Prime Numbers

A Nesin Mathematics Village course

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Preface

Here are notes for the second offering of my course on the Prime Number Theorem. In preparing and delivering my lectures, I used the typeset notes from the first offering. I had arranged those according to topic, though the Preface described what I had done each day. The present notes are given by day and cover only what I actually talked about in class. Therefore the earlier notes are still useful.

The second time around, on the first day of my course, I was given a list of 47 registered students. I was told that more might attend, unofficially. I did not count, but there were a lot of students in the audience of the Nişanyan Library. Their numbers shrunk, day by day. On the seventh day, there was one student, and I talked with him about algebraic number theory, since he expressed curiosity about number theory in general. Other students from previous days were still in the Village, but said they hadn't known there would be class. Had they come too, I would have summarized the material relegated to the appendices.

Two days of material is so relegated, judging by page count. Another time I may skip Bertrand's Postulate, which took up part of Tuesday and the entirety of Wednesday.

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1. Monday

Let p_n be the nth prime number, thus:

n	1	2	3	4	5	6	7	8	9	10	11	12	...
p_n	2	3	5	7	11	13	17	19	23	29	31	37	...

If $x \in \mathbb{R}$, we define

$$\max\bigl(\{k \in \mathbb{N} \colon p_k \leqslant x\} \cup \{0\}\bigr) = \pi(x).$$

Thus $\pi(x)$ is the number of primes that do not exceed x:

$$\pi(x) = |\{p \colon p \leqslant x\}| = \sum_{p \leqslant x} 1.$$

We aim to understand π, whose graph is in Fig. 1.1.

Theorem 1 (Euclid [1, IX.20]). $\lim_{x\to\infty} \pi(x) = \infty$.

Proof. We show that p_k exists for each k in $\mathbb{N}$. We use strong induction. For some n in $\mathbb{N}$, whenever $k < n$, suppose p_k exists. Then

$$p_1 p_2 \cdots p_{n-1} + 1 \equiv 1 \pmod{p_k},$$
$$p_k \nmid p_1 p_2 \cdots p_{n-1} + 1.$$

However, thus sum must *have* a prime factor, since

$$p_1 p_2 \cdots p_{n-1} + 1 \geqslant 2,$$

even when $n = 0$ (for then the product is 1). The least prime factor is p_n. By strong induction, p_k exists for all counting numbers k. □

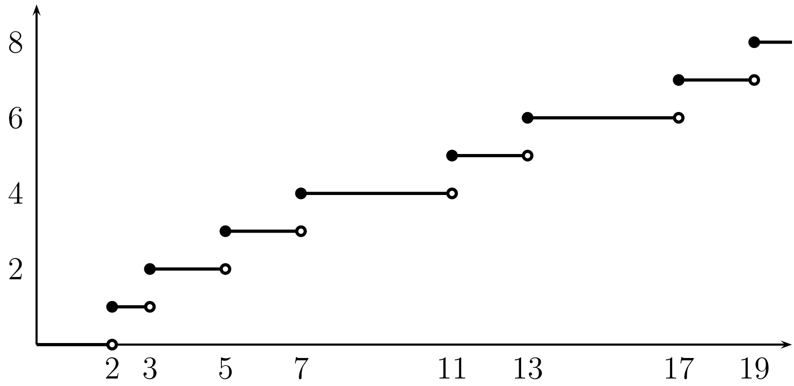

Figure 1.1.: $y = \pi(x)$

We can say a little more, as Hardy and Wright do [2, p. 12]. See Fig. 1.2.

Theorem 2. *For real numbers x exceeding* 1*,*

$$\pi(x) \geqslant \log_2 \log_2 x. \tag{1.1}$$

Proof. Since

$$\log_2 \log_2 x \leqslant 0 \iff \log_2 x \leqslant 1 \iff x \leqslant 2,$$

the claim is true when $x \leqslant 2$. If $x > 2$, then for some k in $\mathbb{N}$,

$$2^{2^k} \geqslant x > 2^{2^{k-1}}.$$

From the two inequalities respectively, we have

$$k \geqslant \log_2 \log_2 x, \qquad \pi(x) \geqslant \pi(2^{2^{k-1}}).$$

To prove (1.1) now, it is enough to show

$$\pi(2^{2^{k-1}}) \geqslant k.$$

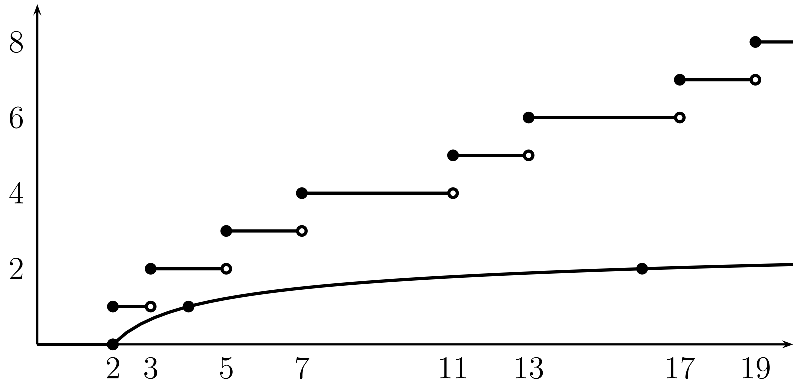

Figure 1.2.: $y = \pi(x)$ and $y = \log_2 \log_2 x$

For *this,* it is enough to show

$$2^{2^{k-1}} \geqslant p_k. \tag{1.2}$$

This follows by strong induction from the proof of Theorem 1. Suppose, for some n in $\mathbb{N}$, that (1.2) is true whenever $k < n$. Then

$$\begin{aligned} p_n &\leqslant p_1 p_2 \cdots p_{n-1} + 1 \\ &\leqslant 2^{2^0} 2^{2^1} \cdots 2^{2^{n-2}} + 1 = 2^{2^0+2^1+\cdots+2^{n-2}} + 1 \\ &= 2^{2^{n-1}-1} + 1 \leqslant 2 \cdot 2^{2^{n-1}-1} = 2^{2^{n-1}}, \end{aligned}$$

so (1.2) holds when $k = n$. □

We aim to prove the **Prime Number Theorem** or **PNT** (Theorem 19, page 43), established in 1896 by Hadamard and de la Vallée-Poussin independently, namely

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1, \tag{1.3}$$

where log is the natural logarithm, $\log_e$. We can rewrite (1.3) as either of

$$\pi(x)\log x \sim x, \qquad \pi(x) \sim \frac{x}{\log x}.$$

We may say then that $\pi(x)$ and $x/\log x$ are **asymptotic** to one another [2, p. 8]. For example, when we define

$$\operatorname{Li}(x) = \int_2^x \frac{\mathrm{d}\,t}{\log t}, \tag{1.4}$$

we obtain

$$\operatorname{Li}(x) \sim \frac{x}{\log x} \tag{1.5}$$

by L'Hôpital's Rule. First we compute

$$\frac{\operatorname{Li}(x) - x/\log x}{x/\log x} = \frac{\log x \cdot \operatorname{Li}(x) - x}{x}.$$

If the latter is f/g, then, since g grows without bound, we compute

$$\frac{f'}{g'} = \frac{(1/x)\operatorname{Li}(x) + \log x/\log x - 1}{1} = \frac{\operatorname{Li}(x)}{x},$$

and if *this* is f/g, then f'/g' is $1/\log x$, which tends to 0 as x grows without bound.

A weaker form of the PNT is **Chebyshev's Theorem** (Theorem 9, page 22), established around 1850, that the functions

$$\frac{\pi(x)\log x}{x}, \qquad \frac{x}{\pi(x)\log x}$$

are bounded above on some interval $[a,\infty)$; we write this also as

$$\pi(x) \asymp \frac{x}{\log x},$$

using the notation of Hardy and Wright [2, p. 7], in whose terminology $\pi(x)$ and $x/\log x$ are **of the same order of magnitude.** For example

$$2 + \sin x \asymp 1.$$

Our first big result will be **Bertrand's Postulate** (Theorem 6, page 15), that for every counting number n,

$$\pi(2n) - \pi(n) > 0. \tag{1.6}$$

In the proof, we shall use that there is a sequence

$$2,\ 3,\ 5,\ 7,\ 13,\ 23,\ 43,\ 83,\ 163,\ 317,\ 631 \tag{1.7}$$

of primes, where each successive term is the greatest prime that is less than twice the previous term. Because of this, any n for which (1.6) fails must be at least 631.

2. Tuesday

As from Theorem 1 we derived Theorem 2, so from Bertrand's Postulate, which is an improvement on Theorem 1, we can derive an improvement of Theorem 2, namely

$$\pi(x) \geqslant \log_2 x - 1,$$

and therefore, by an exercise, for large-enough x,

$$\pi(x) \geqslant \log x.$$

To prove Bertrand's Postulate itself, we define

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

Then for example

$$\begin{aligned}\vartheta(10) &= \log 2 + \log 3 + \log 5 + \log 7 \\ &= \log(2 \cdot 3 \cdot 5 \cdot 7) = \log(210).\end{aligned}$$

We shall often make use of the **binomial coefficients,** defined by

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}, \tag{2.1}$$

where $0 \leqslant k \leqslant n$.

Theorem 3 (Binomial Theorem). *For all natural numbers n,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \tag{2.2}$$

Proof. By direct computation from (2.1),

$$\binom{m}{0} = 1, \qquad \binom{m}{m} = 1,$$

and

$$1 \leqslant k \leqslant m \implies \binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}.$$

Then (2.2) is true when $n = 0$, and if it is true when $n = m$, then

$$\begin{aligned}
(x+y)^{m+1} &= (x+y)\sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k \\
&= \sum_{k=0}^{m}\binom{m}{k}x^{m+1-k}y^k + \sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^{k+1} \\
&= x^{m+1} + \sum_{k=1}^{m}\binom{m}{k}x^{m+1-k}y^k + \sum_{k=0}^{m-1}\binom{m}{k}x^{m-k}y^{k+1} + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m}\binom{m}{k}x^{m+1-k}y^k + \sum_{k=1}^{m}\binom{m}{k-1}x^{m+1-k}y^k + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}x^{m+1-k}y^k + y^{m+1} = (x+y)^{m+1},
\end{aligned}$$

so (2.2) is true when $n = m + 1$. □

Corollary. *Each binomial coefficient is a natural number.*

We shall often use the following bound [2, Thm 441, p. 341].

Theorem 4. *For all positive real numbers x,*

$$\vartheta(x) < 2x \log 2. \tag{2.3}$$

Proof. Since

$$\vartheta(x) = \vartheta\big([x]\big),$$

where

$$[x] = \max\{n \in \mathbb{Z}\colon n \leqslant x\},$$

it is enough to prove the claim when x is a counting number. We shall use strong induction, but need some preliminary work. Primes enter the argument through a standard exercise,

$$0 < k < p \implies p \biggm| \binom{p}{k}. \tag{2.4}$$

The proof uses Euclid's Lemma [1, Proposition VII.30], that a prime measuring a product measures one of the factors; symbolically,

$$p \mid ab \;\;\&\;\; p \nmid a \implies p \mid b.$$

Thus, in particular, since

$$p \biggm| \binom{p}{k} \cdot k! \cdot (p-k)!,$$

but

$$0 < k < p \implies p \nmid k! \cdot (p-k)!,$$

we can conclude (2.4). In the same way,

$$n - k \leqslant k < p \leqslant n \implies p \biggm| \binom{n}{k},$$

and therefore

$$n - k \leqslant k \implies \prod_{k<p\leqslant n} p \biggm| \binom{n}{k} \implies \prod_{k<p\leqslant n} p \leqslant \binom{n}{k}.$$

Taking natural logarithms preserves the last inequality, and

$$\log \prod_{k<p\leqslant n} p = \sum_{k<p\leqslant n} \log p = \vartheta(n) - \vartheta(k),$$

so

$$n-k \leqslant k \implies \vartheta(n) - \vartheta(k) \leqslant \log \binom{n}{k}. \tag{2.5}$$

We can now proceed. We assume (2.3) holds when x is an integer less than n. We have to consider four cases of n.

1. In case $n = 1$, we have

$$\vartheta(n) = \vartheta(1) = 0 = 2n \log 2.$$

2. In case $n = 2$, we compute

$$\vartheta(n) = \vartheta(2) = \log 2 < 4 \log 2 = 2n \log 2.$$

3. In case $n = 2m$, where $m > 1$, then, $2m$ being composite,

$$\vartheta(n) = \vartheta(2m) = \vartheta(2m-1) < 2(2m-1)\log 2 < 2n \log 2.$$

4. We suppose finally $n = 2m+1$, where again $m > 1$. As a special case of (2.5),

$$\vartheta(n) = \vartheta(2m+1) \leqslant \log \binom{2m+1}{m+1} + \vartheta(m+1). \tag{2.6}$$

We have also

$$\binom{2m+1}{m+1} = \binom{2m+1}{m},$$

and these are distinct terms in the expansion of $(1+1)^{2m+1}$, by Theorem (3); so

$$2\binom{2m+1}{m+1} \leqslant 2^{2m+1}, \qquad \binom{2m+1}{m+1} \leqslant 2^{2m}.$$

Plugging this, and the strong inductive hypothesis

$$\vartheta(m+1) < 2(m+1)\log 2,$$

into (2.6), we obtain

$$\vartheta(n) \leqslant 2(2m+1)\log 2 = \log 2n \log 2. \qquad \square$$

We need one more lemma for Bertrand's Postulate.

Theorem 5. *For all positive integers n,*

$$\log(n!) = \sum_{p \leqslant n} \log p \cdot \sum_{j=1}^{\infty} \left[\frac{n}{p^j}\right]. \tag{2.7}$$

Proof. A proof will just express what one sees in an example:

$$\begin{aligned} 10! = 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) \\ = 2^{5+2+1} \cdot 3^{3+1} \cdot 5^2 \cdot 7 \\ = 2^{[10/2]+[10/4]+[10/8]} \cdot 3^{[10/3]+[10/9]} \cdot 5^{[10/5]} \cdot 7^{[10/7]} \\ = 2^{\sum_{j=1}^{\infty}[10/2^j]} + 3^{\sum_{j=1}^{\infty}[10/3^j]} + 5^{\sum_{j=1}^{\infty}[10/5^j]} + 7^{\sum_{j=1}^{\infty}[10/5^j]}, \end{aligned}$$

while $\{2, 3, 5, 7\} = \{p \colon p \leqslant 10\}$. $\square$

One may observe $10! = 2^{2^3} \cdot 3^{2^2} \cdot 5^{2^1} \cdot 7^{2^0}$, but the sequence of exponents is entirely coincidental.

The sum in (2.7) has a term $\log p \cdot [n/p^j]$ precisely for those powers p^j that are no greater than n. Thus we might write

$$\log(n!) = \sum_{p^j \leqslant n} \log p \cdot \left[\frac{n}{p^j}\right] = \sum_{k \leqslant n} \Lambda(k) \cdot \left[\frac{n}{k}\right],$$

where Λ is the **von Mangoldt function,** given by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^j \text{ for some positive } j; \\ 0, & \text{otherwise.} \end{cases}$$

We shall use this function in (**??**) on page **??**. Meanwhile, here is a puzzle; what is $\sum_{d|n} \Lambda(d)$? For example,

$$\begin{aligned}\sum_{d|12} \Lambda(d) &= \Lambda(1) + \Lambda(2) + \Lambda(3) + \Lambda(4) + \Lambda(6) + \Lambda(12) \\ &= 0 + \log 2 + \log 3 + \log 2 + 0 + 0 \\ &= \log(2 \cdot 3 \cdot 2) = \log(12).\end{aligned}$$

This is not an accident: always

$$\sum_{d|n} \Lambda(d) = \log n.$$

We can understand this as a form of the Fundamental Theorem of Arithmetic.

3. Wednesday

Theorem 6 (Bertrand's Postulate). *For every positive integer n there is a prime p such that*

$$n < p \leqslant 2n. \tag{3.1}$$

Proof. No prime factor of $\binom{2n}{n}$ exceeds $2n$. Thus it is enough to show that $\binom{2n}{n}$ has a prime factor exceeding n. There are exponents $n(p)$ such that

$$\binom{2n}{n} = \prod_{p \leqslant 2n} p^{n(p)}.$$

Thus

$$n(p) \geqslant 1 \iff p \,\Big|\, \binom{2n}{n}, \tag{3.2}$$

$$\log \binom{2n}{n} = \sum_{p \leqslant 2n} \log p \cdot n(p). \tag{3.3}$$

Since also, by (2.1),

$$\log \binom{2n}{n} = \log\big((2n)!\big) - 2\log(n!),$$

we have, by Theorem 5,

$$n(p) = \sum_{j=1}^{\infty} \left(\left[\frac{2n}{p^j} \right] - 2\left[\frac{n}{p^j} \right] \right). \tag{3.4}$$

Here, in each case,

$$0 \leqslant \left(\left[\frac{2n}{p^j} \right] - 2 \left[\frac{n}{p^j} \right] \right) \leqslant 1, \tag{3.5}$$

for,

$$[x] = m \iff m \leqslant x < m+1 \iff 2m \leqslant 2x < 2m+2,$$
$$2m \leqslant 2x < 2m+1 \implies [2x] = 2m,$$
$$2m+1 \leqslant 2x < 2m+2 \implies [2x] = 2m+1.$$

Moreover,

$$k > 2n \implies \left[\frac{2n}{k} \right] - 2 \left[\frac{n}{k} \right] = 0, \tag{3.6}$$

while

$$p^j > 2n \iff j \log p > \log(2n). \tag{3.7}$$

Plugging (3.5), (3.6), and (3.7) into (3.4) gives

$$n(p) \leqslant \sum_{j \leqslant \log(2n)/\log p} 1 \leqslant \frac{\log(2n)}{\log p}. \tag{3.8}$$

Therefore

$$2 \leqslant n(p) \implies 2 \log p \leqslant \log(2n) \implies p \leqslant \sqrt{2n}. \tag{3.9}$$

From (3.3) and (3.2),

$$\begin{aligned} \log \binom{2n}{n} &= \sum_{n(p) \geqslant 1} \log p \cdot n(p) \\ &= \sum_{n(p)=1} \log p + \sum_{n(p) \geqslant 2} \log p \cdot n(p). \end{aligned} \tag{3.10}$$

We bound the two terms. Again by (3.2),

$$\sum_{n(p)=1} \log p \leqslant \sum_{p|\binom{2n}{n}} \log p,$$

while by (3.8) and (3.9),

$$\sum_{n(p)\geqslant 2} \log p \cdot n(p) \leqslant \sum_{n(p)\geqslant 2} \log(2n) \leqslant \log(2n)\sqrt{2n}.$$

Thus (3.10) yields

$$\log \binom{2n}{n} \leqslant \sum_{p|\binom{2n}{n}} \log p + \log(2n)\sqrt{2n}. \tag{3.11}$$

Thus we have an upper bound on $\log\binom{2n}{n}$. We introduce a lower bound by noting that, since

$$2^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} = 2 + \sum_{j=1}^{2n-1} \binom{2n}{j},$$

where there are $2n$ terms, the greatest being $\binom{2n}{n}$,

$$2^{2n} \leqslant 2n\binom{2n}{n},$$

$$2n\log 2 - \log(2n) \leqslant \log\binom{2n}{n}. \tag{3.12}$$

Combining (3.11) and (3.12) yields

$$2n\log 2 \leqslant \sum_{p|\binom{2n}{n}} \log p + \log(2n)(1+\sqrt{2n}). \tag{3.13}$$

The left-hand side dominates the second term on the right, that is,

$$\lim_{x\to\infty} \frac{\log(2n)(1+\sqrt{2n})}{2n\log 2} = 0,$$

since generally

$$\lim_{x\to\infty} \frac{\log x}{x^s} = 0$$

when $s > 0$, by L'Hôpital's Rule. We may write this theorem as

$$s > 0 \implies \log x \prec x^s. \tag{3.14}$$

We shall show that $\binom{2n}{n}$ must have prime factors exceeding n, to compensate. By Theorem 4,

$$\sum_{p \left| \binom{2n}{n} \right. \,\&\, p\leqslant n} \log p \leqslant \sum_{p\leqslant n} \log p = \vartheta(n) \leqslant 2n\log 2. \tag{3.15}$$

Since this is just the left-hand side of (3.13), there is no problem so far. However,

$$\begin{aligned} \frac{2n}{3} < p \leqslant n &\implies 2p \leqslant 2n < 3p \\ &\implies \left[\frac{2n}{p}\right] - 2\left[\frac{n}{p}\right] = 2 - 2\cdot 1 = 0, \end{aligned}$$

and also

$$\begin{aligned} \frac{2n}{3} < p \;\&\; n \geqslant 5 &\implies p^2 > \frac{4n^2}{9} = \frac{2n}{9}\cdot 2n > 2n \\ &\implies \left[\frac{2n}{p^2}\right] = 0, \end{aligned}$$

and therefore

$$\frac{2n}{3} < p \leqslant n \;\&\; n \geqslant 5 \implies n(p) = 0.$$

Thus, assuming $n \geqslant 5$, in the manner of (3.15) we have

$$\sum_{p \left| \binom{2n}{n} \right. \,\&\, p \leqslant n} \log p \leqslant \sum_{p \leqslant 2n/3} \log p = \vartheta\left(\frac{2n}{3}\right) \leqslant \frac{4n}{3} \log 2.$$

Combining with (3.13) gives

$$\sum_{p \left| \binom{2n}{n} \right. \,\&\, n<p} \log p \geqslant \frac{2n}{3} \log 2 - \left(1 + \sqrt{2n}\right) \log(2n). \tag{3.16}$$

We already know that the right-hand side is positive when n is large enough. It is enough to show 631 is large enough, because of the sequence (1.7). Since $631 > 512 = 2^9$, let us assume now

$$n > 2^9, \qquad 2n > 2^{10} = 1024, \qquad \sqrt{2n} > 2^5 = 32.$$

Multiplying the right-hand side of (3.16) by what will turn out to be a convenient factor, we compute

$$\begin{aligned} &\frac{3}{\sqrt{2n} \log \sqrt{2n} \log 2} \left(\frac{2n}{3} \log 2 - \left(1 + \sqrt{2n}\right) \log(2n) \right) \\ &\qquad = \frac{\sqrt{2n}}{\log \sqrt{2n}} - \frac{2 \cdot 3}{\log 2} \left(\frac{1}{\sqrt{2n}} + 1 \right), \end{aligned}$$

while

$$\begin{aligned} \frac{2 \cdot 3}{\log 2} \left(\frac{1}{32} + 1 \right) &= \frac{2 \cdot 3 \cdot 33}{2^5 \log 2} \leqslant \frac{2 \cdot 100 \cdot 2^5}{2^{10} \log 2} \\ &\leqslant \frac{2 \cdot 100 \cdot 2^5}{1000 \log 2} = \frac{2^5}{5 \log 2} = \frac{2^5}{\log(2^5)}. \end{aligned}$$

Now the right-hand side of (3.16) is positive, because $x/\log x$ is an increasing function of x on $[\mathrm{e}, \infty)$, since its derivative is $(\log x - 1)/(\log x)^2$. □

4. Thursday

The **Riemann zeta function** is given by

$$\zeta(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}, \tag{4.1}$$

where (in Riemann's own notation)

$$s = \sigma + \mathrm{i}\, t$$

and

$$\sigma > 1.$$

The convergence of the sum is absolute, by the Integral Test, since

$$|n^s| = n^\sigma \left|n^{\mathrm{i}\, t}\right| = n^\sigma \left|\mathrm{e}^{\mathrm{i}\, t \log n}\right| = n^\sigma,$$

and

$$\int_1^\infty \frac{\mathrm{d}\, x}{x^\sigma} = \frac{1}{1-\sigma} \cdot \frac{1}{x^{\sigma-1}} \Big|_1^\infty = \frac{1}{\sigma - 1}. \tag{4.2}$$

Theorem 7. *When* $\sigma > 1$, *then*

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}. \tag{4.3}$$

Proof. By definition,

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{2^s \cdot 2^s} + \frac{1}{5^s} + \frac{1}{2^s \cdot 3^s} + \cdots$$

while, by the rule

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \tag{4.4}$$

for geometric series,

$$\prod_p \frac{1}{1-p^{-s}} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right) \cdot \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right) \cdot \left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \cdots\right) \cdots$$

By the Fundamental Theorem of Arithmetic, the two sides of (4.3) match up. (A proper proof would use the absolute convergence of the sum expansion of $\zeta(s)$.) □

In complex analysis, a **holomorphic function** is a differentiable function, according to the rule from real analysis, with complex numbers used for real numbers. We shall say more later. Meanwhile:

Theorem 8. *The function*

$$\zeta(s) - \frac{1}{s-1}$$

extends holomorphically to $\sigma > 0$.

Proof. When $\sigma > 1$, we have, by (4.2),

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{\mathrm{d}\,x}{x^s} = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) \mathrm{d}\,x,$$

and the last series converges absolutely when $\sigma > 0$, since

$$\frac{1}{n^s} - \frac{1}{x^s} = s \int_n^x \frac{\mathrm{d}\,u}{u^{s+1}},$$

and this, on $[n, n+1]$, is bounded absolutely by $|s|/n^{\sigma+1}$. □

With more complex analysis we shall show

$$x \sim \vartheta(x).$$

Meanwhile, we prove Chebyshev's Theorem (as stated [2, Thm 7, p. 9] and proved [2, §§22.1–4, p. 340–6] by Hardy and Wright), and from this derive the corollary

$$\vartheta(x) \sim \pi(x)\log x.$$

Landau also proves the theorem [3, Thm 112, pp. 88–91].

Theorem 9 (Chebyshev)**.** *For some positive A and B, for large x,*

$$Ax \leqslant \pi(x)\log x \leqslant Bx;$$

that is,

$$\pi(x) \asymp \frac{x}{\log x}.$$

Proof for the first inequality. Since

$$\vartheta(x) = \sum_{p \leqslant x} \log p \leqslant \sum_{p \leqslant x} \log x = \pi(x)\log x, \tag{4.5}$$

it is enough to bound $\vartheta(x)$ below by a positive multiple of x. To do so, we define

$$\psi(x) = \sum_{m=1}^{\infty} \vartheta(x^{1/m}).$$

Since

$$\vartheta(y) > 0 \iff y \geqslant 2,$$

$$x^{1/m} \geqslant 2 \iff \frac{1}{m}\log x \geqslant \log 2 \iff m \leqslant \frac{\log x}{\log 2},$$

we have

$$\psi(x) = \sum_{m=1}^{[\log x/\log 2]} \vartheta(x^{1/m}) = \vartheta(x) + \sum_{m=2}^{[\log x/\log 2]} \vartheta(x^{1/m}).$$

Also

$$m \geqslant 2 \implies \vartheta(x^{1/m}) \leqslant \vartheta(\sqrt{x}) \leqslant \sqrt{x}\log\sqrt{x} \leqslant \sqrt{x}\log x,$$

so that

$$\sum_{m=2}^{[\log x/\log 2]} \vartheta(x^{1/m}) \leqslant \sqrt{x}\log x \cdot \frac{\log x}{\log 2} \prec x$$

by (3.14). Thus

$$\psi(x) - \vartheta(x) \prec x.$$

It is now enough to bound $\psi(x)$ below by a positive multiple of x; for if $Ax \leqslant \psi(x)$, then for x large enough,

$$\psi(x) - \vartheta(x) \leqslant \frac{1}{2}Ax, \qquad \frac{1}{2}Ax \leqslant \vartheta(x).$$

Towards bounding $\psi(x)$, we observe

$$\psi(x) = \sum_{m=1}^{\infty} \sum_{p \leqslant x^{1/m}} \log p$$
$$= \sum_{m=1}^{\infty} \sum_{p^m \leqslant x} \log p = \sum_{p \leqslant x} \log p \cdot \left[\frac{\log x}{\log p}\right].$$

From our proof of Bertrand's Postulate (Theorem 6), specifically (3.3) and (3.8),

$$\log\binom{2n}{n} = \sum_{p \leqslant 2n} \log p \cdot n(p) \leqslant \sum_{p \leqslant 2n} \log p \cdot \left[\frac{\log(2n)}{\log p}\right].$$

Thus

$$\log \binom{2n}{n} \leqslant \psi(2n).$$

Moreover,

$$2^n = \prod_{j=1}^{n} 2 \leqslant \prod_{j=1}^{n} \frac{n+j}{j} = \binom{2n}{n}, \qquad n \log 2 \leqslant \log \binom{2n}{n}.$$

Suppose finally $n = [x/2]$, that is,

$$n \leqslant \frac{x}{2} < n + 1.$$

If also $x \geqslant 6$, so that $x/6 \geqslant 1$, then

$$\frac{x}{3} = \frac{x}{2} - \frac{x}{6} < n, \qquad 2n \leqslant x,$$

so that

$$\frac{\log 2}{3} \cdot x \leqslant n \log 2 \leqslant \psi(2n) \leqslant \psi(x). \qquad \square$$

5. Friday

Letting $s = \sigma + \mathrm{i}\,t$ in $\mathbb{C}$, we have

- defined $\zeta(s) = \sum_{n=1}^{\infty} 1/n^2$ when $\sigma > 1$;
- shown $\zeta(s) = \prod_p 1/(1 - p^{-s})$;
- proved that some *holomorphic* function on $\sigma > 0$ agrees with $\zeta(s) - 1/(s-1)$ on $\sigma > 1$;
- proved that, for some positive A, for large real x, $Ax \leqslant \vartheta(x) \leqslant \pi(x)\log x$.

In our remaining time, we are going to show that:

- for some B, for large x, $\pi(x)\log x \leqslant Bx$;
- $\pi(x)\log x \sim \vartheta(x)$;
- $\zeta(s) \neq 0$ when $\sigma \geqslant 1$;
- some holomorphic function on $\sigma \geqslant 1$ agrees with $\Phi(s) - 1/(s-1)$ on $\sigma > 1$, where $\Phi(s) = \sum_p \log p/p^s$;
- Theorem 17: when f is bounded on $[0,\infty)$ and integrable on every bounded interval, and a holomorphic function $g(s)$ on $\sigma \geqslant 0$ agrees with $\int_0^\infty f(x)\mathrm{e}^{-sx}\,\mathrm{d}\,x$ on $\sigma > 0$, then $g(0) = \int_0^\infty f(x)\,\mathrm{d}\,x$.
- $\int_1^\infty (\vartheta(x) - x)\,\mathrm{d}\,x/x^2$ converges.
- $\vartheta(x) \sim x$.

Proof for the second inequality. We bound $\pi(x)\log x$ above by a multiple of x. From Theorem 4, we have such an upper bound, namely $2x\log 2$, on $\vartheta(x)$. Moreover,

$$\vartheta(x) = \sum_{p\leqslant x} \log p \geqslant \sum_{x\geqslant p>\sqrt{x}} \log\sqrt{p} \geqslant \sum_{x\geqslant p>\sqrt{x}} \log\sqrt{x}$$

$$= \big(\pi(x) - \pi(\sqrt{x})\big) \log \sqrt{x}. \quad (5.1)$$

Since

$$\pi(\sqrt{x}) \leqslant \sqrt{x} = \frac{x}{\sqrt{x}} \leqslant \frac{x}{\log x}, \qquad \log \sqrt{x} = \frac{1}{2} \log x,$$

we obtain

$$\vartheta(x) \geqslant \left(\pi(x) - \frac{x}{\log x}\right) \frac{\log x}{2}, \qquad (5.2)$$

$$2\vartheta(x) + x \geqslant \pi(x) \log x,$$

and this is enough. □

We can now obtain the following as a corollary.

Theorem 10.

$$\vartheta(x) \sim \pi(x) \log x. \qquad (5.3)$$

Proof. As a variant of (5.1) and (5.2), if $0 < \varepsilon < 1$, then

$$\vartheta(x) \geqslant \sum_{x^{1-\varepsilon} < p \leqslant x} \log(x^{1-\varepsilon}) \geqslant (\pi(x) - x^{1-\varepsilon}) \log x \cdot (1 - \varepsilon),$$

$$\vartheta(x) + x^{1-\varepsilon} \log x \cdot (1 - \varepsilon) \geqslant \pi(x) \log x \cdot (1 - \varepsilon),$$

$$\frac{\vartheta(x)}{\pi(x) \log x} + \frac{x}{\pi(x) \log x} \cdot \frac{\log x}{x^{\varepsilon}} \cdot (1 - \varepsilon) \geqslant 1 - \varepsilon.$$

We can make the second term on the left as small as desired, since $x/\pi(x) \log x$ is bounded above by Theorem 9, while $\log x \prec x^{\varepsilon}$ by (3.14). Thus when x is large enough we have

$$\frac{\vartheta(x)}{\pi(x) \log x} \geqslant 1 - 2\varepsilon.$$

For all small ε, for large x, this is true. Since the left side is bounded above by 1 by (4.5), we conclude

$$\lim_{x\to\infty}\frac{\vartheta(x)}{\pi(x)\log x}=1. \qquad \square$$

We turn to holomorphic functions. A subset Ω of $\mathbb{C}$ is **open** if, for every element a of Ω, for some positive ε,

$$\{z\in\mathbb{C}\colon\ |z-a|<\varepsilon\}\subseteq\Omega.$$

If $f\colon\Omega\to\mathbb{C}$ and $g\colon\Omega\to\mathbb{C}$ and, for every a in Ω,

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a}=g(a), \tag{5.4}$$

then g is the **complex derivative**—denoted by f'—of f, and f itself is **holomorphic.**

For example, complex conjugation $z\mapsto\overline{z}$ is *not* holomorphic, since

$$\frac{\overline{z}-\overline{a}}{z-a}=\frac{\overline{z-a}}{z-a}=\frac{\overline{z-a}^{\,2}}{|z-a|^2},$$

and this is on the unit circle and can be anywhere on that circle.

When we consider $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, then complex conjugation is a linear transformation of this space, and then it is is own derivative at each point. Why then conjugation is not holomorphic is that it is not a linear transformation of $\mathbb{C}$ as a complex vector space.

In detail, we can rewrite (5.4) as

$$\lim_{z\to a}\frac{f(z)-f(a)-f'(a)(z-a)}{z-a}=0.$$

Suppose merely

$$\lim_{z\to a} \frac{f(z) - f(a) - T(z-a)}{z-a} = 0,$$

where T is an $\mathbb{R}$-linear function, that is,

$$T(z+w) = T(z) + T(w), \qquad T(uz) = uT(z),$$

where $u \in \mathbb{R}$. Then T is the **real derivative** of f at a. T is $\mathbb{C}$-linear if and only if it is multiplication by a complex number, which in this case would be $f'(a)$, and f would be holomorphic.

In any case, the partial derivatives of f at a are as follows, where h is a real number.

$$\partial_x f(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$
$$\partial_y f(a) = \lim_{h\to 0} \frac{f(a+\mathrm{i}\,h) - f(a)}{h} = -\mathrm{i} \lim_{h\to 0} \frac{f(a+\mathrm{i}\,h) - f(a)}{\mathrm{i}\,h}.$$

If f is holomorphic, we obtain the **Cauchy–Riemann Equation,**

$$\partial_x f(a) = -\mathrm{i}\,\partial_y f(a). \tag{5.5}$$

6. Saturday

We know $\pi(x)\log x \sim \vartheta(x)$, and $\zeta(s) - 1/(s-1)$ is defined holomorphically on $\sigma > 0$, where $s = \sigma + \mathrm{i}\,t$. It remains to prove the following.

1. When $\sigma \geqslant 1$, then $\zeta(s) \neq 0$, and $\Phi(s) - 1/(s-1)$ is holomorphic, where
$$\Phi(s) = \sum_p \frac{\log p}{p^s}. \tag{6.1}$$
2. Theorem 17: When f is bounded on $[0, \infty)$ and integrable on every bounded interval, and a holomorphic function $g(s)$ on $\sigma \geqslant 0$ agrees with $\int_0^\infty f(x)\mathrm{e}^{-sx}\,\mathrm{d}\,x$ on $\sigma > 0$, then $g(0) = \int_0^\infty f(x)\,\mathrm{d}\,x$;
3. $\int_1^\infty (\vartheta(x) - x)\,\mathrm{d}\,x/x^2$ converges.
4. If $\int_1^\infty (f(x) - x)\,\mathrm{d}\,x/x^2$ converges, and f is increasing, then $f(x) \sim x$.

The last theorem is just calculus:

Theorem 11. *If f is an increasing function such that the integral*
$$\int_1^\infty \frac{f(t) - t}{t^2}\,\mathrm{d}\,t$$
converges, then
$$f(x) \sim x.$$

Proof. Let f be an increasing function such that $f(x) \nsim x$. There are two ways this can happen.

1. Suppose first for some λ, $\lambda > 1$ and, for arbitrarily large x,

$$f(x) \geqslant \lambda x.$$

For such x, since f is increasing,

$$\int_x^{\lambda x} \frac{f(t) - t}{t^2} \,\mathrm{d}\, t \geqslant \int_x^{\lambda x} \frac{\lambda x - t}{t^2} \,\mathrm{d}\, t = I$$

for some I. Letting $t = xu$, so that $\mathrm{d}\, t = x \,\mathrm{d}\, u$, we have

$$I = \int_1^{\lambda} \frac{\lambda x - ux}{u^2 x^2} x \,\mathrm{d}\, u = \int_1^{\lambda} \frac{\lambda - u}{u^2} \,\mathrm{d}\, u > 0.$$

However, if $\int_1^\infty g$ converges, then for large x, $\left|\int_x^\infty g\right| < I/2$ and $\left|\int_{\lambda x}^\infty g\right| < I/2$, so $\left|\int_x^{\lambda x} g\right| < I$.

2. In the other case, for some λ, $0 < \lambda < 1$ and for arbitrarily large x,

$$f(x) \leqslant \lambda x.$$

The argument is similar. □

Theorem 12. *The function*

$$\Phi(s) - \frac{1}{s-1},$$

where Φ *is as in* (6.1)*, extends holomorphically to* $\sigma \geqslant 1$*, and* ζ *is non-zero here.*

Proof. From (4.3), namely $\zeta(p) = \prod_p (1 - p^{-s})^{-1}$, we know that ζ is nonzero on $\sigma > 1$. From

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = -\sum_p \log(1 - \mathrm{e}^{-s \log p}),$$

taking the derivative, we compute

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s - 1}.$$

Since

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x(x-1)},$$

we have now

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \left(\frac{\log p}{p^s} + \frac{\log p}{p^s(p^s-1)} \right)$$

$$= \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}. \quad (6.2)$$

The last series converges absolutely when $\sigma > 1/2$, since for large p,

$$\frac{1}{p^{2s} - p^s} < \frac{2}{p^{2s}}.$$

It remains to show $\zeta(s) \neq 0$ when $\sigma = 1$. Then our claim will follow from complex analysis. Specifically, if f is holomorphic, then, by Theorem 16 on page 37, f is **analytic,** which means that, for every point a of its domain, for some coefficients a_n, for z near a,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots = \sum_{n=0}^{\infty} a_n(z-a)^n.$$

- If $a_0 = \ldots = a_{k-1} = 0$, but $a_k \neq 0$, then f has a **zero of order k at a.**
- If $a_0 \neq 0$, and $k > 0$, then $f(z)/(z-a)^k$ has a **pole of order k at a.**

Suppose now

$$g(z) = \sum_{n=k}^{\infty} b_n (z-a)^n = b_k (z-a)^k + \cdots ,$$

where k may be negative. The **residue** of g at a is b_{-1}. Also

$$g'(z) = k b_k (z-a)^{k-1} + \cdots ,$$
$$\frac{g'(z)}{g(z)} = k(z-a)^{-1} + \cdots .$$

If $k \neq 0$, then g'/g has a pole of order 1 at a, and the residue k of g'/g at a is the order of the zero at a, if there is one; otherwise $-k$ is the order of the pole at a.

By Theorem 8, since $1/(s-1)$ has a pole of order 1 at 1, so does $\zeta(s)$. Therefore ζ'/ζ has a pole of order 1, and residue 1, at 1. By (6.2), so has Φ. Thus it remains to show $\zeta(s) \neq 0$ when $\sigma = 1$.

Suppose if possible ζ has a zero of order μ at $1 + \mathrm{i}\,a$. Then μ is the residue of ζ'/ζ at $1 + \mathrm{i}\,a$, so

$$\begin{aligned} \mu &= \lim_{z \to 1+\mathrm{i}\,a} \frac{(z-1-\mathrm{i}\,a)\zeta'(z)}{\zeta(z)} \\ &= \lim_{z \to 0} \frac{z\zeta'(1+\mathrm{i}\,a+z)}{\zeta(1+\mathrm{i}\,a+z)} = -\lim_{z \to 0} z\Phi(1+\mathrm{i}\,a+z). \qquad (6.3) \end{aligned}$$

Likewise

$$1 = \lim_{z \to 0} z\Phi(z), \qquad (6.4)$$

since Φ has a pole of order 1, and residue 1, at 1.

Either ζ has a zero of some order ν at $1 + 2\mathrm{i}\,a$, or else we let $\nu = 0$. For small positive ε, we compute

$$\sum_{j=-2}^{2}\binom{4}{2+j}\Phi(1+j\mathrm{i}\,a+\varepsilon)=\sum_{j=-2}^{2}\binom{4}{2+j}\sum_{p}\frac{\log p}{p^{1+\varepsilon+j\mathrm{i}\,a}}$$
$$=\sum_{p}\frac{\log p}{p^{1+\varepsilon}}\sum_{j=-2}^{2}\binom{4}{2+j}\frac{1}{p^{j\mathrm{i}\,a}}=\sum_{p}\frac{\log p}{p^{1+\varepsilon}}\left(\frac{1}{p^{\mathrm{i}\,a/2}}+\frac{1}{p^{-\mathrm{i}\,a/2}}\right)^{4}$$
$$=\sum_{p}\frac{\log p}{p^{1+\varepsilon}}\left(2\Re\left(\frac{1}{p^{\mathrm{i}\,a/2}}\right)\right)^{4}\geqslant 0. \quad (6.5)$$

Here $\Re(z)=(z+\overline{z})/2$, the real part of z. Since

$$\overline{\Phi(s)}=\Phi(\overline{s}),$$

so that

$$2\Re(\Phi(s))=\Phi(s)+\Phi(\overline{s}),$$

we have,

$$\sum_{j=-2}^{2}\binom{4}{2+j}\Phi(1+j\mathrm{i}\,a+\varepsilon)$$
$$=2\Re(\Phi(1+2\mathrm{i}\,a+\varepsilon))+8\Re(\Phi(1+\mathrm{i}\,a+\varepsilon))$$
$$+6\Re(\Phi(1+\varepsilon)).$$

In the limit at ε, the product of the sum here with ε is $-2\nu-8\mu+6$, by (6.3), and the similar computation for ν, and (6.4). By (6.5), we have

$$-2\nu-8\mu+6\geqslant 0,$$

so μ cannot be positive. □

A. Analytic Functions

From the definition on page 31, analytic functions are holomorphic. We shall prove the converse.

We are considering a holomorphic function f on an open subset Ω of $\mathbb{C}$. Supposing $\gamma\colon [a,b] \to \Omega$, and making the analysis

$$\gamma = \gamma_0 + \mathrm{i}\,\gamma_1, \tag{A.1}$$

where $\gamma_e\colon [a,b] \to \mathbb{R}$, we define

$$\int_a^b \gamma = \int_a^b \gamma_0 + \mathrm{i} \int_a^b \gamma_1, \tag{A.2}$$

provided the integrals on the right exist. Suppose further that γ is one-to-one and **continuously differentiable,** which means that each γ_e has a continuous derivative and

$$\gamma' = {\gamma_0}' + \mathrm{i}\,{\gamma_1}'.$$

Then γ has the **initial point** $\gamma(a)$ and the **terminal point** $\gamma(b)$, and γ itself is an **arc** or **path** or **line.** We define the **line integral** of f along γ using the substitution

$$z = \gamma(t), \qquad \mathrm{d}\,z = \gamma'(t)\,\mathrm{d}\,t,$$

obtaining

$$\int_\gamma f(z)\,\mathrm{d}\,z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}\,t,$$

or simply

$$\int_\gamma f = \int_a^b (f \circ \gamma)\gamma'. \tag{A.3}$$

We say $\int_\gamma f$ is **path independent** if it depends only on f, $\gamma(a)$, and $\gamma(b)$.

Theorem 13. *The line integral of a continuous derivative of a holomorphic function is path independent.*

Proof. If $f = F'$, then by the Chain Rule,

$$(F \circ \gamma)' = (F' \circ \gamma)\gamma' = (f \circ \gamma)\gamma',$$

and so by (A.3) and the Fundamental Theorem of Calculus,

$$\int_\gamma f = \int_a^b (F \circ \gamma)' = F(\gamma(b)) - F(\gamma(a)). \qquad \square$$

An arc whose initial and terminal points are the same is a **closed curve.** The line integrals of a function are path independent if and only if the integrals around closed curves are path independent. The latter case means those integrals are zero.

For example, if $n \neq 0$ and $f(z) = (z-a)^n/n$, then f is holomorphic on its domain, which is $\mathbb{C}$ if $n > 0$, and $\{z \in \mathbb{C} \colon z \neq a\}$ if $n < 0$. Also

$$f'(z) = (z-a)^{n-1},$$

so that $(z-a)^{n-1}$ is the derivative of a holomorphic function of z, unless $n = 0$. Therefore, letting $\oint$ denote an integral along a counterclockwise closed curve in the domain of the integrand, we have

$$n \neq 0 \implies \oint (z-a)^{n-1} \,\mathrm{d}\, z = 0.$$

The case when $n = 0$ is different, as follows.

Theorem 14. *If γ describes a counterclockwise loop around a, then*

$$\int_\gamma \frac{\mathrm{d}\,z}{z-a} = 2\mathrm{i}\,\pi.$$

Proof. We may assume $a = 0$. If δ is $t \mapsto \mathrm{e}^{\mathrm{i}\,t}$ on $[0, 2\pi]$, we compute

$$\int_\delta \frac{\mathrm{d}\,z}{z} = \int_0^{2\pi} \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,t}\,\mathrm{d}\,t}{e^{\mathrm{i}\,t}} = \mathrm{i}\int_0^{2\pi} \mathrm{d}\,t = 2\mathrm{i}\,\pi.$$

The general case follows from Theorem 13, since we can analyze $\delta - \gamma$ as a sum of closed curves, each surrounding a region where $1/z$ is the derivative of a holomorphic function (which we ambiguously call $\log z$). □

In the following, (A.4) is **Cauchy's Integral Formula.**

Theorem 15. *If f is holomorphic on an open neighborhood of a, and γ describes a counterclockwise loop, within that neighborhood, around a, then*

$$f(a) = \frac{1}{2\mathrm{i}\,\pi}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}\,z. \tag{A.4}$$

Proof. Again we may assume $a = 0$. By Theorem 14,

$$f(0) = \frac{1}{2\mathrm{i}\,\pi}\int_\gamma \frac{f(0)}{z}\,\mathrm{d}\,z. \tag{A.5}$$

As in the proof of Theorem 14, we may adjust γ, now shrinking it to a circle of radius δ around 0. Given a positive ε, we may let δ be small enough that

$$|f(z) - f(0)| < \varepsilon$$

on γ. Using (A.5), we compute

$$
\begin{aligned}
\left|\frac{1}{2\mathrm{i}\,\pi}\int_\gamma \frac{f(z)}{z}\,\mathrm{d}\,z - f(0)\right| &= \left|\frac{1}{2\mathrm{i}\,\pi}\int_\gamma \frac{f(z)-f(0)}{z}\,\mathrm{d}\,z\right| \\
&= \frac{1}{2\pi}\left|\int_0^1 \frac{f(\delta \mathrm{e}^{2\mathrm{i}\,\pi t})-f(0)}{\delta \mathrm{e}^{2\mathrm{i}\,\pi t}} 2\mathrm{i}\,\pi\delta \mathrm{e}^{2\mathrm{i}\,\pi t}\,\mathrm{d}\,t\right| \\
&\leqslant \int_0^1 \left|f(\delta \mathrm{e}^{2\mathrm{i}\,\pi t}) - f(0)\right|\mathrm{d}\,t < \varepsilon.
\end{aligned}
$$

This being so for all positive ε, then claim follows. □

Theorem 16. *Holomorphic functions are analytic.*

Proof. Let γ describe a circle in the domain of a holomorphic function f. We may assume the center of the circle is 0. Let w be a point inside the circle. By Theorem 15, and then the rule (4.4) for geometric series,

$$
\begin{aligned}
2\mathrm{i}\,\pi f(w) &= \int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}\,z = \int_\gamma \frac{f(z)}{z(1-w/z)}\,\mathrm{d}\,z \\
&= \int_\gamma \frac{f(z)}{z}\sum_{n\in\omega}\left(\frac{w}{z}\right)^n \mathrm{d}\,z = \int_\gamma \sum_{n\in\omega}\frac{f(z)w^n}{z^{n+1}}\,\mathrm{d}\,z.
\end{aligned}
$$

Since $f(z)$ is bounded on γ (this being compact), the convergence of the series is absolute, so we can interchange the integration and summation:

$$
f(w) = \sum_{n\in\omega}\frac{1}{2\mathrm{i}\,\pi}\left(\int_\gamma \frac{f(z)\,\mathrm{d}\,z}{z^{n+1}}\right)w^n. \qquad \square
$$

B. The Prime Number Theorem

We shall prove the Prime Number Theorem as Theorem 19 below. This will use our earlier results, along with Theorem 18, which will be a consequence of the following "analytic theorem."

Theorem 17 (Newman [4]). *Any function f that is bounded and locally integrable on $[0, \infty)$ is globally integrable, provided the function g given on $\sigma > 0$ by*

$$g(s) = \int_0^\infty f(t) \mathrm{e}^{-st} \, \mathrm{d}\, t$$

extends holomorphically to $\sigma \geqslant 0$. In this case, moreover,

$$g(0) = \int_0^\infty f(t) \, \mathrm{d}\, t.$$

Proof. Defining

$$g_x(z) = \int_0^x f(t) \mathrm{e}^{-zt} \, \mathrm{d}\, t,$$

we want to prove

$$g(0) = \lim_{x \to \infty} g_x(0). \tag{B.1}$$

Given large positive R, we can find positive δ so that g is holomorphic on the region $-\delta \leqslant \sigma \quad \& \quad |s| \leqslant R$ shown in Figure B.1. Let γ be a counterclockwise path around this

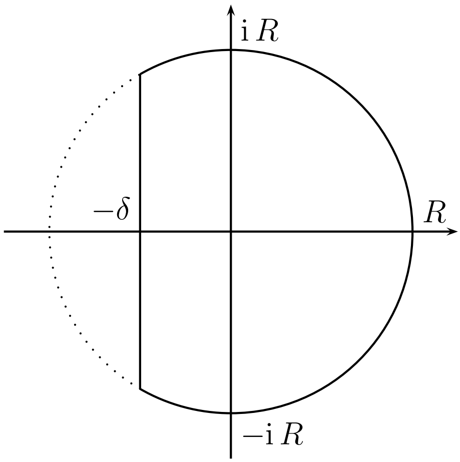

Figure B.1.: A contour for integration

region. By Cauchy's Integral Formula,

$$g(0) - g_x(0) = \frac{1}{2\mathrm{i}\,\pi} \int_\gamma \big(g(z) - g_x(z)\big) \frac{\mathrm{d}\,z}{z}. \tag{B.2}$$

Now let

$$h_x(s) = \mathrm{e}^{sx} \left(1 + \frac{s^2}{R^2}\right),$$

so h_x is holomorphic on $\mathbb{C}$. The innovation of Newman is to multiply $g(z) - g_x(z)$ by $h_x(z)$, so that, since $h_x(0) = 1$, (B.2) becomes

$$g(0) - g_x(0) = \frac{1}{2\mathrm{i}\,\pi} \int_\gamma \big(g(z) - g_x(z)\big) \frac{h_x(z)}{z} \,\mathrm{d}\,z.$$

We shall bound the integral. We have

$$\left|\frac{h_x(s)}{s}\right| = \frac{\mathrm{e}^{\sigma x}}{R} \left|\frac{R}{s} + \frac{s}{R}\right|,$$

and so

$$|s| = R \implies \left|\frac{h_x(s)}{s}\right| \leqslant \frac{2\mathrm{e}^{\sigma x}|\sigma|}{R^2}. \qquad \text{(B.3)}$$

Since f is assumed to be bounded, we may let

$$B = \sup_{0\leqslant t} |f(t)|,$$

so that

$$\sigma > 0 \implies |g(s) - g_x(s)| = \left|\int_x^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}\,t\right| \leqslant B\int_x^\infty \mathrm{e}^{-\sigma t}\,\mathrm{d}\,t = \frac{B}{\mathrm{e}^{\sigma x}\sigma}. \qquad \text{(B.4)}$$

Combining the two estimates (B.3) and (B.4), letting γ_+ be the restriction of γ so that the range is in $\sigma > 0$, and thus the length of γ_+ is πR, we have

$$\left|\int_{\gamma_+}\big(g(z) - g_x(z)\big)\frac{h_x(z)}{z}\,\mathrm{d}\,z\right| \leqslant \pi R \cdot \frac{B}{\mathrm{e}^{\sigma x}\sigma} \cdot \frac{2\mathrm{e}^{\sigma x}\sigma}{R^2} = \frac{2\pi B}{R}. \qquad \text{(B.5)}$$

Letting γ_- be the other part of γ, we have

$$\left|\int_{\gamma_-}\big(g(z) - g_x(z)\big)\frac{h_x(z)}{z}\,\mathrm{d}\,z\right| \leqslant \left|\int_{\gamma_-} g(z)\frac{h_x(z)}{z}\,\mathrm{d}\,z\right| + \left|\int_{\gamma_-} g_x(z)\frac{h_x(z)}{z}\,\mathrm{d}\,z\right|. \qquad \text{(B.6)}$$

For the last integral, since g_x is entire, we can replace γ_- with ${\gamma_-}'$ having the same endpoints, other points having negative real part and absolute value R. Since, as in (B.4),

$$\sigma < 0 \implies |g_x(s)| = \left|\int_0^x f(t)\mathrm{e}^{-st}\,\mathrm{d}\,t\right|$$
$$\leqslant B\int_{-\infty}^x \mathrm{e}^{-\sigma t}\,\mathrm{d}\,t = \frac{B}{\mathrm{e}^{\sigma x}\,|\sigma|},$$

combining with (B.3) gives, as in (B.5),

$$\left|\int_{\gamma_-} g_x(z)\frac{h_x(z)}{z}\,\mathrm{d}\,z\right| = \left|\int_{\gamma_-{}'} g_x(z)\frac{h_x(z)}{z}\,\mathrm{d}\,z\right|$$
$$\leqslant \pi R\cdot\frac{B}{\mathrm{e}^{\sigma x}\,|\sigma|}\cdot\frac{2\mathrm{e}^{\sigma x}\,|\sigma|}{R^2} = \frac{2\pi B}{R}.$$

Thus we have a bound of $4\pi B/R$ on everything so far, and we can make this bound as small as we like, by letting R grow large. One integral remains to consider from (B.6). We have

$$\int_{\gamma_-} g(z)\frac{h_x(z)}{z}\,\mathrm{d}\,z = \int_{\gamma_-} \mathrm{e}^{zx}g(z)\left(1+\frac{z^2}{R^2}\right)\frac{\mathrm{d}\,z}{z}.$$

Here x occurs only in the factor e^{zx}. For some large N, we analyze γ_- into components γ_N and γ_N', according to whether the real part of a point is greater than (that is, to the right of) $-\delta/N$ or not. First,

$$\left|\int_{\gamma_N} \mathrm{e}^{zx}g(z)\left(1+\frac{z^2}{R^2}\right)\frac{\mathrm{d}\,z}{z}\right| \leqslant \left|\int_{\gamma_N} g(z)\left(1+\frac{z^2}{R^2}\right)\frac{\mathrm{d}\,z}{z}\right|.$$

We can make this bound, which is independent of x, as small as we like, by making N large enough. Moreover,

$$\left|\int_{\gamma_N'} \mathrm{e}^{zx}g(z)\left(1+\frac{z^2}{R^2}\right)\frac{\mathrm{d}\,z}{z}\right|$$

$$\leqslant \mathrm{e}^{-\delta x/N} \left| \int_{\gamma'_N} g(z) \left(1 + \frac{z^2}{R^2}\right) \frac{\mathrm{d}\,z}{z} \right|,$$

and we can make *this* as small as we like, given N, by making x large enough. Thus for all large R, for all positive ε, for sufficiently large x,

$$|g(0) - g_x(0)| \leqslant \frac{4\pi B}{R} + \varepsilon.$$

This implies (B.1). □

We immediately apply Theorem 17.

Theorem 18. *The integral*

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2}\,\mathrm{d}\,x$$

converges.

Proof. As in the proof of Theorem 2, we consider the primes as forming the increasing sequence $(p_n \colon n \in \omega)$. Now

$$\log p_0 = \vartheta(p_0), \qquad \log p_{n+1} = \vartheta(p_{n+1}) - \vartheta(p_n),$$

so that, when $\sigma > 1$,

$$\begin{aligned}\Phi(s) = \sum_{n\in\omega} \frac{\log p_n}{{p_n}^s} &= \frac{\vartheta(p_0)}{{p_0}^s} + \sum_{n\in\omega} \frac{-\vartheta(p_n) + \vartheta(p_{n+1})}{{p_{n+1}}^s} \\ = \sum_{n\in\omega} \vartheta(p_n) \left(\frac{1}{{p_n}^s} - \frac{1}{{p_{n+1}}^s}\right) &= \sum_{n\in\omega} \vartheta(p_n) \int_{p_n}^{p_{n+1}} \frac{s\,\mathrm{d}\,x}{x^{s+1}} \\ &= s \sum_{n\in\omega} \int_{p_n}^{p_{n+1}} \frac{\vartheta(x)}{x^{s+1}}\,\mathrm{d}\,x = s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}}\,\mathrm{d}\,x,\end{aligned}$$

and therefore

$$\frac{\Phi(s+1)}{s+1} - \frac{1}{s} = \int_1^\infty \frac{\vartheta(x)}{x^{s+2}}\,\mathrm{d}\,x - \int_1^\infty \frac{\mathrm{d}\,x}{x^{s+1}} = \int_1^\infty \frac{\vartheta(x)-x}{x^{s+2}}\,\mathrm{d}\,x.$$

By Theorem 12, the left-hand side extends holomorphically to $\sigma \geqslant 0$. We want to show that the equation still holds when $s = 0$. To apply Theorem 17, we use the substitution

$$x = \mathrm{e}^t, \qquad \mathrm{d}\,x = \mathrm{e}^t\,\mathrm{d}\,t,$$

obtaining

$$\int_1^\infty \frac{\vartheta(x)-x}{x^{s+2}}\,\mathrm{d}\,x = \int_0^\infty \mathrm{e}^{-st}\left(\frac{\vartheta(\mathrm{e}^t)}{\mathrm{e}^t} - 1\right)\mathrm{d}\,t.$$

Since $\vartheta(\mathrm{e}^t)/\mathrm{e}^t$ is bounded by Theorem 4, we are done. □

Theorem 19 (The Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\log x}. \qquad \text{(B.7)}$$

Proof. By Theorems 18, 11, and 10,

$$x \sim \vartheta(x) \sim \pi(x)\log x.$$

□

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