y = 0, \\ y = -2a^2b\frac{a^2 + bh}{2a^4 + a^2b^2 + 2a^2bh + b^2h^2} \end{aligned}$$

since $y \neq 0$. The expression seems to overwhelm PostScript or else my own powers of programming it. Since $h$ is a derived parameter, or an dependent variable, and $2bh = c^2 - a^2$, we have

$$\begin{aligned} y &= -2a^2b\frac{4a^2 + 2(c^2 - a^2)}{8a^4 + 4a^2b^2 + 4a^2(c^2 - a^2) + (c^2 - a^2)^2} \\ &= -4a^2b\frac{a^2 + c^2}{5a^4 + 4a^2b^2 + 2a^2c^2 + c^4}. \end{aligned}$$

This lets us obtain $AB''$ as in Figure 33.

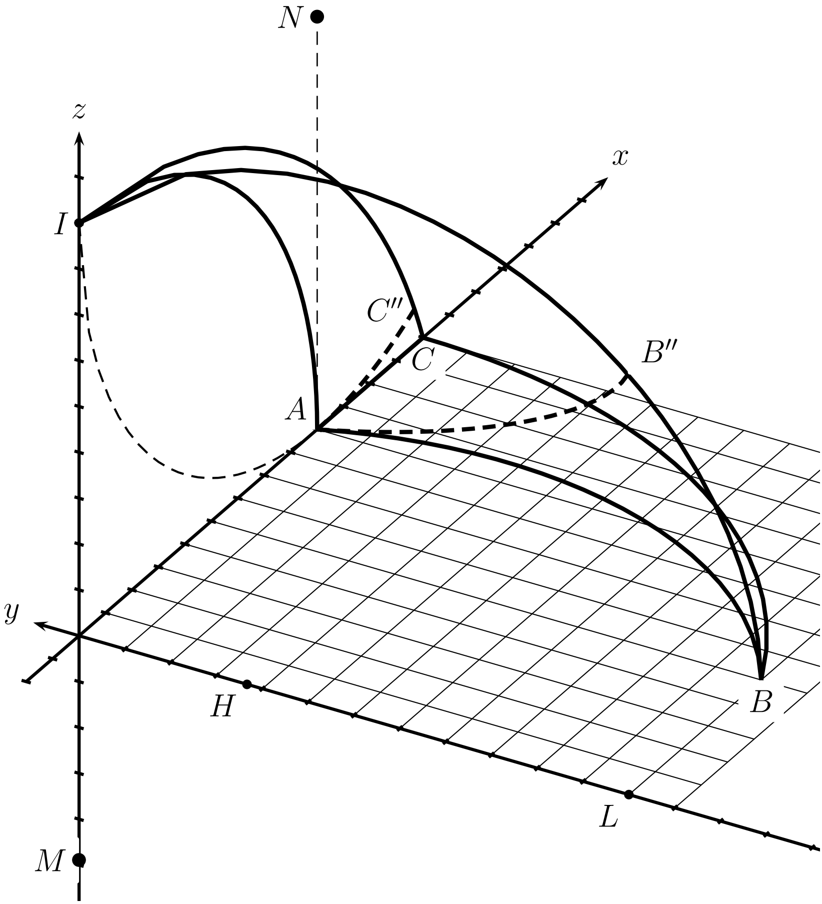



Figure 33: The middle plane

## The third plane

Like $AC''$, $C''B''$ is the intersection of two circles with centers in the $xz$ plane: again the horosphere, whose center is $N$ and that passes through $I$; but now also the sphere with center $M$ that passes through $I$. Since the $z$ coordinate of $M$ is $-(c^2-a^2)/2a$, the plane of intersection of the spheres is

$$ax + \left(a + \frac{c^2-a^2}{2a}\right)(z-a) = 0$$

or rather

$$2a^2x + (a^2+c^2)(z-a) = 0.$$

Eliminating $x$ from the equation of the horosphere gives

$$\begin{gathered}\big((a^2+c^2)(z-a)+2a^3\big)^2 + 4a^4\big(y^2+(z-a)^2\big) = 4a^6,\\ \big((a^2+c^2)^2+4a^4\big)(z-a)^2 + 4a^3(a^2+c^2)(z-a) + 4a^4y^2 = 0,\end{gathered}$$

and so

$$\begin{aligned}&\frac{z-a}{2}\\ &= \frac{-a^3(a^2+c^2) \mp \sqrt{a^6(a^2+c^2)^2 - a^4\big((a^2+c^2)^2+4a^4\big)y^2}}{(a^2+c^2)^2+4a^4}\\ &= a^2\frac{-a(a^2+c^2) \mp \sqrt{a^2(a^2+c^2)^2 - \big((a^2+c^2)^2+4a^4\big)y^2}}{(a^2+c^2)^2+4a^4}\\ &\hfill = a^2\Phi,\end{aligned}$$

say; from the plane equation then,

$$x = -(a^2+c^2)\Phi.$$

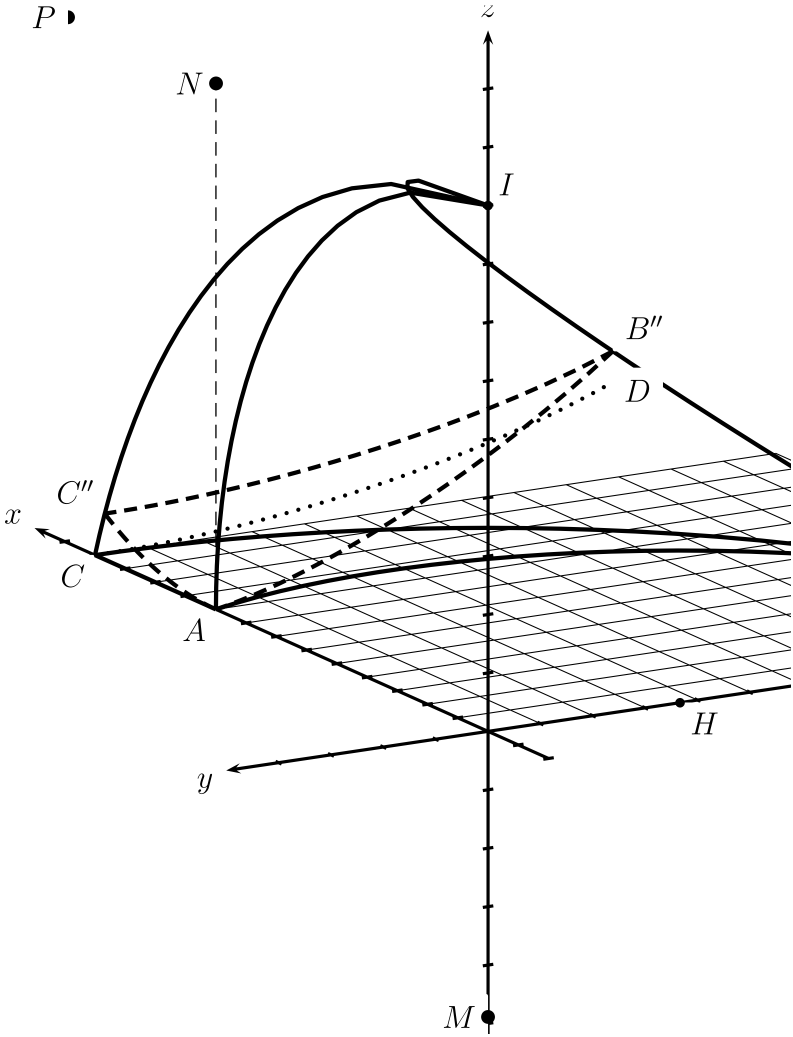



Figure 34: The third plane

Since we already know where $B''$ itself is, we can draw $C''B''$ as in Figure 34.

Finally, we consider a new horocycle, still tangent to the $yz$ plane at $I$, but now passing through $C$. The $x$ coordinate of its center $P$ is thus given by

$$x^2 = (c-x)^2 + a^2,$$
$$2cx = a^2 + c^2,$$

so

$$P = \left(\frac{a^2+c^2}{2c}, 0, a\right),$$

and the new horosphere itself is given by

$$\left(x - \frac{a^2+c^2}{2c}\right)^2 + y^2 + (z-a)^2 = \left(\frac{a^2+c^2}{2c}\right)^2.$$

This cuts $BI$ at a point $D$, and we want to draw the arc $CD$ in the intersection of the new horosphere with the sphere having center $M$ and also passing through $I$ (and $C$). Thus $CD$ is an arc of a circle in the plane given by

$$\frac{a^2+c^2}{2c}x + \frac{a^2+c^2}{2a}(z-a) = 0,$$
$$ax + cz = ac.$$

Eliminating now $z-a$ from the horosphere equation gives

$$\left(2cx - (a^2+c^2)\right)^2 + 4c^2y^2 + 4a^2x^2 = (a^2+c^2)^2,$$
$$4(a^2+c^2)x^2 - 4c(a^2+c^2)x + 4c^2y^2 = 0,$$
$$x = c\frac{a^2+c^2 \pm \sqrt{(a^2+c^2)^2 - 4(a^2+c^2)y^2}}{2(a^2+c^2)} = c\Psi,$$
$$z - a = -a\Psi.$$

The point $D$ lies as before in the plane $BIL$, given by

$$b(z-a)=ay,$$

and so the $y$ coordinate of $D$ satisfies

$$\begin{aligned} y &= -b\Psi, \\ 2y &= -b \mp b\frac{\sqrt{(a^2+c^2)^2-4(a^2+c^2)y^2}}{a^2+c^2}, \\ (a^2+c^2)^2(2y+b)^2 &= b^2\big((a^2+c^2)^2-4(a^2+c^2)y^2\big), \\ (a^2+c^2)(a^2+b^2+c^2)y^2+(a^2+c^2)^2by &= 0, \\ y &= -b\frac{a^2+c^2}{a^2+b^2+c^2}. \end{aligned}$$