GEOMETRY
AND THE
IMAGINATION

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CHAPTER III

PROJECTIVE CONFIGURATIONS

In this chapter we shall learn about geometrical facts that can be formulated and proved without any measurement or comparison of distances or of angles. It might be imagined that no significant properties of a figure could be found if we do without measurement of distances and angles and that only vague statements could be made. And indeed research was confined to the metrical side of geometry for a long time, and questions of the kind we shall discuss in this chapter arose only later, when the phenomena underlying perspective painting were being studied scientifically. Thus, if a plane figure is projected from a point onto another plane, distances and angles are changed, and in addition, parallel lines may be changed into lines that are not parallel; but certain essential properties must nevertheless remain intact, since we could not otherwise recognize the projection as being a true picture of the original figure.

In this way, the process of projecting led to a new theory, which was called projective geometry because of its origin. Since the 19th century, projective geometry has occupied a central position in geometric research. With the introduction of homogeneous coordinates, it became possible to reduce the theorems of projective geometry to algebraic equations in much the same way that Cartesian coordinates allow this to be done for the theorems of metric geometry. But projective analytic geometry is distinguished by the fact that it is far more symmetrical and general than metric analytic geometry, and when one wishes, conversely, to interpret higher algebraic relations geometrically, one often transforms the relations into homogeneous form and interprets the variables as homogeneous coordinates, because the metric interpretation in Cartesian coordinates would be too unwieldy.

The elementary figures of projective geometry are points, straight lines, and planes. The elementary results of projective geometry deal with the simplest possible relations between these entities, namely their *incidence*. The word incidence covers all the following relations: A point lying on a straight line, a point lying in a plane, a straight line lying in a plane. Clearly, the three statements that a straight line passes through a point, that a plane passes through a point, that a plane passes through a straight line, are respectively equivalent to the first three. The term incidence was introduced to give these three pairs of statements symmetrical form: a straight line is incident with a point, a plane is incident with a point, a plane is incident with a straight line.

The theorems relating to incidence are by far the most important theorems of projective geometry. However, we use two other fundamental concepts, which can not be derived from the concept of incidence. First, we have to distinguish between two different ways in which four collinear points may be arranged; second, we need the concept of continuity, which relates the set of all points on a straight line to the set of all numbers. This completes the list of the basic concepts of projective geometry.

We shall study a particularly instructive part of projective geometry—the configurations. This will also reveal certain aspects of various other geometrical problems. It might be mentioned here that there was a time when the study of configurations was considered the most important branch of all geometry.[1]

§ 15. Preliminary Remarks About Plane Configurations

We define a plane configuration as a system of p points and l straight lines arranged in a plane in such a way that every point of the system is incident with a fixed number λ of straight lines of the system and every straight line of the system is incident with a fixed number π of points of the system. We characterize such a configuration by the symbol $(p_\gamma\, l_\pi)$. The four numbers p, l, π, and λ may not be chosen quite arbitrarily. For, by the conditions we have stipulated, λp straight lines of the system, in all, pass through the p points; however, every straight line is counted π times because it passes through π points; thus the number of straight lines l is equal to $\lambda p/\pi$. It is seen, then, that the following

[1] A comprehensive treatment of the subject is given in the book *Geometrische Konfigurationen* by F. Levi (Leipzig, 1929).

relation must be true for every configuration:

$$p\lambda = l\pi.$$

The simplest configuration consists of a point and a straight line passing through it; it has the symbol $(1_1 1_1)$. The triangle forms the configuration next in order of simplicity, $(3_2 3_2)$. Four straight lines in the plane, no two of which are parallel and no three of which have a common point, give us six points of intersection A, B, C, D, E, and F (see Fig. 104). The figure thus obtained, which is the well-known figure of the complete quadrilateral, is a configuration with the symbol $(6_2 4_3)$. (Note that the equation $6 \cdot 2 = 3 \cdot 4$ confirms our general formula.) In this case, as opposed to the first two trivial cases, not all the straight lines joining points of the configuration are lines of the configuration; similarly, in the general case the points at which the straight lines of a configuration intersect need not all belong to the configuration.

In order to obtain all the straight lines connecting points of the configuration of Fig. 104, we need to adjoin the diagonals AD, BE, CF. This also gives us the vertices P, Q, R of the triangle formed by the diagonals as additional points of intersection. One might think that the continued process of connecting points and adjoining new points of intersection of straight lines might ultimately lead to a configuration that shares the property of the triangle that the straight line connecting any two points of the configuration is itself a line belonging to the configuration and the point of intersection of any two straight lines of the configuration is itself a point belonging to the configuration. However, it may be proved that, except for the triangle, no configuration with this property exists. If, starting with a quadrilateral, we keep connecting points by straight lines and adjoining new points of intersection, it can even be shown that there will ultimately be such points of intersection lying as close as we please to every point of the plane. The figure obtained in this way is called a Moebius net; it may be used for defining projective coordinates.

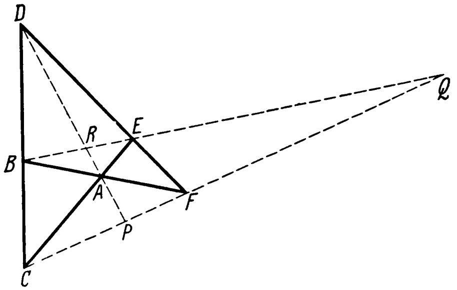

FIG. 104

For the sake of subsequent application, we remind the reader of the significance of the quadrilateral for the construction of harmonic sets of points. Four points C, P, F, Q on a straight line are called a harmonic set—or Q is called the fourth harmonic of P with respect to C and F—if a quadrilateral can be constructed in which these points are determined by the same incidence relations as in Fig. 104. A theorem that is fundamental for projective geometry says that any three points on a straight line have exactly one fourth harmonic. According to this theorem,[2] we may use the points C, P, F as starting points for the construction of two different quadrilaterals but we will come out both times with the same point Q (see Fig. 105).

In the following pages we shall discuss principally those configurations in which the number of points is equal to the number

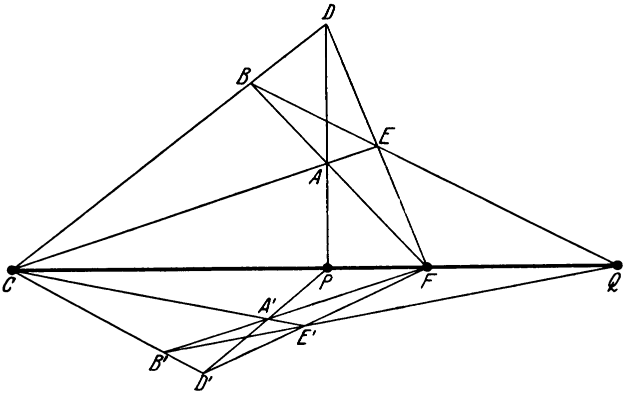

FIG. 105

of lines, i.e. for which $p = l$. Then it follows from the relation $p\lambda = \pi l$ that $\lambda = \pi$, so that the symbol for such a configuration is always of the form $(p_\gamma p_\gamma)$. We shall introduce the more concise notation (p_γ) for such a configuration. Furthermore, we shall make the reasonable stipulation that the configuration be connected and be not decomposable into separate figures.

The cases $\lambda = 1$ and $\lambda = 2$ are unimportant. $\lambda = 1$ yields only the trivial configuration consisting of a point and a straight line passing through it. For, if a configuration with $\lambda = 1$ had several

[2] This theorem is an immediate consequence of the theorem of Desargues discussed in § 19.

points, it would necessarily consist of separate parts, since no straight line of the configuration may contain more than one point. The case $\lambda = 2$ is realized by the closed polygons in the plane; and conversely, the conditions that each point of a configuration (p_2) be incident with two straight lines and each straight line with two points may be seen to imply that every configuration of the form (p_2) consists of the vertices and sides of a p-sided polygon.

On the other hand, the case $\lambda = 3$ gives rise to many interesting configurations. In this case the number of points (and straight lines), p, must be at least seven. For through any given point of the configuration there pass three lines, on each of which there must be two further points of the configuration. We shall go into detail only for the cases where $7 \leqq p \leqq 10$.

§ 16. The Configurations (7_3) and (8_3)

In constructing a configuration with the symbol (p_γ), the following method will be found the simplest: We label the p points with the numbers 1 through p and label the p straight lines, similarly, with the numbers (1) through (p). Then we set up a rectangular scheme of $p\lambda$ points in which the λ points incident with any given straight line are arranged in a column; there will be p columns corresponding to the p straight lines.

In this way, the scheme corresponding to the configuration (7_3) is as follows:

$$\begin{array}{c} \\ \lambda \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \end{array} \overbrace{\begin{array}{ccccccc} (1) & (2) & (3) & (4) & (5) & (6) & (7) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}}^{p}$$

In filling in the spaces, the following three conditions must be satisfied. First, the numbers written in any one column must all be different to ensure that no less than three points are on any given straight line. Second, two different columns cannot have two numbers in common, as this would make the straight lines corresponding to the columns coincide. And third, every number must occur three times in all, since three straight lines are supposed to pass through every point. These three conditions are certainly necessary if a geometrical counterpart for the schematic table is to exist. On the other hand, they are not sufficient, as we shall soon see by some examples. The reason for this is that the geometrical realization of a table also depends on some geometric or algebraic considerations which cannot be directly expressed in terms of the arithmetic scheme. But if a table does represent a configuration, then it admits several alterations that do not affect the configuration in any way. Thus the vertical order of the numbers in any column may be changed. Also, the order of the columns themselves may be changed, as this only corresponds to a renumbering of the straight lines. And finally, the numbering of the points may also be changed at will. Since all these alterations in the schematic representation leave the configuration unchanged, we shall consider all tables differing only by such transformations as identical.

With this understanding, we may construct one, but only one, table having the symbol (7_3). To begin with, we denote the points on the first straight line by 1, 2, and 3. Then two more straight lines pass through the point 1, and they cannot contain the points 2 and 3. Let us denote the points of the second straight line by 4 and 5, and those of the third straight line by 6 and 7. Now all the points are numbered and the table is partly filled in, as follows:

1	1	1	.	.	.	.
2	4	6	.	.	.	.
3	5	7	.	.	.	.

In the remaining columns, each of the numbers 2 and 3 has to appear two more times, subject to the condition that they be not both in one column. Hence we complete the first row as follows:

1	1	1	2	2	3	3
2	4	6	.	.	.	.
3	5	7	.	.	.	.

The numbers 1, 2, and 3 are used up, and only 4, 5, 6, and 7 are available for filling in the remaining eight places. The number 4 has to appear two more times and may not be written under the same number both times. Thus we may place the 4's as follows:

2	2	3	3
4	.	4	.
.	.	.	.

All the other possible arrangements are not essentially different from this one. Again, 5 has to occur twice, but may no longer occur under a 4. Thus we may write

2	2	3	3
4	5	4	5
.	.	.	.

The first two of the four remaining places have to be occupied by 6 and 7 because they are the only numbers left and because we can not use the same one of them in both columns containing a 2. Interchanging the numbers 6 and 7 would not constitute an essential modification, and so we may write

4	5	4	5
6	7	.	.

The remaining places are necessarily filled in the order 7, 6. Thus we have indeed obtained just one possibility for the configuration (7_3), namely

(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

We have already mentioned earlier that the existence of this table does not imply the existence of an actual configuration (7_3). Now it will turn out that such a configuration is indeed impossible. This may be seen by trying to find the equations of the straight lines of the table by the methods of analytic geometry, which leads to an incompatible system of equations. We can also demonstrate the non-existence of the configuration by means of a diagram. To begin with, we draw the straight lines (1) and (2) of Fig. 106, denote their point of intersection by 1 as indicated in the table, and let 2, 3 and 4, 5 be

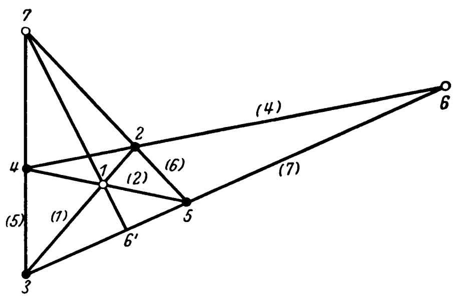

FIG. 106

arbitrary pairs of points on the lines (1) and (2) respectively. Then we draw the straight lines (4) and (7) whose positions are fixed by the pairs of points 2, 4 and 3, 5 and whose point of intersection, according to the table, has to be labeled 6. Similarly, the pairs of points 2, 5 and 3, 4 determine the lines (5) and (6) and their point of intersection 7. All the points of the configuration are now determined. But the three points 1, 6, and 7, which are supposed to be on the remaining straight line (3), are not collinear, so that the intersection of the lines (17) and (7) gives us an additional point 6′. It might be imagined that this is due to an unfortunate choice of the points 2, 3, 4, and 5. But such is not the case. For, our figure is a reproduction of the harmonic construction of Fig. 104; consequently, 6′ is the fourth harmonic of the point 6 with respect to 3 and 5, and it follows by an elementary theorem of projective geometry that 6′ cannot coincide with any of these three points.

We turn to the configuration (8_3). By the same method as before, it can be shown that there is essentially only one possible table, namely

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	1	1	2	2	3	3	4
2	4	6	3	7	4	5	5
5	8	7	6	8	7	8	6

The configuration may be interpreted as consisting of two quadrilaterals 1234 and 5678 each of which is inscribed in and at the same time circumscribed about the other (see Fig. 107; see also the footnote on p. 110). For, the line 12 passes through the point 5, the line 23 through the point 7, and the line 34 through the point 7, and the line 41 through the point 8, and at the same time the sides 56, 67, 78, and 85 are incident with the points 4, 1, 2, and 3, respectively. Obviously, it is not possible to draw a configuration of this kind. Applying analytic methods, we find that the table gives rise to a system of equations which—while it does not contain a contradiction, as in the

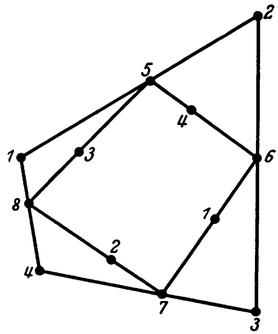

FIG. 107

case of (7_3)—has, however, complex solutions only, and never any real solutions.

The configuration is nevertheless not without geometric interest and has an important role in the theory of third-order plane curves without double points. These curves have nine points of inflection, but at most three of them can be real. Furthermore, it can be demonstrated algebraically that every straight line connecting any two of these points of inflection must pass through a third point of inflection. No four points of inflection, on the other hand, can ever be collinear, because a third-order curve cannot meet a straight line in more than three points. Now, the straight lines connecting points of inflection form a configuration, and we have for this configuration $p = 9$, $\pi = 3$. Also, $\lambda = 4$, which can be seen as follows: If any point of inflection is selected, the remaining eight of them are collinear with it in pairs, so that each point is in fact incident with four straight lines. The formula $l = p\lambda/\pi$ gives the value 12 for l. Thus the configuration is of the type $(9_4 12_3)$. For the table of such a configuration there is essentially only one possibility, namely

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	1	1	2	2	3	3	4	1	2	5	6
2	4	6	3	7	4	5	5	3	4	7	8
5	8	7	6	8	7	8	6	9	9	9	9

If the point 9 and the lines passing through it, viz. (9), (10), (11), and (12), are omitted from this table, what remains is precisely the same as our table (8_3). The configuration (8_3) is also obtained on the omission of any other one of the nine points together with the four straight lines passing through it. For it is found that all the points of the configuration $(9_4 12_3)$ are equivalent.

§ 17. The Configurations (9_3)

While the cases $p = 7$ and $p = 8$ gave rise to only one table each, neither of which could be realized geometrically, the case $p = 9$ gives rise to three essentially different tables, and all of them represent configurations of real points and lines.

By far the most important of these configurations, and indeed the most important configuration of all geometry, is the one known as the Brianchon-Pascal configuration. For the sake of brevity we shall give it the symbol $(9_3)_1$ and use the symbols $(9_3)_2$ and $(9_3)_3$ for the other two configurations of type (9_3).

The table for the configuration $(9_3)_1$ may be written as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	1	1	2	2	3	3	4	5
2	4	6	4	7	6	5	6	7
3	5	7	8	9	8	9	9	8

In drawing such a configuration, we begin with the points 8 and 9, which may be chosen arbitrarily (see Fig. 108), and draw the arbitrary straight lines (4), (6), and (9) through 8, and (5), (7), and (8) through 9. Six of the nine resulting points of intersection belong to the configuration; in accordance with the table, we shall designate them by 2, 3, 4, 5, 6, and 7. These six points fix the positions of the remaining straight lines (1), (2), and (3). First of all, we draw the line (1) through 2 and 3, and the line (2) through 4 and 5. Their point of intersection has to be labeled 1. The straight line (3) is determined by the points 6 and 7. According to the table, this line must pass through 1. Now it is found that this condition is automatically satisfied despite the arbitrary choice of the points 8 and 9 and of the three straight lines through each of these points.

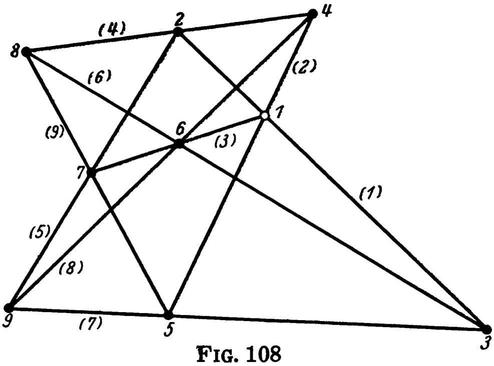

FIG. 108

The geometric reason for this surprising phenomenon lies in the theorems of Brianchon, which we shall now study.

Our point of departure is the hyperboloid of one sheet. As we have seen in Chapter I, the surface contains two families of straight lines such that every straight line of one family intersects every straight line of the other, while two lines of the same family never meet. Let us pick three straight lines of one family (drawn as double lines in Fig. 109) and three of the other (drawn as heavy single lines in the figure), from which we obtain the hexagon $ABCDEFA$ in space, as follows: On a straight line of the first family we move from A to B; a definite line of the second family passes through B, and along this we move to a point C; from C we follow the straight line of the first family passing through that point to another point D; thence we move to E along a line of the second family, and finally follow a line of the first family to that point F where it intersects the line of the second family that goes through A. Thus the sides of the hexagon belong alternately to the first family and the second.

We shall now prove that all three diagonals AD, BE, and CF of the hexagon have a point in common. We begin with AD and BE. The sides AB and DE of the hexagon have a common point because AB belongs to one and DE to the other family of straight lines on the hyperboloid. Therefore the four points A, B, D, E lie in one plane, and so AD and BE also have a point in common. In exactly the same way it can be shown that each of the other two pairs of diagonals also intersects at a point. But three straight lines that intersect each other in pairs are coplanar, or, if not, must all pass through a common point. Now if the three diagonals of the hexagon $ABCDEF$ were all in one plane, the hexagon itself would also have to lie in this plane, and any two of its sides would have a point in common; this is ruled out, since AB and CD (to give one example) are straight lines of the same family and therefore cannot intersect each other. All three diagonals do, accordingly, pass through one point.

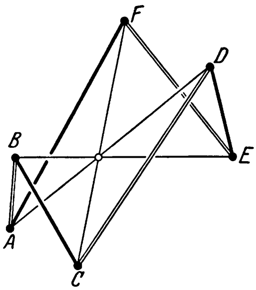

FIG. 109

This theorem of the geometry of space leads to the Brianchon theorems of plane geometry. To obtain them, we look at the hyperboloid of one sheet from a point P, which for the time being we shall assume not to lie on the surface. The contour of the hyperboloid as seen from this point is a conic section which may be either a hyperbola (Fig. 110) or an ellipse (Fig. 111). The area on one side of the contour appears empty, while the region on the other side appears doubly covered, what appear to be two layers in the picture being connected along the conic forming the contour. The straight lines of the surface are partly visible in the picture, and partly covered. Thus, they extend from one layer into the other and must therefore meet the contour. On the other hand, they can not intersect that curve, since one side of it is empty. Hence our hexagon in space has become a plane hexagon whose sides are tangent to a conic; this gives us the following theorem of plane geometry:

The diagonals of a hexagon that is circumscribed about a conic intersect at one point.

So far we have not proved the theorem except for those conics that can be obtained as the outline of a hyperboloid of one sheet, that is, only for certain ellipses and hyperbolas. But we shall immediately see that the outline can also be a parabola. For, the lines of sight which give rise to the outline—or, more technically,

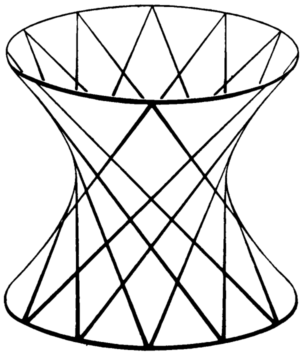

FIG. 110

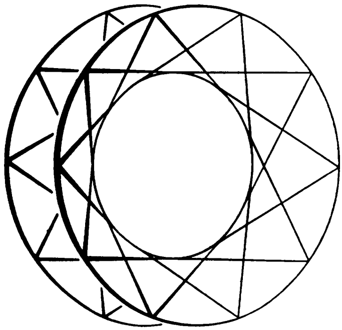

FIG. 111

a central projection—form the tangent cone of the surface with the vertex P, i.e. a second-order cone (see p. 12); but the outline, or central projection, is the curve in which this cone intersects the image plane, and this is a parabola if we choose as the image plane a plane parallel to one of the generators of the cone (see pp. 12, 13, 8).

We shall now go over to the case where the surface is observed from a point P (the center of projection) that is on the surface itself. Here, the two straight lines of the surface that pass through P are seen as two points, while the other straight lines are still seen as straight lines. And since every line of one family intersects the line of the other family that passes through the center of projection, the first family is seen as a pencil of lines whose vertex is the image of the straight line g of the other family that passes through P. Similarly, the other family is also seen as a pencil of lines. The vertices of the two pencils are distinct, being the images of two different straight lines passing through P. The following theorem is accordingly a consequence of the theorem about the space hexagon:

The diagonals of a plane hexagon whose sides pass alternately through two fixed points, meet at a point.

These theorems about the tangent hexagons of one of the three types of conics or of a degenerate conic consisting of a pair of points are called Brianchon's theorems, after their discoverer. The point at which the three diagonals meet is called the Brianchon point.

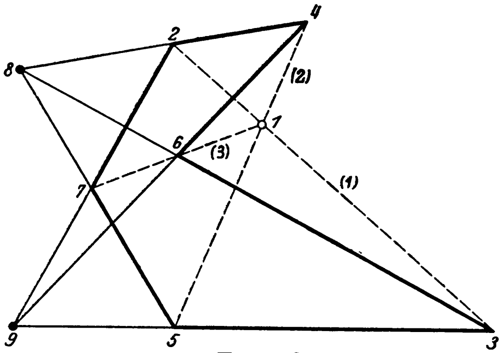

FIG. 112

Our space construction does not, to be sure, complete the proof of Brianchon's theorems, as it might be possible that not every Brianchon hexagon can be obtained as a projection of a space hexagon of the type we have considered. It can be proved, however, that it is indeed possible to start with any hexagon that satisfies the Brianchon assumptions and construct from it a spatial figure of the sort we have been considering.

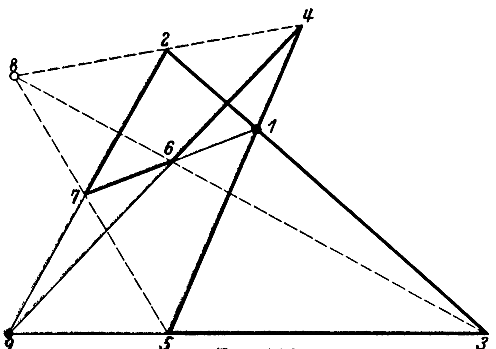

FIG. 113

Now the last of the Brianchon theorems is closely connected with the configuration $(9_3)_1$ and explains the fact that the last incidence condition is automatically satisfied in the construction of this configuration. Indeed we see that, in the notation of Figs. 112 and 108, the points 2, 4, 6, 3, 5, 7 form a hexagon whose sides pass alternately through the points 8 and 9, and the straight lines (1), (2), and (3) are the diagonals 23, 45, and 67 of this hexagon. So (3) must pass through the point of intersection 1 of the straight lines (1) and (2), and 1 is the Brianchon point of the hexagon:

In our construction, the points of the configuration $(9_3)_1$ do not all play the same roles: the points 2, 4, 6, 3, 5, 7 form the hexagon; 8 and 9 are the points through which the sides pass; and 1 is the Brianchon point. But this lack of symmetry is not inherent in the configuration but is due to an arbitrary choice on our part. For we may also assign the role of the Brianchon point to 8 or 9. It is sufficient to make this clear as regards the point 8 (see Fig. 113), since we see from Fig. 112 that 8 and 9 are alike. Similarly, we may choose any one of the points 2, 4, 6, 3, 5, 7 for the role of Brianchon point. Again, it is sufficient to show this for the point 2 (see Fig. 114), since all the points 2, 4, 6, 3, 5, 7 are alike in their relation to the rest of the figure.

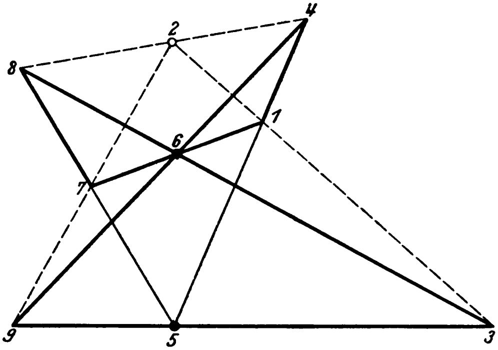

FIG. 114

Owing to this inherent symmetry, $(9_3)_1$ is called a *regular* configuration. In much the same way as in the study of point systems and polyhedra, we arrive at the concept of regularity by the study

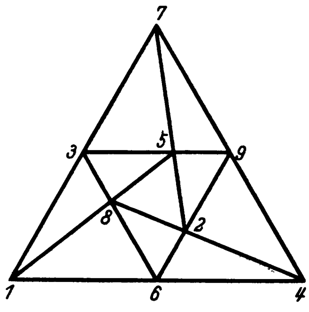

FIG. 115

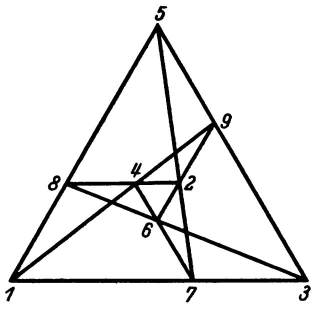

FIG. 116

of certain mappings of a configuration into itself which are called "automorphisms" and are analogous to the symmetry transformations in the case of point systems and polyhedra. We obtain an automorphism of a configuration if we can permute its points and its lines in such a way that no incidence is lost and no new incidence added. It is easy to see that the automorphisms form a group. Now a configuration is called regular if the group of its automorphisms is "transitive," i.e. if it contains enough transformations so that every point of the configuration can be transformed into every other point of the configuration by one of them.

For the study of the automorphisms of a configuration it suffices to consider its abstract scheme. In this way it may be shown that the tables for (7_3) and (8_3) are regular. The same is true for $(9_4 12_3)$ (see p. 102).

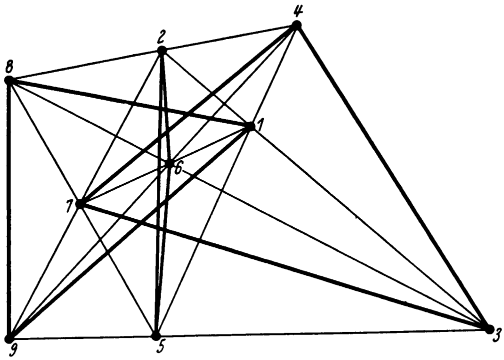

FIG. 117

Let us now turn to the other two configurations (9_3). They are shown in Figs. 115 and 116. In order to see what it is that differentiates the three configurations of the type (9_3), we may proceed as follows. Since every point in a configuration (p_3) is connected with exactly six others by lines of the configuration, it follows in the case $p = 9$ that for every point of the configuration there are exactly two others not connected with it. For example, in $(9_3)_1$ the points 8 and 9 are not connected with 1. Also there is no line connecting 8 with 9. Hence 1, 8, and 9 form a triangle of unconnected points. Similarly 2, 5, 6 and 3, 4, 7 form such triangles

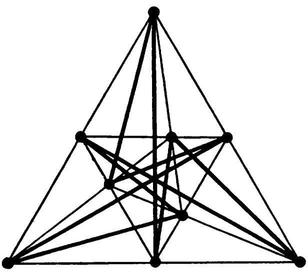
FIG. 118

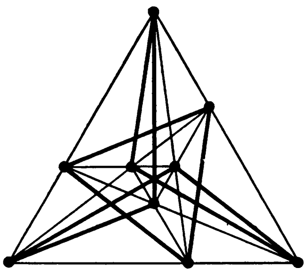
FIG. 119

(Fig. 117). Let us use the same procedure for $(9_3)_2$ and $(9_3)_3$, combining the paths between unconnected points to form polygons. In the case $(9_3)_2$ we get a nonagon (Fig. 118), and in $(9_3)_3$ a hexagon and a triangle (Fig. 119). This tells us, first, that the three figures 108, 115, and 116 do not merely differ in the positions of their points but are essentially different configurations. Furthermore, we may conclude that the configuration $(9_3)_3$ cannot possibly be regular. For, an automorphism can transform points of the hexagon only into points of the hexagon, and never into points of the triangle. In the case $(9_3)_3$, on the other hand, the regular arrangement of the unconnected points leads us to conjecture that the configuration is regular. This is confirmed by further inspection of the table.

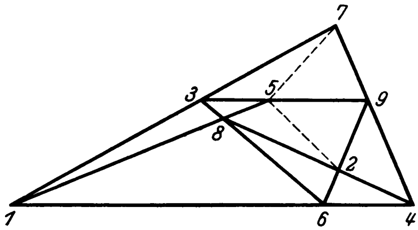

FIG. 120

We may try to construct the other two configurations step by step in much the same way as we constructed $(9_3)_1$. But we then find that the last incidence condition is no longer satisfied automatically but is satisfied only if special provisions have been made in the preceding steps. This is the reason why $(9_3)_2$ and $(9_3)_3$ are not of such fundamental importance as $(9_3)_1$; they do not express a general theorem of projective geometry. Fig. 120 illustrates a case in which the last straight line of $(9_3)_2$ cannot be drawn.

The auxiliary constructions that are necessary to make possible the construction of $(9_3)_2$ and $(9_3)_3$ are, however, distinguished by a special property: they can be carried out by means of a ruler alone, so that all three of the configurations (9_3) can be constructed without any instruments except a ruler This is expressed analytically by the fact that all the elements of the configuration can be determined by the successive solution of linear equations in which the coefficients of each equation are rational functions of the characteristic quantities of the configuration that have already been determined from the preceding equations. It is quite true, of course, that the equations of straight lines are always linear. But in obtaining the system of

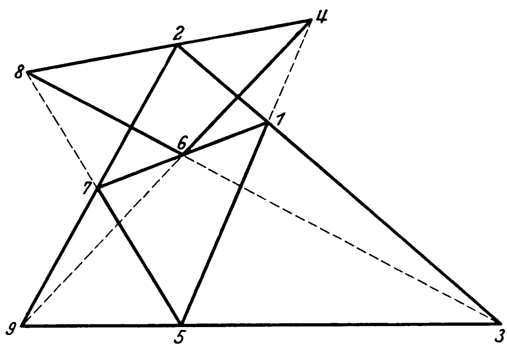

FIG. 121

equations of a configuration, the coefficients of some of the equations have to be computed from other equations by elimination, since some of the straight lines are fixed by the straight lines that have been constructed before. In the general case, this elimination

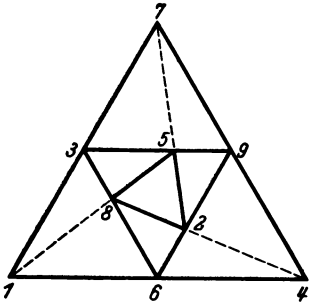

FIG. 122

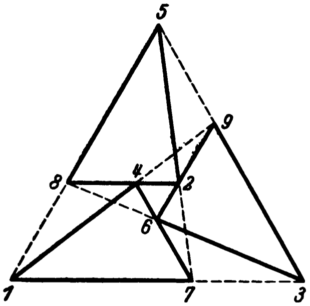

FIG. 123

gives rise to equations of higher degree; this must be the case in (8_3) since we could not otherwise get any complex elements. Now the special property of the configurations (9_3) is that all the auxiliary equations are linear, with the result that all three configurations can be constructed in the real plane and with the sole use of a ruler.

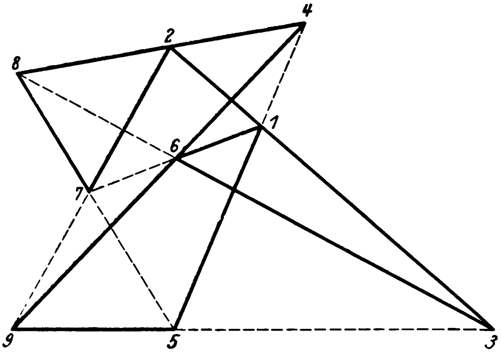

FIG. 124

The arrangement of the elements in the configurations (9_3) may be interpreted in a variety of different ways. For example, each of the configurations can be considered as forming three triangles of which the first is inscribed in the second, the second in the third, and the third in the first.[1]

[1] The word "inscribed" is used here in a generalized sense; thus in Fig. 121, the triangle 468 is said to be inscribed in triangle 157 because 4 is on the straight line 15, 6 on the straight line 17, and 8 on the straight line 75, although 4 and 8 are not on the segments 15 and 75 but on their continuations. "Circumscribed" is used in the corresponding general sense, triangle *A* being "circumscribed" about triangle *B* if triangle *B* is "inscribed" in triangle *A*. The same remarks apply to the use below of "inscribed" and "circumscribed" in reference to general polygons. [*Trans.*]

The triangles 157, 239, 468 of Fig. 121, the triangles 258, 369, 147 of Fig. 122, and the triangles 147, 258, and 369 of Fig. 123 are examples of such systems of triangles. Similarly we interpreted (8_3) as a pair of mutually inscribed and circumscribed quadrilaterals (see Fig. 107, p. 101). The three configurations (9_3) can also be interpreted as nonagons inscribed in and circumscribed about themselves; examples of such nonagons are 2361594872 of Fig. 124, 1627384951 of Fig. 125, and 1473695281 of Fig. 126. In the configuration $(9_3)_1$ we can find several additional nonagons with the same properties, by applying suitable automorphisms.

The construction of p-sided polygons that are inscribed in and circumscribed about themselves necessarily leads to configurations

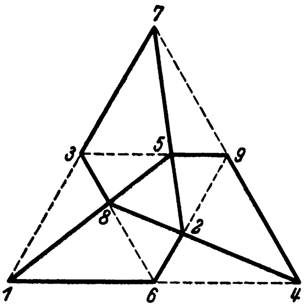

FIG. 125

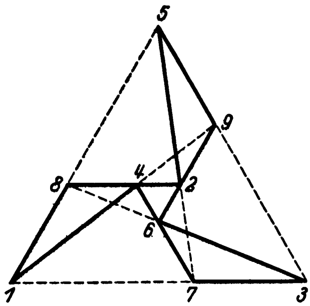

FIG. 126

of the type (p_3). For, every side of the polygon contains one vertex of the polygon in addition to the vertices it connects, and every vertex, likewise, must be incident with three sides of the polygon. The only assumption needed in this argument was that all the sides and all the vertices of the polygon play the same role. If this assumption were not made, one side could contain two or more extra vertices; but then some other side of the polygon would have to be empty.

(7_3) and (8_3) may also be interpreted as being p-sided polygons of this type. In the notation of the configuration tables, the heptagon 12457361 and the octagon 126534871 are inscribed in and circumscribed about themselves.

In order to understand another important property of configurations, we must study the principle of duality. It is this principle that confers upon projective geometry its special clarity and symmetry. It may be derived in visual terms from the method of projecting, which we have already used in arriving at Brianchon's theorems.

§ 18. Perspective, Ideal Elements, and the Principle of Duality in the Plane

If we draw the picture of a flat landscape on the blackboard, the landscape being a horizontal plane and the blackboard a vertical plane, then the image of the horizontal plane appears to be bounded

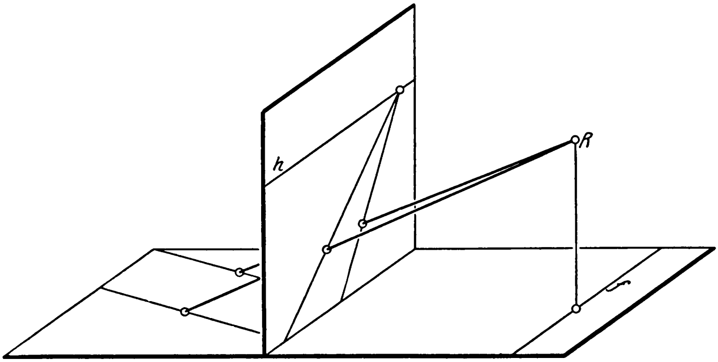

FIG. 127

by a straight line h, the horizon (see Fig. 127). Two parallel straight lines in the horizontal plane which are not parallel to the plane of the blackboard appear in the picture as straight lines that meet on the horizon. In painting, the point of intersection of the two lines in the image is called the vanishing point of the parallel lines.

We see, then, that the images of parallel lines under central perspective are not usually parallel. We see furthermore that the mapping effected is not one-to-one. The points of the horizon on the image plane do not represent any points of the original plane. Conversely, there are points of the plane which do not have an image. These are the points of the straight line f that is vertically below the observer R and parallel to the image plane (Fig. 127).

The description of this phenomenon can be simplified by replacing each point of the plane by the line of sight passing through the point. Thus we replace every point P of the plane e (Fig. 128) by the straight line $AP = p$ connecting P with A, the point where the observer's eye is located. Then the image of P on an arbitrarily placed board t is the point P' at which the straight line p meets the board; thus the mapping is determined once P is given. If P describes a curve in e, then p sweeps out a cone with A as vertex. The image of the curve on t is the intersection of t and the cone. In particular, if P moves along a straight line g in e, the cone becomes the plane γ that contains A and g. Thus, while the points of e become straight lines through A, the straight lines of e give rise to planes through A. The image on t of the straight line g is the intersection of t and γ, i.e. another straight line g'. This property of transforming straight lines into straight lines is the most important property of a central perspective.

We have expressed the perspective mapping as the resultant of

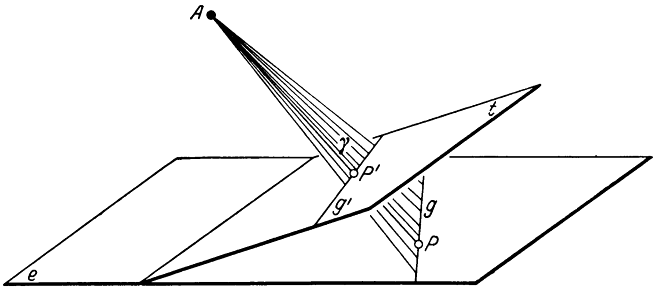

FIG. 128

two transformations that are of types that may be considered as the inverses of each other. First the points (P) and straight lines (g) of a plane are replaced by the straight lines (p) and the planes (γ) passing through A, and then the straight lines and planes through A are transformed into the points (P') and straight lines (g') of another plane. For reasons of symmetry, it therefore suffices to study only the first step.

This transformation $e \rightarrow A$ is fully defined only in the given direction, not in the reverse direction $A \rightarrow e$. The transformation assigns a special role to those straight lines through A that are parallel to e; they do not correspond to any point of e, while each of the remaining straight lines through A belongs to a definite point of e, namely the point at which it intersects e. The straight lines p_u through A parallel to the plane e fill out a plane γ_u, the plane through A that is parallel to e (see Fig. 129). Of all the planes containing A, γ_u is also the one that plays an anomalous role in the transformation $A \rightarrow e$. For, each of the other planes through A is associated with a definite straight line g of e, the line in which it cuts e, but no such straight line corresponds to the plane γ_u, since it does not meet e.

Now it is expedient to eliminate these exceptions conceptually by assigning additional points P_u to the plane e, as "infinitely distant" or "ideal" points. These "points" are defined by the stipulation that they shall be the images of the rays p_u in the transformation $A \to e$. They are regarded as constituting, in their totality, the image of the plane γ_u. In order to divest this plane of its anomalous position in relation to the other planes passing through A, we have to call its image a straight line. We therefore say that the infinitely distant points of e form a straight line g_u, the so-called infinitely distant[1] or "ideal" line of e. Clearly the mapping of the points and straight lines of e into the straight lines and planes through A is fully defined and one-to-one once we have supplemented the plane e in the manner described.

The suitability of the definitions we have introduced becomes apparent on examining the central perspective of e onto any other plane t. The plane t must also be supplemented by ideal points constituting the ideal line of this plane. But unless e and t happen to be parallel, the plane that goes into the ideal line l_u of t under the transformation $A \to t$ is not γ_u but some other plane λ through A. λ meets e in a straight line l. Hence the perspective mapping $e \to t$ associates the points of the infinitely distant line of the second plane with the points of an ordinary straight line in the first plane. It is only the introduction of the ideal points that makes the central perspective a one-to-one mapping of the points and straight lines of one plane into the points and straight lines of another plane. In this mapping, the infinitely distant points are on a par with the finite points.

We shall now look into the question of how the concept of incidence between points and straight lines must be extended to accommodate the ideal elements we have added. As before, we begin with the transformation $e \to A$. An *ordinary* point P and an *ordinary* straight line g of e are incident if and only if the corresponding p and γ are incident. Let us generalize this to cover

[1] The term "infinitely distant" stems from the fact that the ray from a point of e to the eye approaches one of the straight lines p_u if the point of e recedes indefinitely in a fixed direction.

arbitrary points and straight lines of e. An infinitely distant point P_u and a straight line g shall be called incident if the ray p_u is incident with γ. If γ coincides with γ_u, i.e. if g is the ideal line of e, this does not tell us anything new. But if g is an ordinary straight line, then γ and γ_u intersect in a definite straight line p_u. Hence every ordinary straight line has exactly one infinitely distant point, its point of intersection with g_u. If g' and g are parallel, this means that the plane γ' belonging to g' passes through p_u (see Fig. 129). Accordingly, two straight lines are parallel if and only if they have the same infinitely distant point; this is the meaning of the occasionally used mode of expression "parallels meet at infinity," which in itself, and when stated without further explanation, would be meaningless. At the same time we recognize the reason for the fact mentioned at the beginning of this section, that two parallel straight lines appear to meet at their vanishing point on the horizon.

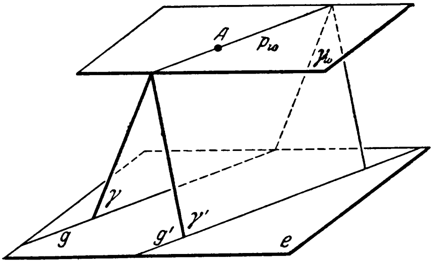

FIG. 129

As an example of the way geometrical notions are simplified by the introduction of the ideal elements, we may cite the conics. Since, as we have proved in Chapter I, they can be obtained as the plane sections of a circular cone, they may all be regarded as perspective images of a circle. According to whether no projecting ray, one ray, or two rays are parallel to the image plane, we obtained an ellipse, a parabola, or a hyperbola, respectively. We may now formulate this as follows: A conic section is an ellipse, a parabola, or a hyperbola according to whether it meets the ideal line in no point, in one point, or in two points, respectively. A central projection onto another plane transforms the conic under consideration

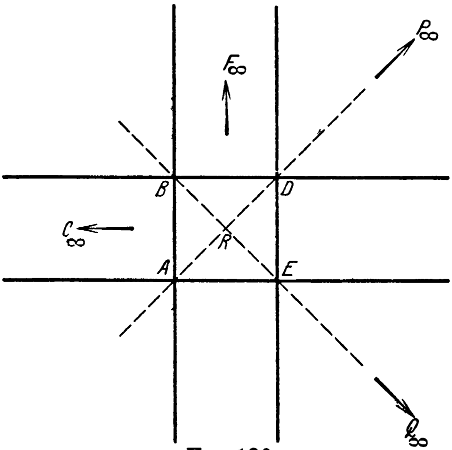

FIG. 130

into another conic that either does not meet the horizon, or touches it, or intersects it in two points, as the case may be. What type of conic the image will be depends on the position of the image plane.

In other cases too, central projection is an important tool for getting much more general figures from special figures. For example, the complete quadrilateral (p. 96) can always be derived from the simple construction of the adjoining figure (Fig. 130).

The importance of the ideal elements, however, lies mainly in the fact that they enable us to modify and considerably simplify the axiomatic foundation of plane geometry. If we confine ourselves to the finite points of the plane, the incidence of points and straight lines is subject to the following axioms:

1. Two distinct points define a straight line with which they are incident.
2. Two distinct points define only one straight line with which they are incident.

From the second axiom it follows that two straight lines in a plane either have one point or no point in common. For if they had two or more common points, they would necessarily be one and the same straight line.

The case where two straight lines have no point in common is elucidated by and subject to the Euclidean axiom of parallels:

If there is given in a plane any straight line a and any point A, where a and A are not incident, there is in the plane one and only one straight line b that passes through A and does not intersect a; the straight line b is called the *parallel* to a through A.

Now if we no longer consider only finite points but enlarge the plane into the "projective plane" by adding the ideal line, then we are in a position to use the two following axioms as a basis instead of the three axioms above.

1. Two distinct points determine one and only one straight line.
2. Two distinct straight lines determine one and only one point.

These two axioms determine the incidence of points and straight lines in the projective plane. Ideal points and the ideal straight line are in no way distinguished here from other points and straight lines. If it is desired to represent the projective plane by a real structure where the equivalence of all points and of all straight lines can be recognized visually, we may refer back to the bundle of straight lines and planes through a fixed point, regarding the straight lines as "points" and the planes as "straight lines." In this model the validity of the two axioms last mentioned is easily verified.

Now this pair of axioms has the purely formal property of remaining unchanged if the word "straight line" is replaced by "point" and the word "point" by "straight line." On closer inspection we see that the remaining axioms of plane projective geometry are also left unchanged when these two words are interchanged. But the two words must then also be interchangeable in all the theorems deduced from these axioms. The interchangeability of points and lines is called the principle of *duality* in the projective plane. According to this principle, there belongs to every theorem a second theorem that corresponds to it dually, and to every figure a second figure that corresponds to it dually. Under this dual correspondence, the points of a curve correspond to a collection of straight lines that in general envelop a second curve as tangents. A more detailed study reveals that the family of straight lines corresponding dually to the points of a conic always envelops another conic.

By the principle of duality we can deduce a number of other theorems from Brianchon's theorems. They are called Pascal's theorems, after their discoverer. In order to bring out the duality of the two groups of theorems more clearly, we shall write them side by side in exactly corresponding forms.

Brianchon Theorems	Pascal Theorems
1, 2, 3. Let there be given a hexagon formed by six straight lines that are tangent to a conic (hexagon circumscribed about a conic). Then the three lines joining opposite vertices intersect at one point.	1, 2, 3. Let there be given a hexagon formed by six points that lie on a conic (hexagon inscribed in a conic). Then the three points of intersection of opposite sides lie on one straight line.
4. Let there be given six straight lines of which three are incident with a point A and three are incident with a point B. Choose six points of intersection, which together with the appropriate connecting lines form a hexagon whose sides pass alter-	4. Let there be given six points of which three are incident with a straight line a and three are incident with a straight line b. Choose six connecting lines, which together with the appropriate points of intersection form a hexagon whose ver-

nately through A and B. Then the straight lines connecting opposite vertices intersect at one point (the Brianchon point of the hexagon).

tices lie alternately on a and b. Then the points of intersection of opposite sides lie on one straight line (the Pascal line of the hexagon).

Evidently the figure corresponding to the last theorem of Pascal must be the dual of the configuration $(9_3)_1$. Now the dual figure of a configuration $(p_\gamma\, l_\pi)$ is always another configuration, and its symbol is $(l_\pi p_\gamma)$. The special configurations we have denoted by the symbol (p_γ), and they only, have as duals configurations with the same symbol. It is conceivable that the configuration of Pascal's theorem, i.e. the dual of $(9_3)_1$, might be one of the other two con-

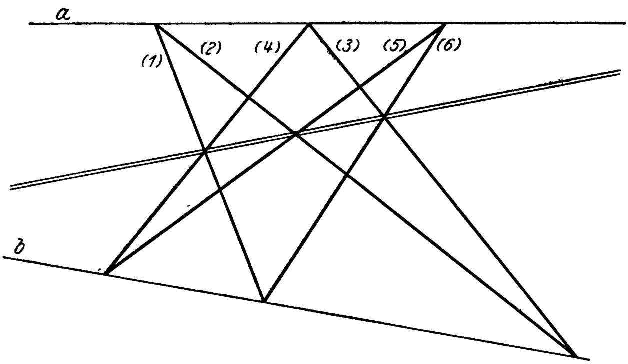

FIG. 131

figurations (9_3). It is found, however, that Pascal's theorem is also represented by the symbol $(9_3)_1$ (see Fig. 131). This is the reason why we have called the configuration the Brianchon-Pascal configuration from the very beginning. Thus $(9_3)_1$ is "dually invariant" or "self-dual." Just as the Brianchon point could be chosen arbitrarily, so we can also choose an arbitrary straight line of the configuration to serve as the Pascal line.

By using the ideal elements we can arrive at a special case of the last Pascal theorem which would not otherwise seem to have any connection with the original theorem. For, by moving the Pascal line to infinity we get the following theorem (Fig. 132): If the vertices of a hexagon lie alternately on two straight lines, and if two pairs of opposite sides are respectively parallel, then the third pair of opposite sides is also parallel.

This special case of Pascal's theorem is called Pappus' theorem.[2]

Having seen that $(9_3)_1$ is self-dual, it is easy for us to conclude that $(9_3)_2$ and $(9_3)_3$ must also be self-dual. For, the only other possibility would be that the figure obtained from $(9_3)_2$ by applying the duality principle is $(9_3)_3$. But since $(9_3)_2$ is a regular con-

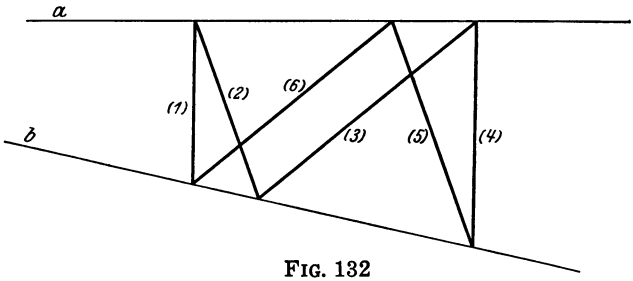

FIG. 132

figuration and $(9_3)_3$ is not, neither of these figures can be the dual of the other.

We shall now take up the configurations (10_3). In order to understand the most important one of these, Desargues' configuration, it is necessary to extend the method of introducing ideal elements, and the principle of duality, from the plane to three-dimensional space.

§ 19. Ideal Elements and the Principle of Duality in Space. Desargues' Theorem and the Desargues Configuration (10_3)

We have arrived at the concept of the projective plane by studying projection in space. Now projective geometry also changes the space as a whole, by the addition of ideal elements, into "projective space," an entity that is in many ways simpler. Only, it is not possible in this case to justify the procedure in visual terms; it is purely abstract. To begin with, we introduce the ideal elements in all the planes of ordinary space according to the principle discussed earlier. Then it appears reasonable to interpret the entity formed by all the ideal points and straight lines as a plane, the "infinitely distant" or "ideal" plane of the space. For, this entity shares with the ordinary planes in space the property that any given plane intersects it in a straight line, the ideal straight line

[2] Frequently the more general theorem, which is called here the fourth Pascal theorem, is also referred to as Pappus' theorem. [*Trans.*]

of the given plane. Every ordinary straight line has only one point, its ideal point, in common with the ideal plane, just as it has only one point in common with any other plane that does not contain the line. Moreover two planes are parallel if and only if they have the same ideal line.[1]

A great many phenomena of the geometry of space are simplified by this point of view. Thus parallel projection can be regarded as a special case of central projection in which the center of projection is an infinitely distant point. Furthermore, to give another example, the difference between the hyperboloid of one sheet and the hyperbolic paraboloid may be characterized by the property that the hyperboloid intersects the ideal plane in a non-degenerate conic whereas the paraboloid intersects it in a pair of generating straight lines of the surface; this distinction amounts to the same thing as the fact explained on page 15, that three skew straight lines lie on a paraboloid rather than on a hyperboloid if and only if they are parallel to a fixed plane; for, this is equivalent to the condition that the three straight lines meet one ideal line, which consequently lies on the surface since it has three points in common with it.

It is clear that all planes of projective space must be regarded as projective planes, so that the principle of duality in the plane is true for them. But the space as a whole is also governed by a different principle of duality as well.

To arrive at this, we proceed as in the plane, compiling the list of axioms by which the incidence of points, straight lines, and planes in space must be regulated if finite and infinitely distant elements are treated alike. The axioms may be formulated as follows:

1. Two planes determine one and only one straight line; three planes that do not pass through a common straight line determine one and only one point.

2. Two intersecting straight lines determine one and only one point and one and only one plane.

3. Two points determine one and only one straight line; three points not on one straight line determine one and only one plane.

This system of axioms remains unaltered if the words "point" and "plane" are interchanged. (The first axiom is interchanged

[1] For, the property of two planes being parallel, and also the property of their having the same ideal line, are each equivalent to the property that parallels to every straight line of one plane can be drawn in the other.

with the third, and the second is unchanged.) The set of remaining axioms of the projective geometry of space is also left unaltered by this interchange. Thus the point and the plane correspond to each other dually, and the straight line corresponds to itself. The set of all points of a surface corresponds dually to the set of all tangent planes to another surface. As was the case with the conics in the plane, the second-order surfaces in space are self-dual.

The simplest and at the same time most important theorem of three-dimensional projective geometry is named after Desargues. Desargues' theorem may be stated as follows (see Fig. 133):

Two triangles ABC and $A'B'C'$ in space being given, let them be so placed that the lines connecting corresponding vertices pass

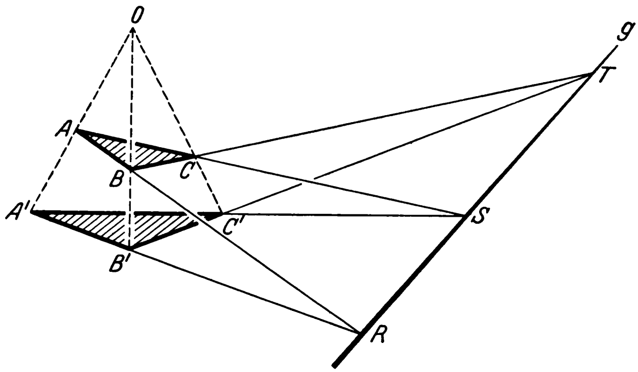

FIG. 133

through a single point O. Then the three pairs of corresponding sides have points of intersection, R, S, and T, and these points of intersection are, moreover, collinear.

The first part of the theorem is easy to prove. By the second axiom for space, the two intersecting straight lines AA' and BB' define a common plane. The straight lines AB and $A'B'$ also lie in this plane, whence it follows, by the second axiom for incidence in the plane, that these two straight lines have a point of intersection R. (R may be a finite or an ideal point.) The existence of the two other points of intersection, S and T, is proved analogously.

The truth of the second part of the theorem is easy to see in the case where the triangles are in different planes. In this case the planes of the triangles determine a common—ordinary or ideal—straight line of intersection (by Axiom 1 for space). Of every pair of corresponding sides of the triangles one lies in one of these planes and the other lies in the other plane. Since we have seen that the sides of such a pair intersect, their point of intersection must be on the straight line that the two planes have in common. This proves Desargues' theorem for the general case.

But it is precisely the special case where the triangles are coplanar that is of particular importance. Here we may apply a method of proof similar to the proof for Brianchon's theorem, in which we project a spatial figure onto the plane. We only need show that every plane Desargues figure is a projection of a three-dimensional Desargues figure. To this end, we connect all the points and straight lines of the plane Desargues figure with a point S outside the plane of the figure (see Fig. 134). We then pass a plane

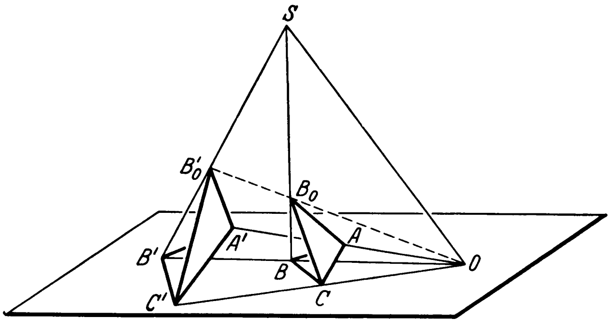

FIG. 134

through the straight line AC intersecting BS at a point B_0 distinct from S, and draw OB_0. The straight lines OB_0 and $B'S$ are coplanar and therefore have a point of intersection B_0'. But now the triangles AB_0C and $A'B_0'C'$ form a three-dimensional Desargues figure, since all the straight lines connecting corresponding vertices pass through O. Projecting the line in which the planes of these triangles intersect from S onto the original plane, we get a straight line on which the pairs of corresponding sides of the original triangles ABC and $A'B'C'$ must intersect. This completes the proof of Desargues' theorem.

The principle of duality for the plane and the one for space both lead to interesting consequences of Desargues' theorem. To begin with, it is readily seen that the converse of the theorem is also true; i.e. the existence of a Desargues line containing the points of intersection of pairs of corresponding sides of the two triangles implies the existence of the Desargues point through which the lines connecting corresponding vertices pass. In the case where the triangles are coplanar, the converse of Desargues' theorem proves to be the same as the theorem we obtain from Desargues' theorem by applying the principle of duality in the plane. We can elucidate this by writing the two theorems side by side, as follows:

Let three pairs of points AA', BB', CC' be given, such that the three lines determined by the pairs pass through a common point. Then the three points of intersection of the pairs of straight lines AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, lie on one straight line.

Let three pairs of straight lines aa', bb', cc' be given, such that the points of intersection of the pairs lie on one straight line. Then the lines joining the pairs of points (ab) and $(a'b')$, (bc) and $(b'c')$, (ca) and $(c'a')$, pass through a common point.

Let us examine the figure (Fig. 135) consisting of the vertices and sides of two coplanar Desargues triangles together with the lines joining pairs of corresponding vertices, the points where pairs of corresponding sides meet, the Desargues point O, and the

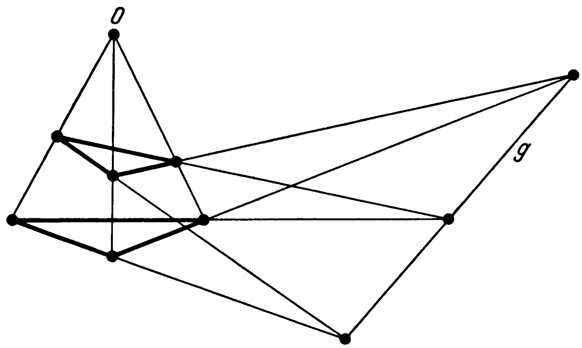

FIG. 135

Desargues line g. It is a simple matter of counting to see that the figure is a configuration of type (10_3). It is called the Desargues configuration. This configuration shares with Pascal's configuration the property that the last incidence condition is automatically satisfied when the figure is constructed step by step from its table. Furthermore, the Desargues configuration, like Pascal's, is self-dual. This is seen to be true because the configuration represents both Desargues' theorem and its converse, and the latter is the dual of the former.

We next consider the result obtained from the three-dimensional case of Desargues' theorem on applying the principle of duality *in space*. We get the following juxtaposition:

Let three pairs of points AA', BB', CC', be given such that the three lines determined by the pairs pass through a common point. Then the three points of intersection of the pairs of straight lines AB and $A'B'$, BC and $B'C'$, CA and $C'A'$, lie on one straight line.	Let three pairs of planes $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, be given such that the three lines of intersection determined by the pairs lie in one plane. Then the three planes containing the pairs of straight lines $(\alpha\beta)$ and $(\alpha'\beta')$, $(\beta\gamma)$ and $(\beta'\gamma')$, $(\gamma\alpha)$ and $(\gamma'\alpha')$, pass through one straight line.

Fig. 136 illustrates the theorem that appears in the right-hand column. In this theorem the two triangles are replaced by two trihedral angles formed by the planes α, β, γ and α', β', γ', respectively. Paralleling what we have done in the case of the plane Desargues figure, we shall now examine the three-dimensional figure consisting of the two Desargues trihedra together with the planes determined by pairs of corresponding edges, the lines of intersection of corresponding pairs of faces, the "Desargues plane" ($\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ in Fig. 136), and the "Desargues line" (VW in the figure). The intersection of this three-dimensional figure with any plane that does not contain any of the points V, W, X, Y, Z is a plane Desargues configuration, since the Desargues trihedra intersect the plane in Desargues triangles. To the planes and straight lines of the space figure there correspond the straight lines and points of the plane configuration. However, the three-dimensional figure has an intrinsic symmetry that is not reflected in the plane figure. The space figure consists of all the connecting straight lines and plane of the

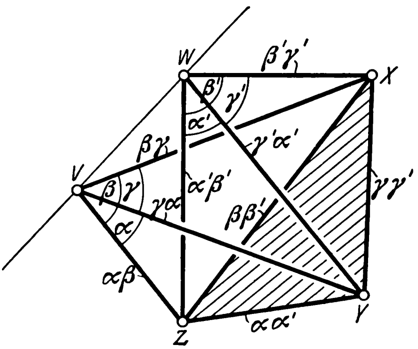

FIG. 136

five points V, W, X, Y, Z, and the roles of the five points are completely equivalent. Conversely, every complete five-point in space becomes a three-dimensional Desargues figure if two of the vertices are arbitrarily chosen as vertices of the Desargues trihedra.[2] From the fact that all the straight lines and all the planes of the spatial figure play the same role, it follows that the same is true for the points and the straight lines of the plane Desargues configuration. This proves that the Desargues configuration is regular, so that the choice of the Desargues point or the Desargues line in the configuration can be made quite arbitrarily.[3]

We shall now represent the Desargues configuration as a pair of mutually inscribed and circumscribed pentagons. To this end, we first look for any pentagons at all in the configuration, where it is required that all the vertices and sides of the polygon be elements of the configuration and no three consecutive vertices be collinear. The problem is considerably simplified by going back to the five-point in space. The vertices of the plane polygon are associated with the corresponding edges of the five-point in space. Since it is required that any two consecutive vertices of the plane polygon lie on a straight line of the configuration, the corresponding edges must be in one plane and must therefore intersect. To ensure that no three consecutive vertices are collinear, we need only see to it that the corresponding edges are not coplanar; this would happen if and only if three consecutive edges formed a triangle. By passing through the vertices V, W, X, Y, Z of the three-dimensional five-point in any order, say in the order in which they are written, we obtain a closed polygonal path of the kind we need; in the plane

[2] The only condition the five points must satisfy is that they be in general position, i.e. that no four of them be coplanar and hence no three of them collinear.

[3] By a complete n-point in space we mean the set of all the straight lines and planes connecting n points in general position in space. As in the case $n = 5$, the section of the complete n-point, for any value of n, by a plane that does not pass through any of the vertices is a configuration. These configurations are regular and of type $p = \frac{n(n-1)}{2}$, $\gamma = n - 2$, $g = \frac{n(n-1)(n-2)}{6}$, $\pi = 3$.

It follows that a configuration of the special type where $p = l$ is only obtained in the case $n = 5$. Other regular configurations can be obtained by using n-points in general position in higher-dimensional spaces. All these configurations are called "polyhedral."

configuration it furnishes a pentagon of the required type. But the edges of the three-dimensional five-point that were not used in this path constitute a second three-dimensional polygon of the same kind. For, two unused edges pass through every vertex of the five-point in space, since every vertex is incident with four edges in all,

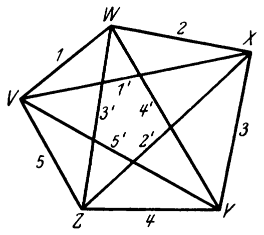

FIG. 137a

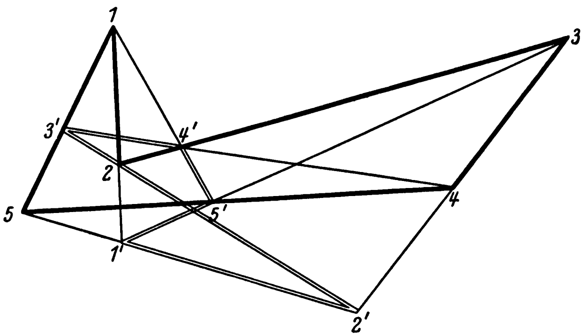

FIG. 137b

two of which were used up for the first path. This second polygonal path corresponds to a second pentagon in the configuration, and a simple enumeration reveals that this must be inscribed in the first pentagon. Because of symmetry, the first pentagon is also inscribed in the second pentagon. Figs. 137a and 137b illustrate

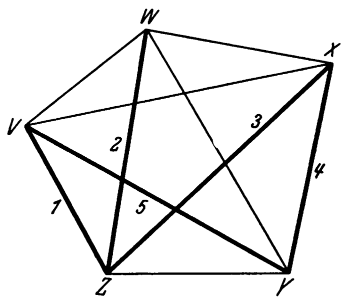

FIG. 138

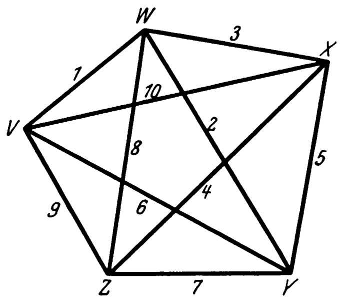

FIG. 139

the way in which the three-dimensional arrangement and the plane pair of pentagons are related.

We can also find other types of systems of five edges of the five-point in space corresponding to pentagons contained in the plane configuration. An example is given in Fig. 138. But it can be verified that it is then impossible to arrange the five remaining edges cyclically in such a way that any two consecutive edges have a common point and no three consecutive edges form a triangle. Hence the construction given in the beginning exhausts all the possibilities. Since an automorphism of the configuration corresponds to every permutation of the vertices and since the decomposition of the five-point in space into two polygonal paths is completely determined by the order of the vertices in the first path, we see that, leaving aside automorphisms, there is only one possible decomposition of the Desargues configuration into two mutually inscribed pentagons.

The question of whether, and in how many ways, the Desargues configuration can be considered as a self-inscribed and self-circumscribed decagon, can be settled by the same method. It is

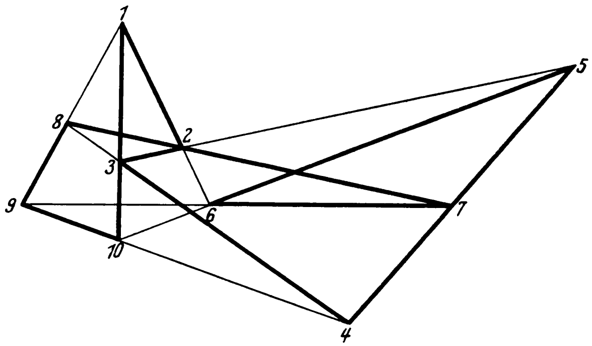

FIG. 140

found that the arrangement of edges in space corresponding to such a decagon can always be chosen as indicated in Fig. 139. Accordingly there is one way, and except for automorphisms only one way, of interpreting the Desargues configuration as a ten-sided polygon inscribed in and circumscribed about itself (Fig. 140). The figure exhibits a certain regularity; if we move along the sides of the decagon from the point 1 to the point 2, from 2 to 3, etc., in order, then one vertex is omitted on each side, and the numbers of the omitted vertices form a sequence in which pairs of successive numbers differ alternately by 1 and 3 (the vertex 5 is omitted on side 23, 8 on 34, 7 on 45, 10 on 56, etc.). Another feature of the decagon revealed by the three-dimensional arrangement is that the sides belong alternately to two mutually inscribed pentagons.

Desargues' configuration is not the only configuration with the symbol (10_3). In fact, there are nine other possibilities for the schematic table of such a configuration. One of these tables has the same property as the table for (7_3), namely that its configuration cannot be realized either in the real plane or in terms of complex coordinates, because its equations are incompatible. On the other hand, the remaining eight configurations of the form (10_3), like the configurations (9_3), can all be constructed with a ruler alone. But they are differentiated from the Desargues configuration by the fact that the last incidence condition is not automatically satisfied in their construction. Thus they do not express a geometrical theorem and are therefore not as important as the configuration of Desargues. One of these configurations is drawn in Fig. 141. It also represents a self-inscribed and self-circumscribed decagon if the points are taken in the numerical order given in the figure, but here the numbers of the vertices successively omitted on the sides of the polygon always differ by 1. In this arrangement all the vertices play the same role, and the sides are interchangeable with the vertices. It follows that the configuration is regular and self-dual.

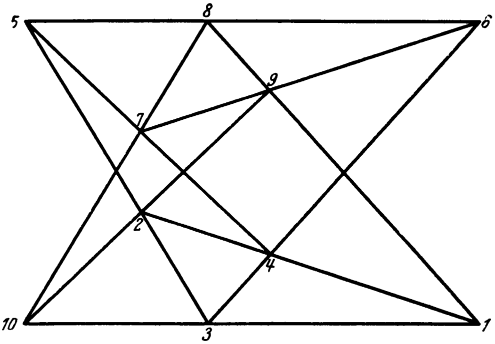

FIG. 141

§ 20. Comparison of Pascal's and Desargues' Theorems

We have found Desargues' theorem and the last of Pascal's theorems to be analogous in many ways. Both theorems were proved by the projection of three-dimensional figures. Both theorems gave rise to configurations, and quite similar configurations at that, both configurations were regular and self-dual, both could be constructed with a ruler alone, the last incidence in both occurred automatically, and both could be regarded as self-inscribed and self-circumscribed polygons.

Nevertheless there is a fundamental difference between the two theorems. The space figure used in the proof of Desargues' theorem can be constructed on the basis of the given axioms for incidence in space, without the assumption of any additional axioms. The Pascal-Brianchon configuration, on the other hand, was obtained by studying a second-order surface. To be sure, the core of the proof appears to be purely a consideration of the incidence relations between the points, straight lines, and planes of a hexagon in space, but on closer examination it is found that the construction of such hexagons in space is essentially equivalent to the construction of a ruled surface of the second order and that the possibility of such a construction cannot be proved from the axioms of incidence alone.

In the first chapter we introduced the conic sections and quadric surfaces on the basis of metric considerations. It might therefore be thought that Pascal's theorem could not be proved without comparisons of lengths and angles. But the curves and ruled surfaces of the second order can also be generated without the help of metric methods, by using the method of projection. By this method, the points of a given straight line can be mapped into the points of any other straight line in such a way that any three pre-assigned points on the first line go into three pre-assigned points on the second line and all harmonic sets of points on the first line become harmonic sets on the second. The first straight line is then said to be mapped projectively onto the second straight line. The construction of such a mapping (or "projectivity") requires only the axioms of incidence in the plane and in space. But the proof that the mapping is uniquely determined for all the points of the straight lines by the two given conditions—that harmonic sets become harmonic sets and that the mapping of three points is given—requires more than just these axioms. We need for this purpose an axiom of continuity which we shall formulate presently. But once the uniqueness of the projectivity in the given sense is proved, we can define the most general ruled surface of the second order as the surface swept out by a variable straight line that always connects corresponding points in a projectivity of two fixed skew straight lines. It then follows from the uniqueness property of the projectivity that a second family of straight lines also lies on the surface defined in this way. If the straight lines related by the projectivity are not skew but intersecting, then the straight line connecting pairs of corresponding points moves in a plane and envelops a curve of the second order. All the properties of the second-order curves that matter in projective geometry can be derived from this definition.

For the complete comprehension of the concept of continuity, two different axioms are needed. But only one of these, the *Archimedean* axiom, is used in the proof of the uniqueness of the projective mapping. In arithmetical terms, this axiom is formulated as follows: Let a and A be any two positive numbers; then—no matter how small a may be and no matter how large A may be—if we add a to itself a sufficient number of times we can always reach a point after a finite number of steps where the sum exceeds A;

$$a + a + a + \ldots + a > A.$$

This axiom is necessary if it is required to measure one length in terms of another length; the axiom in this form thus constitutes an essential part of the foundation of metric geometry. Independently of metric concepts, we can formulate the axiom as follows:

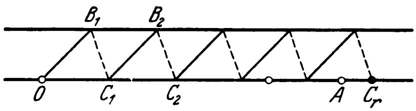

FIG. 142

Let two parallel straight lines be given (as in Fig. 142) and let O and A be two different points on one of them. Draw the line connecting O with an arbitrary point B_1 on the other straight line, and the line connecting B_1 with a point C_1 lying between O and A on the first straight line. Now draw the line parallel to OB_1 through C_1, cutting the other line at a point B_2; then draw the line parallel to B_1C_1 through B_2, cutting the first line at a point C_2, and in this way continue drawing lines parallel to OB_1 and B_1C_1. The Archimedean axiom then states that after a finite number of steps a point C_r on the straight line OA will be reached that does not lie between O and A. In this formulation of the Archimedean axiom we have made use of the notion of a point on a straight line lying between two other points of the straight line. For statements of this sort to be made more precise we need another set of axioms, the axioms of order, which we shall not discuss in detail here. The notion of parallels, on the other hand, was only used to make possible a more concise and readily understood formulation of the axiom. For the purposes of projective geometry it is sufficient that a construction of the kind indicated by Fig. 143 be possible. The figure is obtained from Fig. 142 by a central projection onto another plane.

The axioms of incidence in the plane and in space, together with the axioms of order and the Archimedean axiom, are sufficient to prove the uniqueness of the projectivity that maps three specified points into specified images, albeit the proof is exceedingly lengthy and tedious. From the uniqueness of the projective mapping in the plane we can then prove the last of the theorems of Pascal and Brianchon listed earlier (and the proof proceeds without the aid of any constructions in space).

Desargues' theorem can be proved in space by using only the axioms of incidence. But in order to prove the two-dimensional

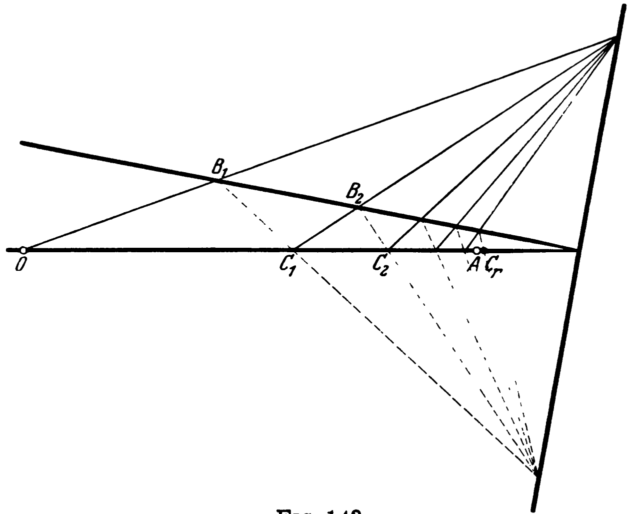

FIG. 143

form of the theorem without three-dimensional constructions, even the axioms of incidence combined with the Archimedean axiom and the axioms of order will not suffice. On the other hand, the axioms of incidence in the plane together with the axioms of order and the axioms of congruence will do, and we can dispense with the Archimedean axiom.

Omitting the axioms of incidence in space affects Pascal's theorem in the same way as it does Desargues', making the plane axioms of incidence, order, and congruence necessary for the proof. Nevertheless a significant difference between the two theorems can also be observed in the plane without the aid of spatial constructions. Pascal's theorem can not be proved from the axioms of incidence together with the validity of Desargues' theorem in the plane.

But Desargues' theorem can be proved from the axioms of incidence in the plane together with Pascal's theorem. We shall prove this for the special case where the Desargues line is the ideal line of the plane. As in the statement of the Archimedean axiom, this additional assumption only serves to make the formulation of the proof shorter and more readily comprehended. Thus we assume the following (see Fig. 144):

The three straight lines AA', BB', CC' pass through a single point O. Furthermore $AB \| A'B'$ and $AC \| A'C'$. It is to be proved by means of Pascal's last theorem that $BC \| B'C'$ follows.

In proof, let us draw the parallel to OB through A, intersecting $A'C'$ at a point L and OC at a point M. Let the straight lines LB' and AB intersect at N. We shall apply Pascal's theorem three times to this figure, always using the special form referred to as Pappus' theorem on page 119. First of all, $ONALA'B'$ is a Pascal hexagon since the six points lie alternately on two straight lines. Also $NA \| A'B'$ by assumption, and $AL \| B'O$ by construction. Hence it follows from Pappus' theorem that the third pair of opposite sides of the hexagon is also parallel, i.e. that $ON \| AC$. Next we consider the Pascal hexagon $ONMACB$. Here $ON \| AC$ as we have just proved, and $MA \| OB$ by assumption. It follows by Pappus' theorem that $NM \| CB$. Finally, we consider the Pascal hexagon $ONMLC'B'$. In this hexagon, $ON \| LC'$ and $ML \| B'O$, and it follows as before that $NM \| C'B'$. And since we have just proved in the previous step that $NM \| CB$, the proof of our assertion that $BC \| B'C'$ is complete.

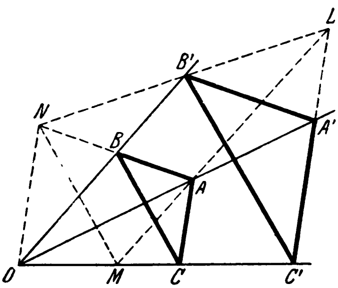

FIG. 144

(From *Grundlagen der Geometrie* by D Hilbert, 7th ed., p. 111. English translation in prep. (Chelsea Publishing Company).)

Any theorems concerned solely with incidence relations in the plane can be derived from the theorems of Desargues and Pascal. And we have now seen that Desargues' theorem is a consequence of Pascal's. Therefore we may say that Pascal's theorem is the only significant theorem on incidence in the plane and that the configuration $(9_3)_1$ thus represents the most important figure in plane geometry.

§ 21. Preliminary Remarks on Configurations in Space

The concept of a configuration can be generalized from the plane to three-dimensional space. A set of points and planes is called a configuration in space if every point is incident with the same number of planes, and every plane with the same number of points. A simple example of such a configuration is furnished by the three-dimensional Desargues theorem. Here we use the same ten points as we did in the corresponding plane configuration. As planes of the configuration we use the two planes of the triangles and the three planes containing the Desargues point and pairs of corresponding sides of the triangles. Then three planes pass through each point, and six points lie on each plane. For the same reason as for plane configurations, the four characteristic numbers for this configuration satisfy the equation $5 \times 6 = 10 \times 3$.

Apart from configurations of points and planes, we can also consider configurations in space which, like plane configurations, consist of points and straight lines, each point being incident with the same number of lines and each line with the same number of points. These two different points of view are often applicable to the same figure. Thus the three-dimensional Desargues figure we have just been considering gives rise to a combination of points and straight lines in space that is essentially identical with the plane Desargues configuration. Analogously, many of the more complicated configurations of points and planes give rise to configurations of points and straight lines consisting of some of the lines in which the planes intersect, together with the points of the original configuration; conversely, a configuration of points and straight lines can often be converted into a configuration of points and planes by adding to it some of the planes common to the intersecting straight lines of the configuration.

In analogy to what we did in the plane, we shall at first confine our attention to configurations in which the number of points equals the number of planes, so that we are dealing with a configuration of p points and p planes. If every point is incident with n planes it follows for the same reason as before that every plane of the configuration must also be incident with n points. We shall denote such a configuration by the symbol (p_n).

In order to exclude the trivial cases, we must take n to be at least 4. For $p \leqq 7$, a configuration (p_4) cannot exist. But for

$p = 8$, five different tables can be set up, and all of them can be realized geometrically. One of these configurations (8_4), the so-called Moebius configuration, is geometrically important because it satisfies the last incidence condition automatically and thus expresses a geometric theorem. This configuration consists of two mutually inscribed and circumscribed tetrahedra.

Going on to higher configurations, the number of possibilities keeps growing, and it soon becomes impossible to get an over-all view of them. Thus there are no less than 26 configurations of the type (9_4) that can be realized geometrically. Accordingly, we shall examine in greater detail only two three-dimensional configurations that are particularly important and play a role in other parts of mathematics as well. These are Reye's configuration and Schaefli's double six.

§ 22. Reye's Configuration

Reye's configuration consists of twelve points and twelve planes. It embodies a theorem of projective geometry, so that the last incidence always follows automatically, regardless of the positions of the points and planes. For the time being, however, we shall arrange the points in a special symmetrical order, so as to facilitate the visualization of the configuration.

We shall use as points of the configuration the eight vertices of a cube together with the center of the cube and each of the three ideal points where four parallel edges of the cube meet (Fig. 145). As planes of the configuration we shall use the planes of the six faces and each of the six diagonal planes passing through a pair of opposite edges. In the figure defined in this way, there are six points lying on each plane: four vertices and two ideal points on each of the planes containing a face of the cube, and four vertices, the center of the cube, and an ideal point on each of the diagonal planes. There are six planes through each point: the six diagonal planes pass through the center of the cube, three face planes and three diagonal planes through each vertex, and four face planes and two diagonal planes through each of the ideal points. Thus we have indeed constructed a configuration of points and planes, and its symbol is (12_6).

But the construction may also be interpreted as being a configuration of points and straight lines. To this end, we select some of the straight lines of intersection of the planes, namely the twelve edges and the four diagonals of the cube. There are three points of the configuration on each of these straight lines: two vertices and one ideal point on each edge, two vertices and the center on each diagonal. Furthermore, there are four straight lines through each point: three edges and one diagonal through each vertex, four diagonals through the center of the cube, and four edges through

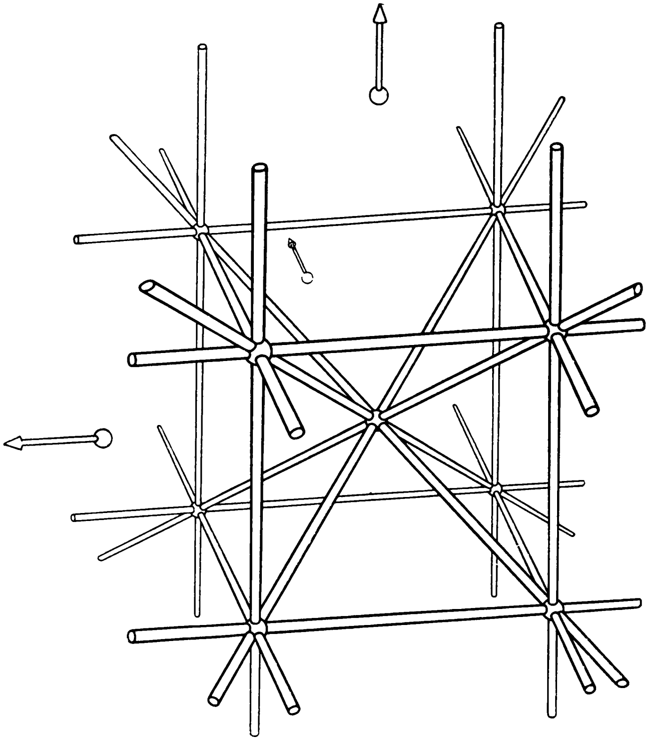

FIG. 145

each ideal point. Hence the points and straight lines of Reye's configuration form a configuration of the type $(12_4 16_3)$.

We can also see, if we count them, that three planes pass through each of the lines and that four lines lie on each plane. The straight lines on any one of the planes together with the six points of the configuration lying in the plane constitute a complete quadrilateral.

Reye's configuration appears in various geometrical contexts. An example is the system of centers of similitude of four spheres, which we shall now study.

The term *centers of similitude* of two circles or spheres denotes the two points that divide the line joining the centers of the circles or spheres in the ratio of their radii. The point on the segment that lies between the centers is called the *internal* center, the one on the extension of the segment the *external* center of similitude. If we are dealing with circles, and each of them lies outside the other, the internal center of similitude is the point of intersection of the two straight lines tangent to the circles on opposite sides, and the external center of similitude is the point of intersection of the straight lines tangent to the circles on the same side

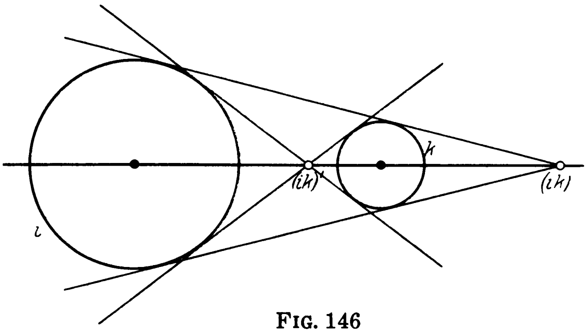

FIG. 146

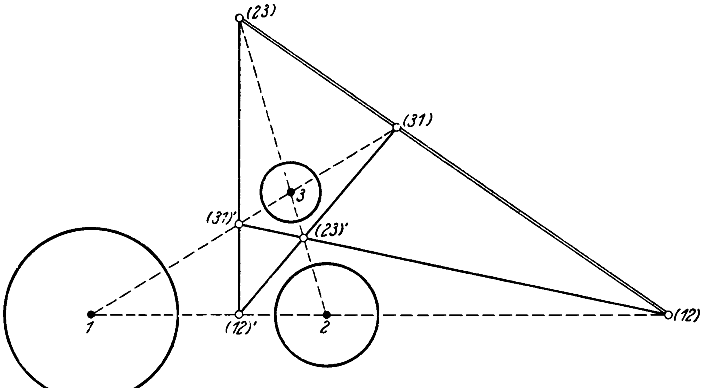

FIG. 147

(see Fig. 146). By rotating this figure about the straight line containing the centers we get an analogous property relating the centers of similitude of two spheres with common tangents to the spheres. (But in addition the spheres have many common tangents that do not pass through a center of similitude.) We shall use the symbols (ik) and (ik') respectively for the external and internal centers of similitude of two circles or spheres i and k.

Let us now consider three circles or spheres, 1, 2, and 3. They have three internal centers of similitude and three external centers of similitude, making six in all. We shall assume that the centers of the circles or spheres are not collinear but form a triangle; no two of the centers of similitude can then coincide, and the six can not all be collinear. By a theorem of Monge, the three external centers of similitude, (12), (23), and (31), are collinear, and each external center of similitude is collinear with the two internal centers of similitude that belong to different pairs of circles or spheres, e.g. (31) with (12)′ and (23)′ (see Fig. 147).[1] Accordingly, all the centers of similitude lie on four straight lines, which are called the axes of similitude of 1, 2, and 3. Monge's theorem may be summarized by saying that the centers of similitude and axes of similitude constitute the six points and four lines of a complete quadrilateral in which the centers of 1, 2, and 3 form the diagonal triangle. We shall denote the axes of similitude by the following symbols: (123) for the straight line containing the external centers of similitude, (1′23) for the straight line on which (23), (12)′, and (13)′ lie, etc.

With this preparation we turn to the consideration of four spheres 1, 2, 3, 4 whose centers are not all in one plane, so that, moreover, no three of the centers can be on one straight line (cf. Fig. 148, p. 140). We shall see that all the centers of similitude and axes of similitude of these spheres collectively constitute the points and straight lines of a Reye configuration. Since six different pairs can be selected from the spheres 1, 2, 3, 4, and since each pair gives rise to an external and an internal center of similitude, there are twelve centers of similitude in all. Also we have the right number, 16, of axes of similitude, for there are four different ways we can select three out of the four spheres, and each set of three spheres gives rise to four different axes of similitude, e.g. (123), (1′23), (12′3), and (123′). Each axis is incident with three points, e.g. (123) is incident with (12), (23), and (13). Similarly, every point is incident with four different axes, e.g. (12)

---

[1] *Proof*: Let the radii of 1, 2, and 3 be equal to r_1, r_2, and r_3, respectively. Then the external centers of similitude divide the sides of the triangle formed by the centers in the ratios $-\frac{r_1}{r_2}, -\frac{r_2}{r_3}, -\frac{r_3}{r_1}$. The product of these ratios is -1, and it follows by a theorem of Menelaus that the external centers of similitude are collinear. If two of the external centers of similitude are replaced by the corresponding internal centers of similitude, two of the ratios change their sign. The product is therefore still -1, so that we once more have three collinear points.

is incident with (123), (123′), (124), and (124′), and (12)′ with (1′23), (12′3), (1′24), and (12′4).

We thus see that the centers and axes of similitude do indeed form a configuration and that its type is $(12_4, 16_3)$. To see that it is identical with Reye's configuration, we need to find twelve suitable planes. First we take the four planes containing the centers of three spheres each. The points and axes lying on any one of these planes form a complete quadrilateral, as in Reye's configuration. To get eight more planes with this property, we simply take all the remaining planes spanned by any two axes that intersect at a point of the configuration. Two axes of this kind must certainly belong to different number triples, for, any two axes associated with the same set of three numbers, e.g. (123) and (1′23), define the plane containing the centers of three spheres (1, 2, and 3 in our case), so that nothing new is obtained. Let us begin with two axes containing only external centers of similitude, e.g. (123) and (124). They span a plane that contains (12). In addition, this plane contains the other four points of those axes, i.e. (13), (23), (14), and (24). But (23) and (24) also lie on the axis (234) which contains as well the remaining external center of similitude (34). Hence all six external centers of similitude lie on the single plane we have been considering. This plane also contains the remaining "external" axes (134) and (234); thus it is incident with six points and four straight lines, as it should be. We proceed to the case of two intersecting axes one of which is "external" and one "internal" and which are associated with two different number triples. Since their point of intersection must be an external center and since all the numbers play the same role, we may pick the axes (123) and (124′) as a representative pair. Apart from their point intersection, (12), these axes contain the points (13), (23), (14)′, and (24)′. By the same reasoning as before, we see that the axes (134′) and (234′) and the point (34)′ are also in the plane of (123) and (124′). Thus the three internal centers of similitude defined by the sphere 4 together with the three other spheres are in a single plane with the three external centers of similitude of the spheres 1, 2, and 3. There must be altogether four planes of this kind. Only the case based on two intersecting internal axes of similitude remains to be considered. Of course the last plane considered above contains three internal axes which intersect in pairs; but the points of intersection are always internal centers of similitude, so that the case of two axes intersecting at an external center of similitude is still open. Let us begin, then, with two internal axes, say (123′) and (124′), which intersect at an external center of similitude—(12) in this case. Apart from the point of intersection, the plane of these axes contains the points (13)′, (23)′, (14)′, and (24)′. Hence this plane also contains the axes (1′34) and (2′34) and the point (34). Thus there are four internal axes of similitude in this plane, and it meets the opposite edges 1, 2 and 3, 4 of the tetrahedron 1, 2, 3, 4 at the external centers of similitude and the remaining edges at the internal centers of similitude. There are three planes of this type, since a tetrahedron has three pairs of opposite edges. Thus we have obtained altogether $1 + 4 + 3 = 8$ planes.

For the sake of clarity, we shall set up the two tables that give the incidence relations between the points and the planes and between the points and the lines, respectively. The faces of the tetrahedron are labelled I, II, III, and IV, where I is the face opposite the point 1. The plane of the external centers of similitude is called e_a, the four planes containing three external and three internal centers are called e_1, e_2, e_3, e_4 respectively, according to the number of the exceptional sphere, and the three remaining planes are denoted by (12, 34), (13, 24), and (14, 23) respectively, according to the exceptional pair of opposite edges of the tetrahedron. For the sake of brevity, parentheses are omitted in the notation for points and straight lines.

	Planes											
	I	*II*	*III*	*IV*	e_a	e_1	e_2	e_3	e_4	(12,34)	(13,24)	(14,23)
Points	23	13	12	12	12	23	13	12	12	12	13	14
	24	14	14	13	13	24	14	14	13	34	24	23
	34	34	24	23	14	34	34	24	23	13′	12′	12′
	23′	13′	12′	12′	23	12′	12′	13′	14′	14′	14′	13′
	24′	14′	14′	13′	24	13′	23′	23′	24′	23′	23′	24′
	34′	34′	24′	23′	34	14′	24′	34′	34′	24′	34′	34′

	Planes											
	I	*II*	*III*	*IV*	e_a	e_1	e_2	e_3	e_4	(12,34)	(13,24)	(14,23)
Lines	234	134	124	123	123	234	134	124	123	123′	12′3	1′23
	2′34	1′34	1′24	1′23	124	1′23	12′3	123′	124′	124′	1′24	12′4
	23′4	13′4	12′4	12′3	134	1′24	12′4	13′4	134′	1′34	134′	13′4
	234′	134′	124′	123′	234	1′34	2′34	23′4	234′	2′34	23′4	234′

The configuration is depicted in Fig. 148.[2] That this configuration is identical with that of Fig. 145 becomes manifest on moving the three points (12), (12)′, and (34) to infinity in mutually perpendicular directions; the three points then assume the positions of the

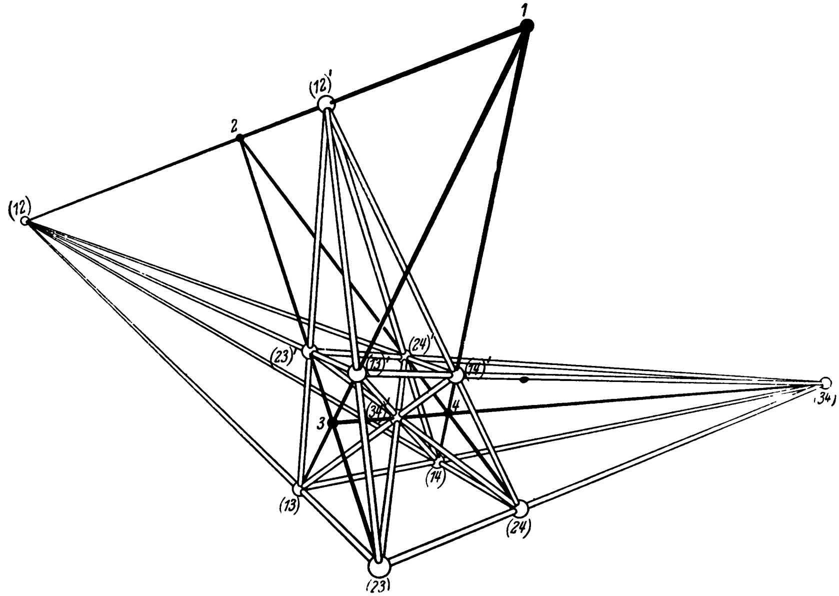

FIG. 148

ideal points of the configuration given in Fig. 145. The eight points (13), (14), (23), (24), (13)′, (14)′, (23)′, and (24)′ become the vertices of the cube, and (34)′ becomes the center of the cube. But the points 1 and 2 also move to infinity. In order to find the four spheres belonging to Fig. 145 it is consequently necessary to extend the definition of center of similitude by the addition of limiting cases. First, the external center of similitude of two equal circles or spheres must be defined as the ideal point on the line connecting the centers (see Fig. 149). Furthermore, the centers of similitude of a sphere k

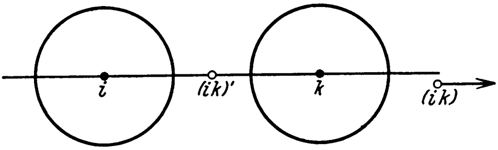

FIG. 149

[2] Viewed as a *plane* figure, Fig. 148 represents a plane configuration of type $(12_4\,16_3)$ consisting of the centers and axes of similitude of four coplanar circles. The centers of the circles are also at 1, 2, 3, and 4, and the radii may be chosen to be the same as in the three-dimensional case.

and a plane e (Fig. 150) must be defined as the extremities (ke) and $(ke)'$ of the diameter of k that is perpendicular to e. For, if e is replaced by a family of spheres K tangent to e at the point P where the extension of the diameter meets e, it is seen that the centers of similitude of k and K approach (ke) and $(ke)'$ as the diameter of K increases to infinity. Finally we consider the case of two planes e and f intersecting in a straight line g (Fig. 151). The centers of similitude must be defined in this case as the ideal points having directions that are perpendicular to g and bisect the two angles formed by e and f. This definition too may be justified by a limiting process, as follows: Replace g by the circle of intersection of two congruent spheres tangent at a fixed point of g to e and f respectively, and then let the radius of the spheres increase to infinity.

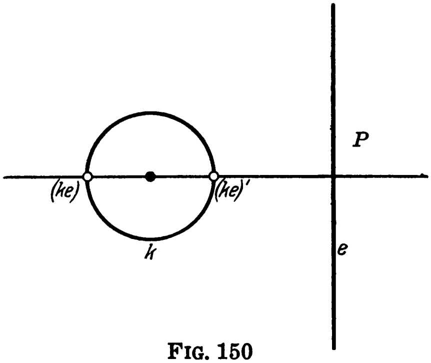

FIG. 150

With these definitions, we are in a position to interpret Reye's configuration in its original version also, as a system of centers of similitude. Let the spheres 3 and 4 have their centers at the midpoints of the front and back faces of the cube in Fig. 145. Let the radii be equal and of such length that each sphere goes through the four corners of the face on which its center lies. Let 1 and 2 be any two planes that are respectively perpendicular to the two diagonals of the faces under consideration. Then the points of the configuration are the centers of similitude of 1, 2, 3, and 4, arranged in the same order as in Fig. 148.

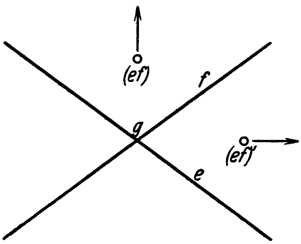

FIG. 151

Instead of this limiting case, we may consider the configuration based on four equal spheres with their centers at the vertices of a regular tetrahedron. Here the external centers of similitude must be at the ideal points of the six edges of the tetrahedron, so that the ideal plane belongs to the configuration and constitutes, in our notation, the plane e_a. The internal centers of similitude are the midpoints of the edges; they form the six vertices of a regular octahedron (see Fig. 152). All the face-planes of the octahedron belong to the configuration, being the face-planes I, II, III, and IV, of the tetrahedron and the planes called e_1, e_2, e_3, and e_4 in our notation. The three remaining planes of the configuration are the three planes of symmetry of the octahedron. The straight lines of the configuration are the four ideal lines of the face-planes of the tetrahedron

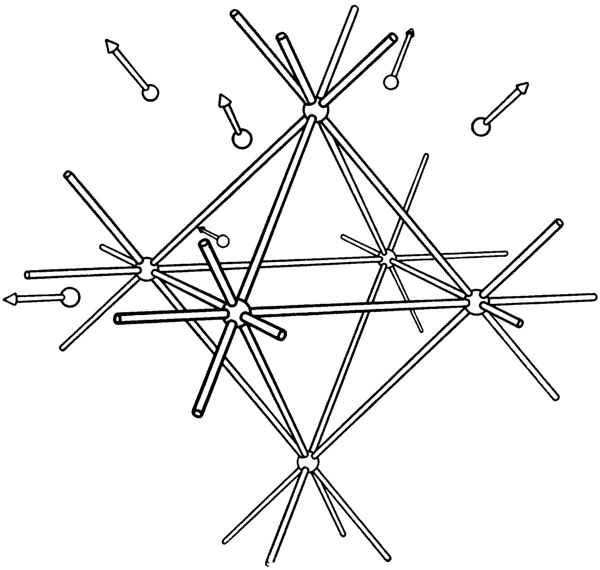

FIG. 152

(external axes of similitude) and the twelve edges of the octahedron (internal axes of similitude).

In the second chapter we have already pointed out how the cube and the octahedron are related. In accordance with § 19, we may say that the cube and the octahedron correspond dually to each other. Similarly, it can be shown more generally that the points and planes of Fig. 152 correspond dually to the planes and points of Fig. 145; the vertices and faces of the cube correspond to the faces and vertices respectively of the octahedron, the center of the cube and the six planes through it correspond to the ideal plane and the six points on it in Fig. 152, and the three ideal points associated with the cube correspond to the three planes of symmetry of the octahedron.[3] It follows that Reye's configuration of points

[3] This correspondence is produced by a polarity with respect to the inscribed sphere of the cube.

and planes is self-dual. Of course the two dual Reye configurations obtained from the cube and octahedron look quite different, but for the purposes of projective geometry, all Reye configurations must be considered as identical.[4]

We shall now show that Reye's configuration also has the other important property of symmetry that we observed in some plane configurations, viz., that it is regular. This is by no means evident from the foregoing discussion. Indeed, the planes belong to four different classes relative to the system of centers of similitude, and in the realization of the configuration either by a cube or an octahedron, both the points and the planes play different sorts of roles. In the following section, we shall obtain Reye's configuration by a method that reveals the equivalence of all the elements. To this end, we need to learn more about the regular polyhdra of three-dimensional and four-dimensional space. For, the figures of four-dimensional space can be projected into three-dimensional space in the same way that the figures of three-dimensional space can be projected into a plane, and a suitable projection of one of the four-dimensional figures gives us Reye's configuration.

§ 23. Regular Polyhedra in Three and Four Dimensions, and their Projections

In Chap. II we listed the five regular polyhedra of three-dimensional space. Among these, the tetrahedron plays an anomalous role in that it is self-dual, whereas the four remaining polyhedra are mutually dual in pairs—the octahedron with the cube, and the dodecahedron with the icosahedron. Possibly this singularity of the tetrahedron is connected with a second phenomenon that distinguishes it from the other polyhedra; the others are symmetrical with respect to a point, which means that the vertices come in pairs that are symmetrical about the center, and the same is true for the edges and the faces (e.g. the straight line connecting any vertex of a cube with the center meets the cube at a second vertex). The tetrahedron, however, is not symmetrical with respect to a point, (does not have "central symmetry") ; the straight line connecting

[4] We obtain a projective generalization of the octahedron by starting with any system of projective coordinates in space; in every case the unit points on the six coordinate axes and the six points of intersection of these axes with the unit plane are the points of a Reye configuration.

a vertex with the center cuts the tetrahedron at the midpoint of one of its faces.

A study similar to the one made at the end of the second chapter proves that the number of regular polytopes[1] that are possible in four-dimensional space is also finite and is equal to six.[2] Of course the boundary of such a polytope comprises three-dimensional regions (called *cells*) in addition to points, edges, and plane faces. Just as we stipulated for regular polyhedra that the faces be regular polygons, so we must stipulate for the regular polytopes in four dimensions that the boundary cells be regular polyhedra. The polytope is called an *n-cell* if it is bounded by n polyhedra. The essential data for the regular polytopes of four-space are given in the following table:

4-Dimensional Space

	Number and Type of Boundary Polyhedra	Number of Vertices	Duality
1. 5-cell	5 Tetrahedra	5	self-dual
2. 8-cell	8 Cubes	16	mutually dual (2, 3)
3. 16-cell	16 Tetrahedra	8	
4. 24-cell	24 Octahedra	24	self-dual
5. 120-cell	120 Dodecahedra	600	mutually dual (5, 6)
6. 600-cell	600 Tetrahedra	120	

The duality relations listed in the last column can be readily deduced from the table. For in four-space, points correspond dually to three-dimensional spaces and straight lines to planes.

We see from the table that the 5-cell is analogous to the tetrahedron, while the 8-cell, 16-cell, 120-cell, and 600-cell take the place of the cube, octahedron, dodecahedron, and icosahedron, respectively. The 24-cell has a singular role; it is not only self-dual but also centrally symmetric, while the other self-dual polytope, the regular 5-cell, shares the property of its analogue, the regular tetrahedron, of having no symmetry about a point.

[1] The polyhedra of n-dimensional space for $n \geqq 4$ are called *polytopes* (or, in the earlier literature, *polyhedroids*). [*Trans.*]

[2] Cf. the book *Die Vierte Dimension* by H. de Vries (Leipzig and Berlin, 1926).

Cf., also, *Regular Polytopes* by H. S. M. Coxeter (Methuen & Co. Ltd., London, 1947) and the last chapter of *Geometry of Four Dimensions* by H. P. Manning (MacMillan, New York, 1914). [*Trans.*]

Analogous studies have also been made for spaces of higher dimensionalities. Here we find a greater simplicity and regularity, as only three regular polytopes can be found in any such space. We again give the most important data in the form of a table.

n-Dimensional Space, $n \geqq 5$

	Number and Type of Boundary $(n-1)$-Dimensional Cells	Number of Vertices	Duality
1. $(n+1)$-cell	$n+1$ n-cells	$n+1$	self-dual
2. $2n$-cell	$2n$ $(2n-2)$-cells	2^n	mutually dual
3. 2^n-cell	2^n n-cells	$2n$	mutually dual

The three-dimensional polyhedra corresponding to these three types of polytopes are the tetrahedron, the cube, and the octahedron ($n+1=4$, $2n=6$, $2^n=8$). The four-dimensional analogues are the 5-cell, the 8-cell, and the 16-cell. Thus the dodecahedron and the icosahedron of three-space as well as the 24-cell, 120-cell, and 600-cell of four-space have no analogues in spaces of higher dimensionality.

We shall now study the projections of the regular polyhedra and polytopes into spaces whose dimensionality is smaller by one than that of the spaces in which the polyhedra and polytopes lie. We begin with the projections of the regular polyhedra into a plane. Of course, the appearance of these projections will vary greatly with the choice of the center of projection and of the image plane. In Figs. 95 through 99 of page 91 we used parallel projections, i.e. projections with the center at an ideal point. This has the advantage of representing parallel lines by parallel lines. But it has the disadvantage of making pieces of faces overlap. The disadvantage can be eliminated by moving the center of projection to a point very close to one of the faces. For the sake of symmetry we move it to a point at a small distance from the center of one of the faces and project into the plane of that face. In this way the five regular polyhedra give us the projections drawn in Figs. 153 through 157. This is the way we see the polyhedra when we remove one of the faces and look at the interior through the hole.

If the center of projection is located on the surface of the polyhedron, the faces passing through it appear as straight lines, so that the image becomes quite unsymmetrical.

If the center of projection is located inside the polyhedron, the image is significantly altered; then it must extend to infinity irrespective of the choice of the image plane. This is so because every plane through the center of projection intersects the polyhedron. This applies, in particular, to the plane going through the center

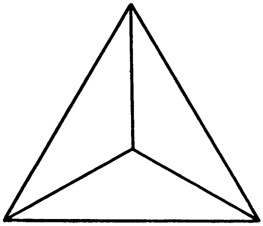

FIG. 153 TETRAHEDRON

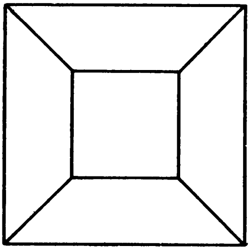

FIG. 154 CUBE

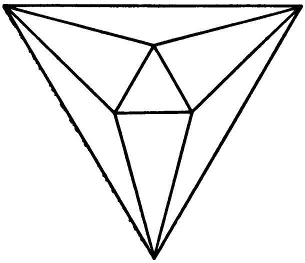

FIG. 155 OCTAHEDRON

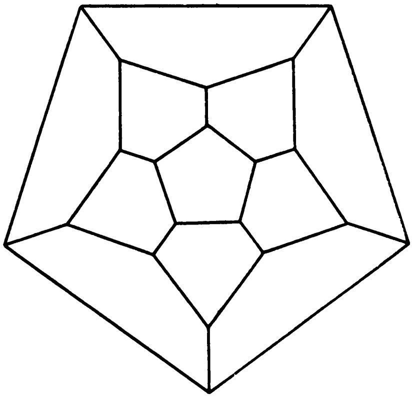

FIG. 156 DODECAHEDRON

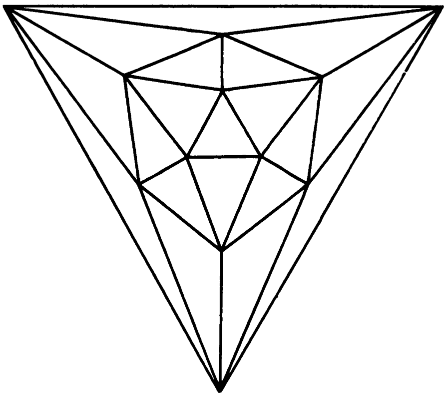

FIG. 157 ICOSAHEDRON

which is parallel to the image plane and which therefore gives rise to the ideal points of the projection (cf. p. 114). Nevertheless, this type of projection leads to a phenomenon of geometric interest in the special case where the center of projection is at the center of the polyhedron. For, in this case—and in this case only—the bundle of rays through the center is arranged symmetrically. As was already noted on page 116, the bundle of rays can be looked on as a model of the projective plane by interpreting the straight lines of the bundle as "points" and the planes of the bundle as "straight lines." Thus the regular polyhedra induce regular partitions of the projective plane. But only in the case of centrally symmetric polyhedra can this partition cover the projective plane simply; in the case of the tetrahedron, every straight line through the center yields two different image points corresponding to the two points where it meets the surface of the polyhedron. so that the projective plane is covered twice. But on all other regular polyhedra every pair of diametrically opposite elements produces one single piece of the projective plane. If we consider the intersection of the bundle of rays with a plane, i.e. if a projection in the proper sense is under consideration, we cannot preserve all the symmetry. The image is particularly simple, however, if its plane is chosen so as to contain a vertex of the polyhedron and to be perpendicular at that vertex to the line connecting the vertex with the center (see Fig. 158 for the octahedron). Figs. 159 through 163 show the five projections obtained in this way. One of the regions extending to infinity is shaded in each diagram. In the projection of the tetrahedron, the image plane is covered twice. In the remaining figures, every polygon in the image plane represents exactly two diametrically opposite faces of the polyhedron.

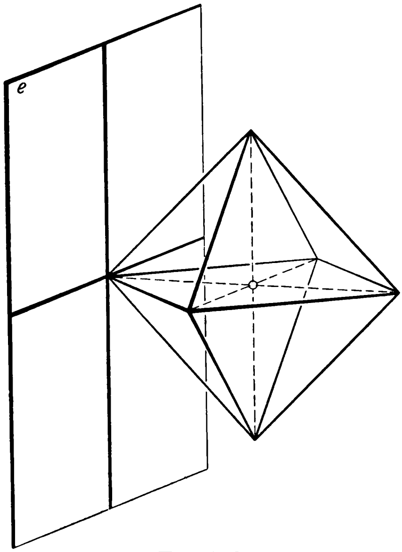

FIG. 158

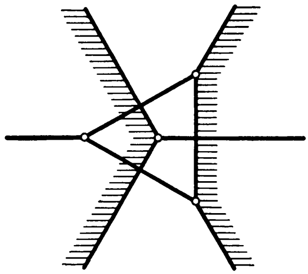

FIG. 159 TETRAHEDRON

Another series of simple figures is obtained from the symmetrical polyhedra by using a face plane as image plane, as shown in Fig. 164 for the cube. (For the tetrahedron this does not give us a new figure.) The projections are shown in Figs. 165 through 168.[3]

Using analogous methods of projection, we can depict the regular polytopes of four-space by figures in three-space. Parallel projec-

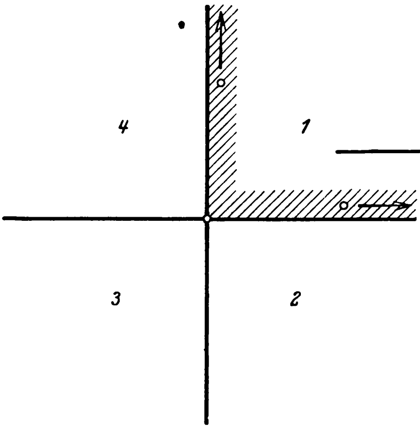

FIG. 160 OCTAHEDRON

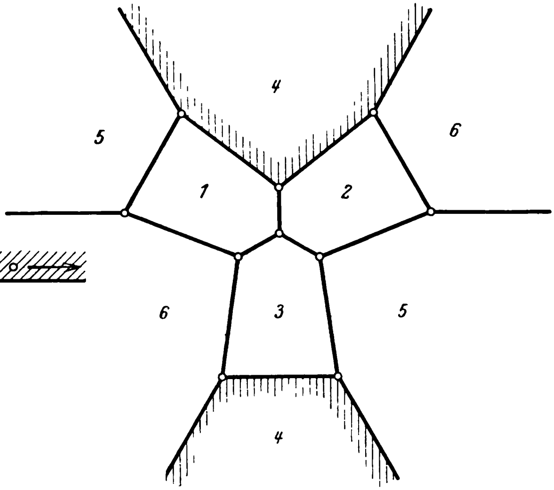

FIG. 162 DODECAHEDRON

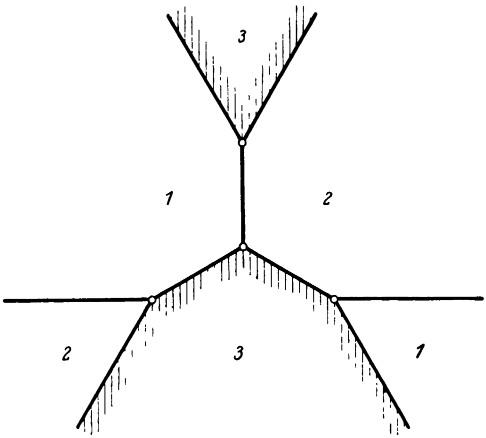

FIG. 161 CUBE

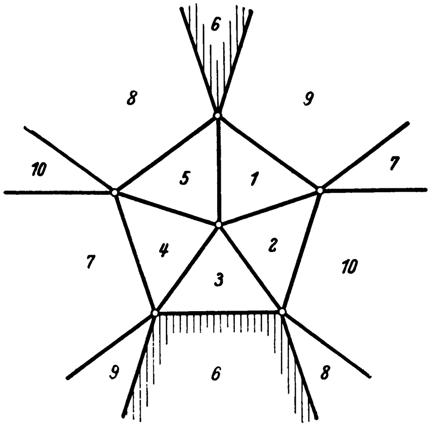

FIG. 163 ICOSAHEDRON

tion is not found to be suitable, as it represents the boundary polyhedra of the polytopes by polyhedra in space which partly overlap and intersect each other. On the other hand, the procedure followed in obtaining Figs. 153 through 157 can be used to give us clear

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[3] In this case, the projection of the octahedron is equivalent to the division of the plane into four triangles by a projective coordinate system.

pictures of the four-dimensional polytopes. The boundary polyhedra of the polytope are represented by a set of polyhedra in space

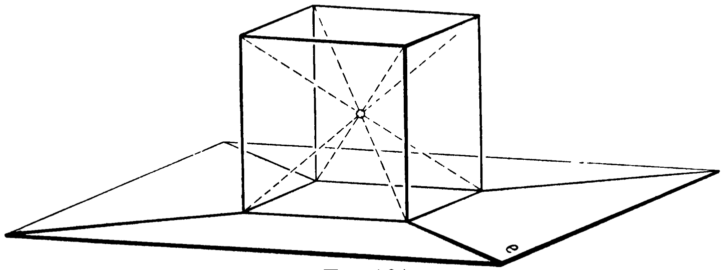

FIG. 164

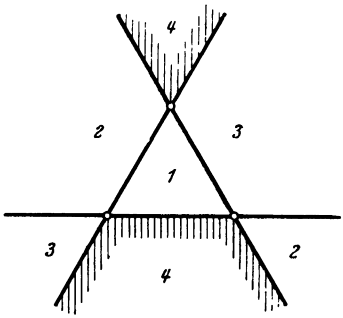

FIG. 165 OCTAHEDRON

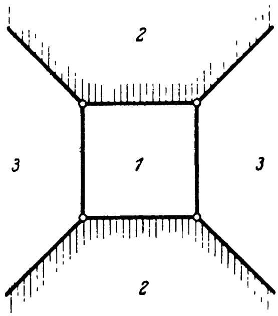

FIG. 166 CUBE

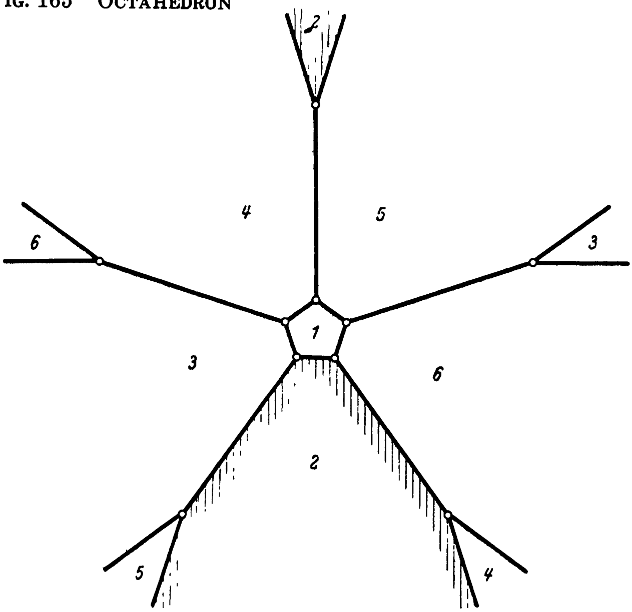

FIG. 167 DODECAHEDRON

of which one plays a special role and is filled up simply by the others. If these models are in turn projected into the plane, we get

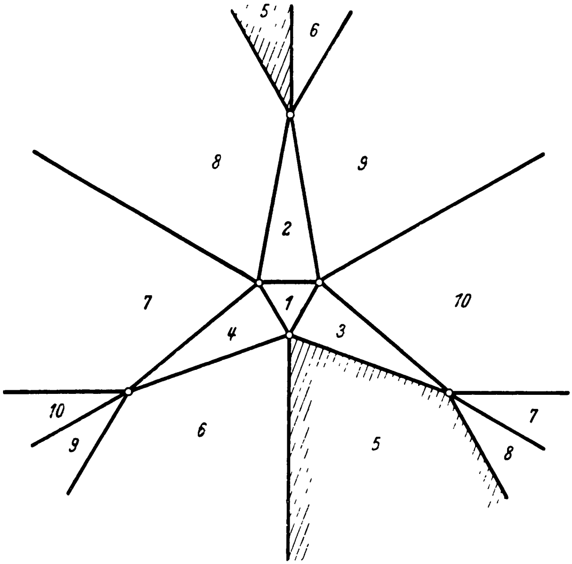

FIG. 168 ICOSAHEDRON

four pictures as shown in Figs. 169 through 172. In Fig. 172 it may be ascertained, though somewhat laboriously, that the large octahedron is filled by 23 smaller octahedra (which are of four different forms) making 24 polyhedra in all. The figures for the 120-cell and the 600-cell would get too confusing.

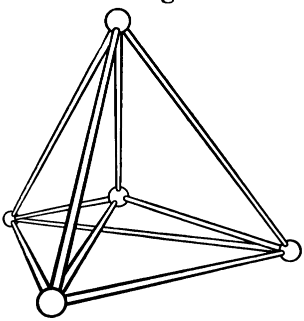

FIG. 169 5-CELL

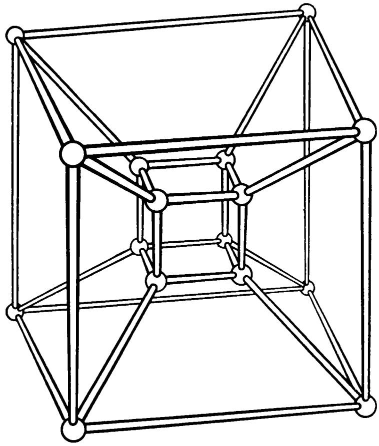

FIG. 170 8-CELL

If the center of projection is moved to the center of the polytope, the result has to be a regular partition of the projective space.

We cannot produce a model for the projective space that is as symmetrical as the bundle of lines representing the projective plane; for, this would involve consideration of a four-dimensional figure. It is necessary, therefore, to single out a particular three-space as image space, and some of the symmetry is lost in the process. But in order to preserve part, at any rate, of the symmetry, we let the image space assume positions analogous to those of the image plane in the case where the dimensionality is one less: either we use one of the boundary spaces, in analogy to the arrangement of Fig. 164,

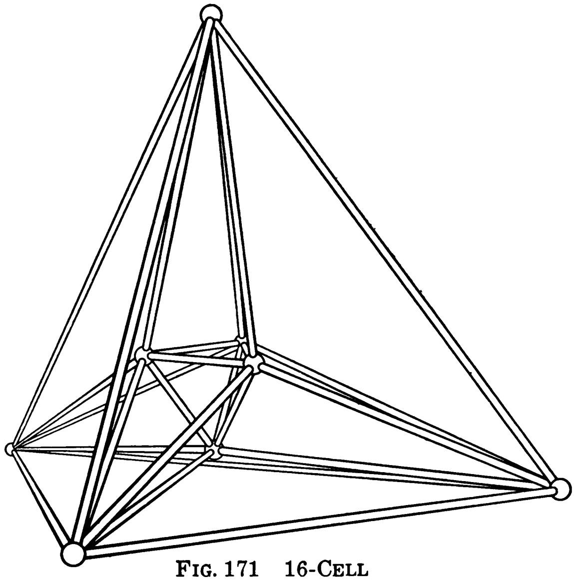

FIG. 171 16-CELL

or we choose a space passing through one of the vertices of the polytope and having the position corresponding to that of the image plane in Fig. 158. In the first case, the boundary polyhedron we select will be reproduced without any distortion, because it is in the image space to begin with; in the second case, the projection is symmetrical with respect to the chosen vertex, which is its own image. First we shall consider the pictures of the 16-cell and the 8-cell obtained by these two methods of projection (Figs. 173 and 174).[4] Here the space is partitioned into eight and four parts respectively, and each part corresponds to two diametrically opposite boundary cells of the polytope. In Fig. 173a, the three-dimensional seg-

[4] This method of projection is not suitable for the 5-cell, as this polytope does not have central symmetry.

ments that extend to infinity are of two different forms. Four of them have one boundary face (e.g. 1, 3, 4) that is wholly confined to the finite part of space and from which they extend across the ideal plane to the opposite vertex (e.g. 2). On the other hand, three

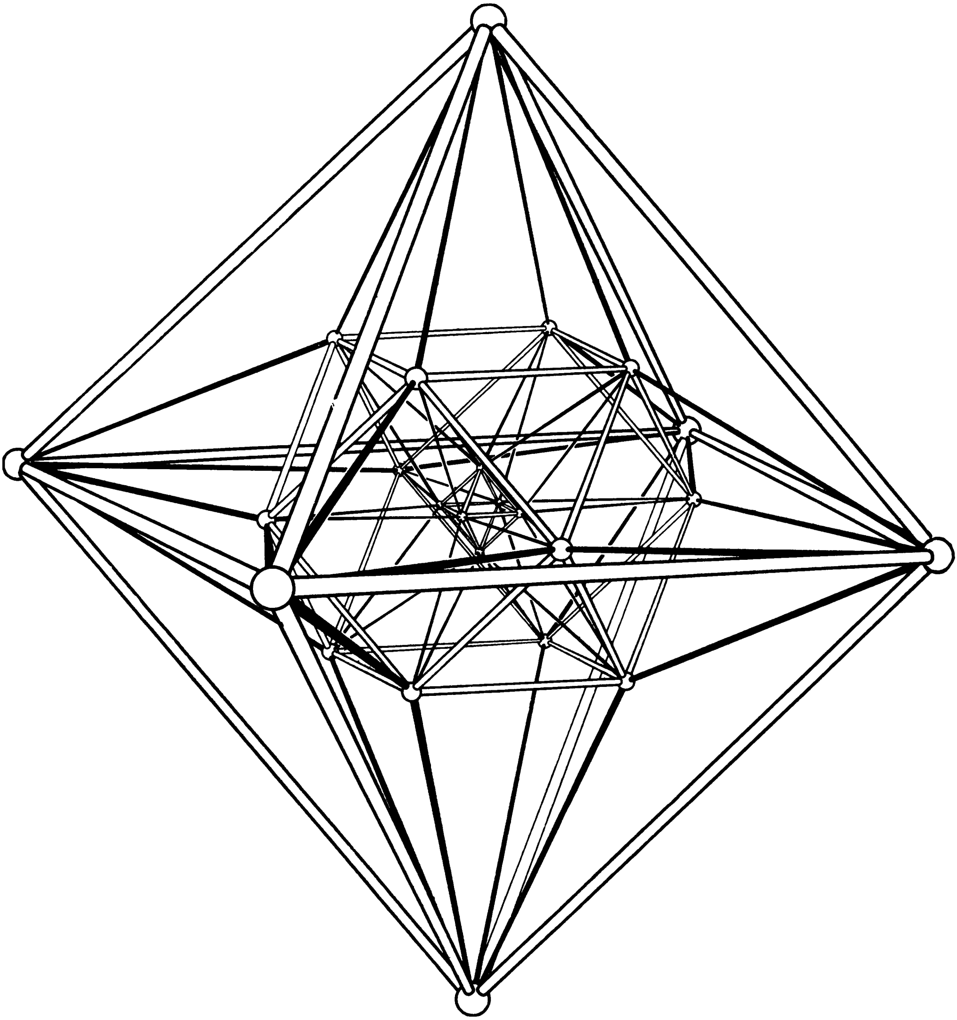

FIG. 172 24-CELL

of the regions have a pair of opposite edges that are finite (e.g. 1, 2, and 3, 4), but no faces that do not extend across infinitely distant elements. In Fig. 173b, the ideal plane itself is a boundary plane. We note that the 16-cell leads to familiar partitions of space—the division into octants by a projective or a Cartesian coordinate system. In the representation of the 8-cell shown in Fig. 174a, all the regions that extend to infinity are of the same form. In Fig.

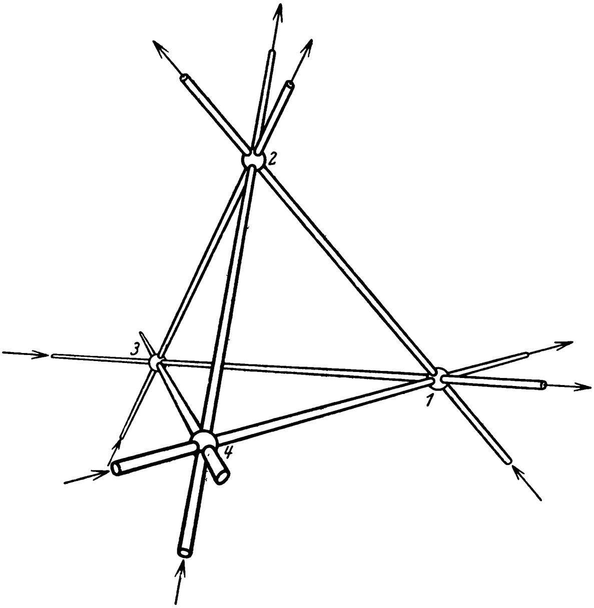

FIG. 173a 16-CELL

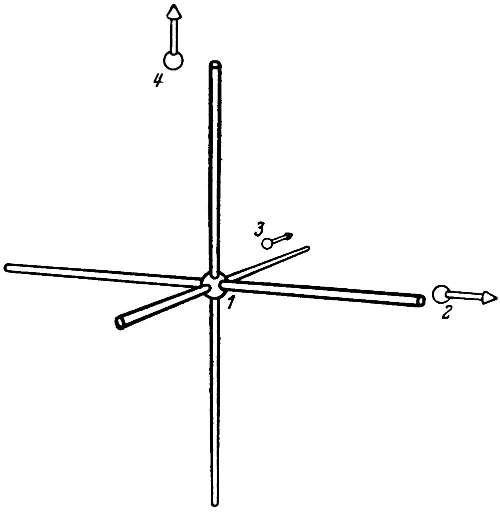

FIG. 173b 16-CELL

174b, arrows marks off the edges of the region that corresponds to the finite cube of Fig. 174a; the edges of this region include the finite edges containing the point 1 with the exception of the edge 1, 6.

We next apply the same two methods of projection to the 24-cell. The results are shown in Figs. 175 and 176. We thus get a partition of the space into twelve octahedra, all of which, with the exception

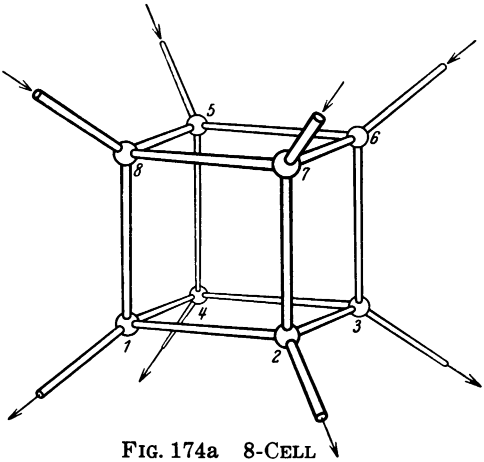

FIG. 174a 8-CELL

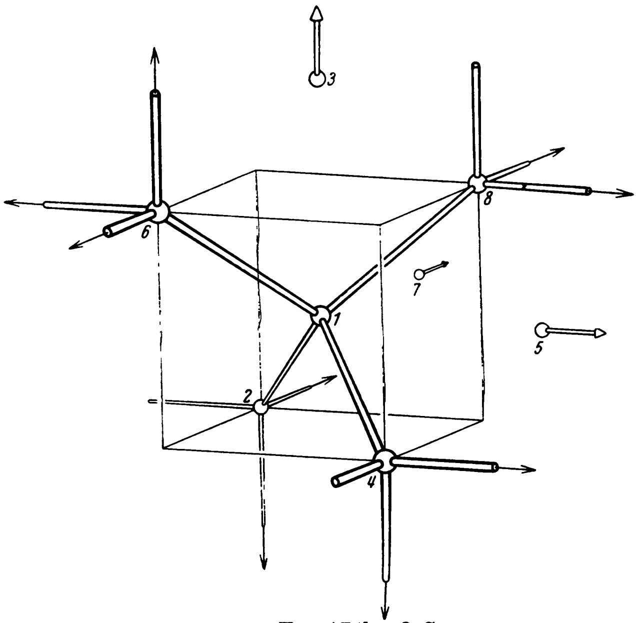

FIG. 174b 8-CELL

of the octahedron in the center of Fig. 175, extend to infinity. It is seen that Figs. 175 and 176 reproduce the two symmetrical forms of Reye's configuration that we studied in the preceding sec-

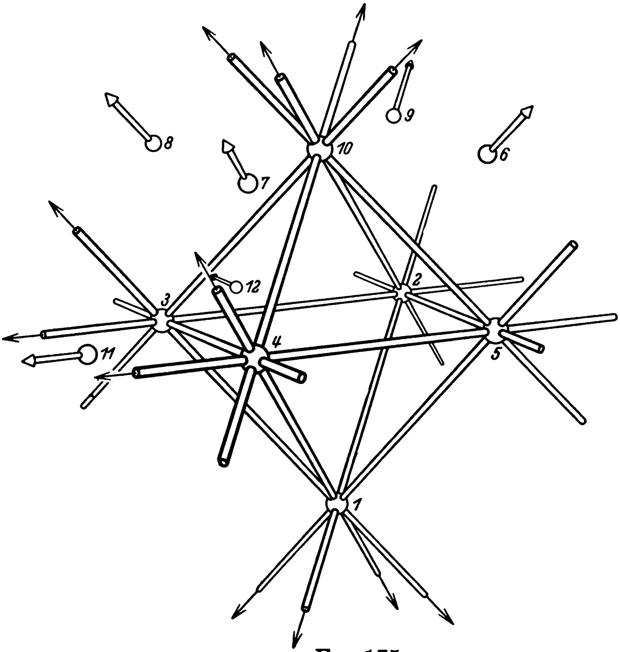

FIG. 175

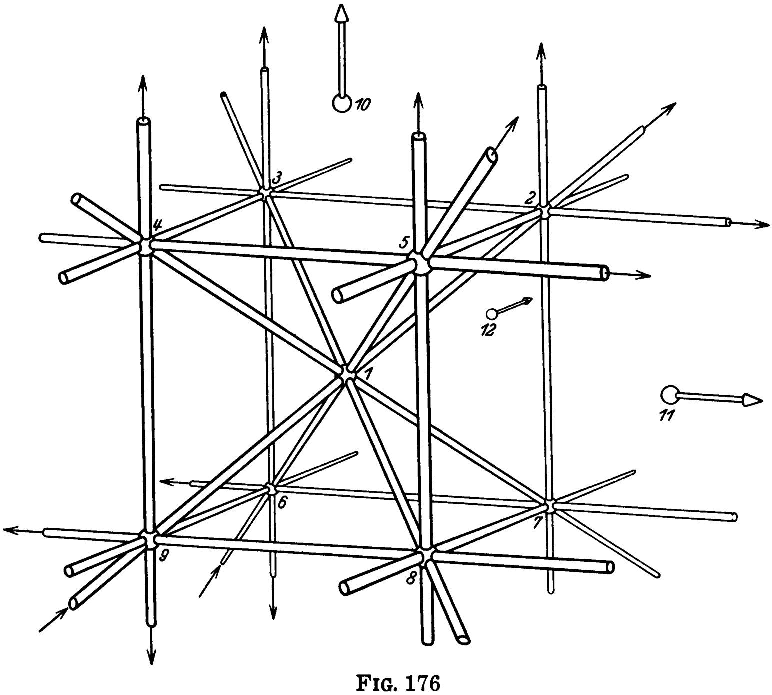

FIG. 176

tion.[5] We see from the finite octahedron in Fig. 175 that the planes of the configuration serve both as the boundary planes and as the planes of symmetry of the twelve octahedra. A closer study reveals the underlying reason for this; a complete quadrilateral divides the projective plane into three quadrangles and four trianges (in Fig. 177, the quadrangles 1, 2, 3 and the triangles I, II, III, IV). In Reye's configuration the straight lines partition each of the planes in this way; and since the faces of the octahedra are triangles, while the planes of symmetry intersect the octahedra in quadrangles, it is seen that each plane of the configuration serves as symmetry plane in three octahedra and as common boundary in 2·4 octahedra, while one of the twelve octahedra is not incident with it; thus the ideal plane is a configuration plane in Fig. 175, and one of the octahedra is located in the finite part of the space.[6]

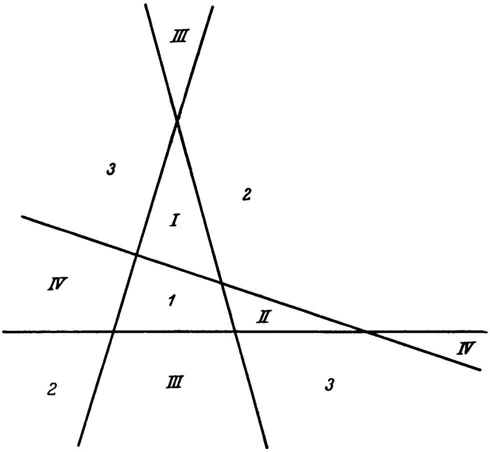

FIG. 177

[5] We had seen there that the two figures are related by a polarity with respect to a sphere. Now we see them as projections of one and the same four-dimensional figure each of which can be changed into the other by moving the three-dimensional image space.

[6] In analogy to the three planes of symmetry of the octahedron which pass through the center and intersect the boundary in a square, the 24-cell has twelve three-dimensional spaces of symmetry that pass through its center and intersect it in a cubo-octahedron. (The cubo-octahedron is illustrated in Fig. 178; a cubo-octahedron is also marked out in Fig. 172.) In the projection we are studying, the spaces of symmetry, like all spaces containing the center of the polytope, become planes. And these planes are precisely the planes of the Reye configuration. The three diametrically opposite pairs of squares and the four diametrically opposite pairs of equilateral triangles of the cubo-octahedron correspond to the three quadrilaterals and four triangles in each plane of Reye's configuration.

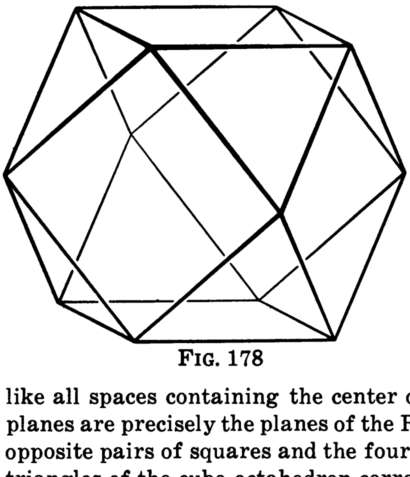
FIG. 178

Fig. 176 is simpler than Fig. 175 in that only two different kinds of octahedra occur in Fig. 176 (where six octahedra are congruent with the octahedron 1, 2, 3, 4, 5, 10 and the other six with 2, 5, 6, 9, 10, 11) while three different kinds of octahedra are present in Fig. 175—here one of the octahedra is regular, in three of them the ideal plane is a plane of symmetry (e.g., 1, 6, 7, 8, 9, 10), and in eight of them the ideal plane belongs to the boundary (e.g., 3, 4, 7, 8, 10, 11).

From this approach to the configuration the assertion made at the end of the last section follows immediately: *Reye's configuration is regular.*

The foregoing discussion suggests the idea of projecting the n-dimensional regular polytopes onto a space of the lowest possible dimensionality, i.e. onto a straight line. Let us study the projection of the n-dimensional cube onto one of its principal diagonals by the method of orthogonal parallel projection. The extremities A and B of such a diagonal are projected into themselves. Let us call the images of the other vertices of the cube V_1, V_2, . . . in the order of their positions on AB beginning with the point nearest to A. From A there emanate n edges, all forming the same angle with AB; hence all their endpoints must be projected into the point V_1 on AB. Furthermore, every edge of the cube is parallel to one of the edges through A, and it follows that the distance $V_k V_{k+1}$ between consecutive points is always equal to the distance AV_1 and is thus constant. Accordingly, the principal diagonal is divided into equal segments. It can be shown that there are exactly n of these segments and that the point V_k is the image of ${}_nC_k$ vertices for all k between 1 and $n - 1$, where ${}_nC_k$ is the well-known symbol denoting the binomial coefficients. For, V_k is the image of all those vertices, and only those, that can be connected with A by k, but not by less than k, edges of the cube, and we see, by counting, that there are exactly ${}_nC_k$ such vertices. In the case of the square and of the ordinary (three-dimensional) cube, these facts can be readily verified.

§ 24. Enumerative Methods of Geometry

The last three-dimensional configuration that we shall consider is Schläfli's double-six. The study of this configuration leads us to a special geometrical method called enumerative geometry. We shall discuss the method first, because we wish to avoid interrupting the study of the double-six and also because the enumerative methods are of great intrinsic interest.

The plane contains infinitely many straight lines and infinitely many circles. In order to characterize the multiplicity of all straight lines in the plane, we begin by fixing a Cartesian coordinate system in the plane. Then a straight line is in general completely determined by the sign and magnitude of its two intercepts with the coordinate axes. Hence any straight line—with exceptions we shall mention presently—can be analytically defined by two numbers. The straight lines that are parallel to one of the axes can also be included in this scheme by assigning the value infinity to the appropriate intercept. On the other hand, all the straight lines through the origin, and they only, are not defined by the intercepts; all of them give us the same data, namely zero, for both intercepts.

The straight lines that do not pass through the origin are said to form a two-parameter family; this means that every member of the family is determined by two numbers (the "parameters" of the family) and that a continuous change in the parameters is accompanied by a continuous change in the entity defined by them. According to this definition, the straight lines through the origin form a one-parameter family, as they can be determined by the angle that they form with one of the axes. Now it is usual to think of a two-parameter family as being, roughly speaking, not significantly enlarged by the addition of a one-parameter family which can be continuously imbedded into the first family. In this sense the set of *all* straight lines in the plane is also called a two-parameter family. We shall soon recognize the usefulness of this point of view.

The straight lines in the plane can also be determined in a variety of other ways, e.g., by a point through which they pass and the angle they make with an arbitrary fixed straight line. Since it takes two coordinates to define a point in the plane, we need altogether three parameters to characterize a straight line in this manner. However, the defining point may be picked arbitrarily on the straight line, and the points of a straight line obviously form a one-parameter family. We find much the same phenomenon when we define a straight line by two of its points. We need four parameters in this case, but a two-parameter family of pairs of points defines one and the same straight line. To get the correct number of parameters it will therefore be necessary to subtract two parameters in the latter example, or one parameter in the former; then we find, as we did by the first method, that the straight lines of the plane form a two-parameter family. This procedure, which is only sketched here, can be given a precise analytic formulation, and it can then be proved that the number of parameters associated with a family of geometrical figures is independent of the way in which the parameters are chosen. By using the symbol ∞ we can write this kind of argument more concisely. We shall say that there are ∞^2 straight lines in the plane, ∞^1 points on a straight line, and ∞^2 pairs of points on a straight line. In this way, enumeration becomes analogous to dividing one power of a number by another; to get the correct "number" ∞^2 of straight lines in the plane, we must "divide" the "number" ∞^4 of pairs of points in the plane by the "number" ∞^2 of pairs of points on a straight line.

Let us apply the procedure to the characterization of the size of the family of all circles in the plane. A circle is defined by its center and radius, i.e. by three numbers, and at the same time only one such number-triple is associated with every circle. The plane thus contain ∞^3 circles. Since the family of all straight lines in the plane has only two parameters and every straight line may be considered as a limiting case of a circle, the family of all circles *and* straight lines also has three parameters. This is in accord with the fact that through any three points of the plane one circle or one straight line can be drawn, as there are ∞^6 triples of points, and any one curve contains ∞^3 of them. Similarly, it can be shown that in any n-parameter family there is always a curve that passes through an arbitrarily chosen n-tuple of points of the plane but, in general, none that passes through $n + 1$ arbitrary points of the plane. This is only true, however, if all the limiting cases are included in the family, just as a unique correspondence between circles and number-triples becomes possible only on including straight lines as limiting cases in the family of circles. To make a rigorous formulation of these statements possible, analytic and algebraic methods are necessary, and in particular it is necessary to consider the imaginary elements along with the real ones.

Let us find the "number," in the above sense, of all the conics. An ellipse is defined by its two foci (four parameters) along with the constant sum of the distances from these points, i.e. by five parameters, and every ellipse is associated with only one such set of five numbers. Hence there are ∞^5 ellipses in the plane. Similarly it is shown that there are ∞^5 hyperbolas in the plane. The ellipses can also be fixed by the lengths of the two axes along with the position of their center and the direction of the major axis; this makes five parameters again, consonant with the general theory. It follows that the family of all parabolas in a plane has four parameters, for, by the construction given on page 4, we get the parabolas from the ellipses by a limiting process in which a one-parameter family of ellipses always determines a single parabola and each ellipse belongs to finitely many—two, to be specific—of the families.

If the values given for the lengths of the two axes of an ellipse are equal, we get a circle. At this point it would be easy to come to the erroneous conclusion that there are ∞^4 circles, rather than ∞^3, for if the axes are to be equal, we are still left with the choice of four numbers. The contradiction is resolved on noting that the equality of the axes makes it unnecessary to know the directions of the axes, since any given pair of perpendicular diameters of a circle can be regarded as constituting the limiting case of the axes of ellipses.

The above discussion does not entitle us to expect that we can always draw an ellipse through an arbitrary set of five points in the plane. At best, this might be the case if the ellipses are supplemented by inclusion of their limiting cases, the parabolas and the circles. It is found, however, that the hyperbolas must be included as well. The totality of all the conics in the plane, i.e. the set of all hyperbolas, parabolas, ellipses, circles, pairs of straight lines, and doubly-counted straight lines, constitutes a single family in the sense of enumerative geometry. In accordance with the above, this must be a five-parameter family; for, each of the different types of conics belongs to a family with five parameters or less. For the totality of conics it is indeed true that a member of this family passes through any set we may choose of five points of the plane. A closer study by methods outside the realm of enumerative geometry reveals that the conic is uniquely determined by the five given points except when four of them are on a straight line. In this exceptional case it is clear that the conic is not uniquely defined; through four points lying on one straight line l and a fifth point P

we can draw ∞^1 special conics consisting of the pair of straight lines l and m where m is an arbitrary straight line passing through P. If, in addition, P is also on l we can even draw ∞^2 pairs of straight lines, since the choice of the straight line m is then completely arbitrary.

We proceed to the application of enumerative methods to three-dimensional figures. By characterizing a plane by its three intercepts in a fixed coordinate system in space, we see that the space contains ∞^3 planes; for, the only planes that can not be defined by their intercepts are the planes that pass through the origin, and these latter are only a two-parameter family. By the method of enumeration we verify the elementary theorem that a plane ca be found which passes through any three given points in space; indeed, there are ∞^9 triples of points in space and ∞^6 such triples on every plane, so that the triples of points in space define "∞^9/∞^6," i.e. ∞^3, planes.

In determining a straight line by means of two points, we find that in space there are ∞^4 straight lines; for, there are ∞^6 pairs of points in space and ∞^2 on a straight line.

The spheres can be characterized by their center and radius. It follows that there are ∞^4 spheres in space. Adding the planes as limiting cases to the family of spheres, we can use enumeration to verify the well-known fact that a sphere or plane can be drawn through any four points in space. Just as in the case of the conics, the determination of the sphere is not always unique although it is unique if—and only if—the four points are not on a common straight line or circle. Analogous conditions govern the general case. If an n-parameter family of surfaces is defined so as to be sufficiently inclusive (like the family of all conics as opposed to the family of ellipses, in the plane), then there is a surface of the family through every set of n points in space. The surface is not always uniquely defined by the n points. It is, however, uniquely defined if the points are "in general position," i.e. if they do not satisfy certain geometrical relations whose nature depends on the given family of surfaces.

A ruled surface of the second order is defined by three skew straight lines. Space contains $\infty^{4\cdot 3} = \infty^{12}$ triples of straight lines. But since every straight line on a ruled quadric is a member of a one-parameter family, ∞^3 triples of straight lines define the same surface. Hence there are ∞^4 ruled quadrics.

Likewise there are ∞^9 general ellipsoids. This follows from the fact that we get every ellipsoid once, and only once, by varying the choice of the center (three parameters), the lengths of the axes (three parameters), the direction of the major axis (two parameters), and—the minor axis lying in the plane through the center perpendicular to the major axis—the direction of the minor axis within that plane (one parameter).

From analytic considerations we learn that there are ∞^9 quadrics altogether. We have, for this family, the theorem that every set of nine arbitrary points in space lies on a surface belonging to the family. In order that the definition of a quadric by nine points be unique, i.e. that the position of the points be sufficiently general for the family of quadrics, it is necessary to stipulate that the points shall not lie on certain space curves of the fourth order; for, these can be obtained as the curves of intersection of pairs of quadrics, so that naturally it would be impossible that any number of points on such a curve could define a quadric uniquely.

We shall now establish the plausibility of the fact that there are infinitely many straight lines on every second-order surface. To this end we begin with the fact, immediately deducible from the analytic definition of second-order surfaces, that every straight line having three points in common with such a surface is wholly embedded in it. Evidently there are ∞^6 triples of points on a quadric (and, for that matter, on any surface). Let us select only those triples of points that are collinear. Enumerative geometry yields the result that there are ∞^4 of them, i.e. that two parameters are lost. For, it takes two analytic relations to express the incidence of one of the points with the straight line defined by the other two; and there is a general theorem that the number of parameters associated with a family is diminished by n if we select only the members satisfying a certain set of n independent relations (where n relations are called independent if they cannot be replaced by less than n equivalent relations). Hence it is true that ∞^4 triples of collinear points lie on any given quadric. And it was pointed out before that every straight line that is incident with such a triple of points must lie on the surface. But there are ∞^3 triples of points on a straight line. Hence the triples of collinear points on a second-order surface lie on ∞^1 straight lines belonging to the surface.

On the ellipsoid, the elliptic paraboloid, and the hyperboloid of two sheets, these straight lines are imaginary.

In conclusion, we add a few remarks on the third-order surfaces, since these surfaces are intimately connected with the properties of Schläfli's double-six to be studied in the next section. Analytically, the third-order surfaces are characterized by the property of having an equation of the third degree in Cartesian coordinates. Now, the general third degree equation in three unknowns has twenty coefficients, and they are determined up to a common factor by the surface associated with the equation. It follows that there are ∞^{19} third-order surfaces and that through any set of 19 points arbitrarily chosen in space there passes a surface of the family. It is necessary here, however, to include certain degenerate cases in the family of third-order surfaces, e.g. a second-order surface and a plane, taken together.

In general, a straight line has three points in common with a third-order surface, and a straight line having four points in common with such a surface must lie on the surface. This is easily deduced from the fact that the surface has an equation of third degree. We shall show by enumeration that the most general third-order surface can only contain a finite number of straight lines. On every surface there are ∞^8 quadruples of points. It takes four conditions to insure that such a quadruple of points be collinear—two conditions for the third point and two for the fourth point to lie on the straight line common to the first two points. Hence there are ∞^4 collinear quadruples of points on a general third-order surface. Every straight line containing such a quadruple lies on the surface and contains ∞^4 other such quadruples. The existence of an infinity of straight lines on the surface would imply that more than ∞^4 quadruples of collinear points could be found on it.

But the third-order surfaces also include a great many ruled surfaces. These surfaces, then, contain ∞^5 or even more quadruples of collinear points. Accordingly, the equation of a ruled surface of the third order must have the special property that this equation together with the four conditions for the collinearity of four points can be replaced by an equivalent system of fewer equations. It may be shown that such a reduction is possible only if the twenty coefficients of the third-degree equation satisfy certain special relations. This also shows the truth of the statement that the general third-order surface contains at most a finite number of straight lines.[1]

An enumeration similar to the above shows that the general surfaces of order higher than the third do not in general contain any straight lines.

§ 25. Schläfli's Double-Six

We begin with some simple considerations concerning the possible positions of straight lines in space. Three skew straight lines a, b, and c define a hyperboloid H. In general, an arbitrary fourth straight line d intersects H at two points, although it may also be tangent to H or lie on H. In the general case, each of the points at which d and H intersect is incident with a straight line lying on H that does not belong to the same family as a, b, c and therefore intersects a, b, and c. Conversely, every straight line that intersects a, b, c, and d, is on H and is incident with one of the points at which d intersects H. Hence there are in general two, and not more than two, straight lines that intersect four given straight lines. In the case where d is tangent to H there is only one (double) straight line that intersects a, b, c, and d. If, on the other hand, there are more than two straight lines that intersect a, b, c, and d, then d must lie on H, and then there are infinitely many straight lines intersecting a, b, c, and d. In this case we say that the four straight lines are in a hyperboloidal position.

In the construction of Schläfli's double-six we start with any straight line 1 and draw three mutually skew straight lines intersecting 1, which we shall call 2′, 3′, and 4′, for reasons that will become apparent later (see Fig. 179). Then we draw another straight line 5′ through 1, which is to have the most general possible position relative to 2′, 3′, and 4′: 5′ will not intersect any of the straight

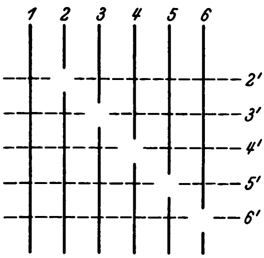

FIG. 179

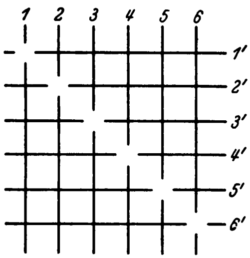

FIG. 180

[1] E.g., there is no straight line on the surface $xyz = 1$ which passes through a finite point of the surface.

lines 2′, 3′, and 4′, and there will be besides 1 just one straight line—we shall call it 6—that intersects 2′, 3′, 4′, and 5′. Finally we draw a straight line 6′ through 1 which must not intersect 6, 2′, 3′, 4′, or 5′, and which must furthermore be such as to make the positions of the quadruples 2′3′4′6′, 2′3′5′6′, 2′4′5′6′, and 3′4′5′6′ as general as possible. Then there is exactly one straight line 5 in addition

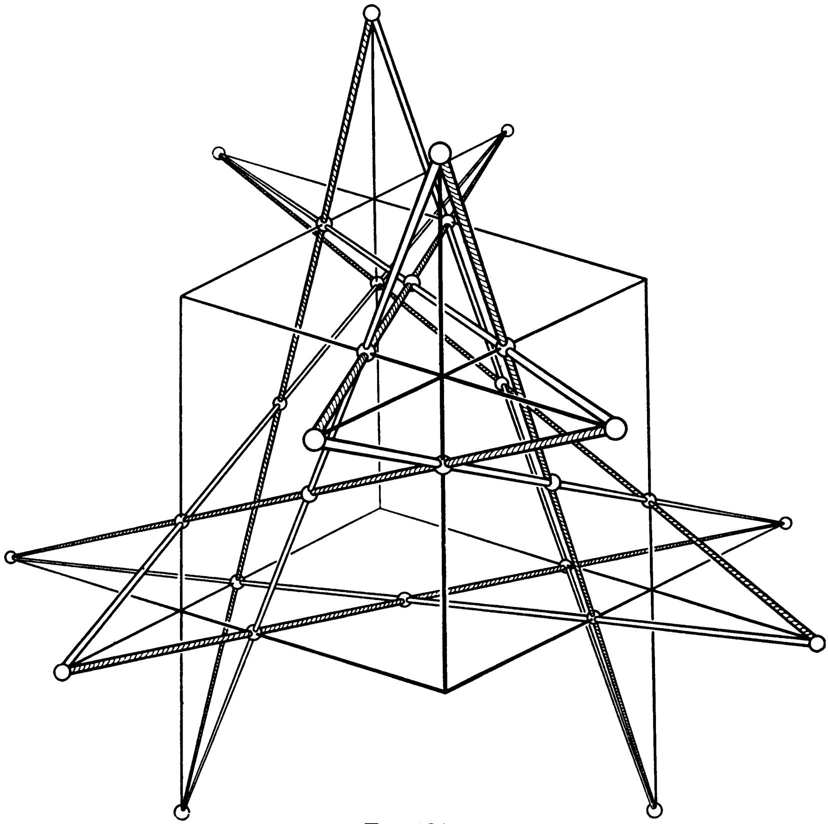

FIG. 181

to 1 which intersects 2′, 3′, 4′, and 6′, and the straight lines 4, 3, and 2 are defined analogously (e.g. 4 is distinct from 1 and intersects 2′, 3′, 5′, and 6′, etc.). In this way we obtain the system of intersections represented schematically in Fig. 179. It is easily seen that our choice of the straight lines 2′, 3′, 4′, 5′, 6′ precludes the possibility of additional intersections. Turning now to the four straight lines 2, 3, 4, and 5, we shall show that they cannot be in a hyperboloidal position. For if they were, every straight line that intersects three of them would also intersect the fourth, and in particular, this would apply to each of the straight lines 2′, 3′, 4′, and 5′, according to our scheme. Then these four straight lines would also be in a hyperboloidal position, contradicting the conditions of our construction. Thus there are at most two straight lines that meet 2, 3, 4, and 5. But according to our construction, 2, 3, 4, and 5 all intersect 6′. Let us denote the second straight line that intersects 2, 3, 4, and 5, by 1′; we assert that 1′ does not coincide with 6′ and that it cuts 6. Pending the proof of this assertion (to be given below), we may supplement the arrangement represented by Fig. 179, changing it into that of Fig. 180. The latter scheme represents the double-six. It is immediately seen that we are dealing with a regular configuration of points and straight lines whose symbol is $(30_2 12_5)$. A particularly clear and symmetrical form of the double-six can be constructed by suitably choosing one of the straight lines of each set of six on each face of a cube. The arrangement should be apparent from Fig. 181 (cf. also Fig. 102, p. 93).

We must now prove the assertion made above that there is a straight line 1′ distinct from 6′ which meets 2, 3, 4, and 5, and that this 1′ must meet 6. Let us tentatively assume that the first part is already proved and prove on the basis of this assumption that 1′ intersects 6. To this end, we select four points on the straight line 1 and three points on each of the straight lines 2′ to 6′, making sure that none of the points of intersection of the lines under consideration are included among the nineteen points thus chosen. According to the argument of the last section, a third-order surface F_3 can be drawn through these nineteen points. Now F_3, having four points in common with the straight line 1, must contain the entire straight line. Furthermore, F_3 has four points in common with each of the straight lines 2′ to 6′—the three points chosen in the beginning and the point (distinct from these) where the line meets 1; thus F_3 contains 2′ to 6′ as well. From this it follows in turn that F_3 also contains the straight lines 2 to 6, as each of them intersects four straight lines lying on the surface. And finally, F_3 contains 1′ for the same reason. Supposing now that 1′ did not intersect 6, let us consider the straight line l which, like 5′, intersects 2, 3, 4, and 6. As in the construction of 1′, we shall rule out for the time being the case where l coincides with 5′. l cannot coincide with 1′, since it was assumed that 1′ does not meet 6. Since l

meets four straight lines lying on F_3, namely 2, 3, 4, and 6, l itself lies on F_3. By our construction, each of the four straight lines l, 1′, 5′, 6′ meets 2, 3, and 4. Hence the four straight lines are in a hyperboloidal position. Then the entire associated hyperboloid must be a part of F_3; this follows directly from the fact that every straight line that intersects l, 1′, 5′, and 6′, lies on F_3, while the set of all such straight lines covers the hyperboloid.

Now, it is easy to prove algebraically that a third-order surface that contains all the points of a second-order surface must consist of the second-order surface and a plane: If $G = 0$ and $H = 0$ are the equations of the third-order and second-order surface respectively, the polynomial G of the third degree must be divisible by the polynomial H of the second degree, and this can only be the case if G is the product of H and a linear expression. From the conclusion that the surface F_3 defined by our nineteen points must be a degenerate case of this sort, we can easily deduce a contradiction. For, no four of the straight lines 2′, 3′, 4′, 5′, 6′ have a hyperboloidal position; hence at most three of them could be on the hyperboloid that forms a part of F_3. Hence at least two would have to be on the plane that constitutes the other component of F_3 and these two would therefore have a point of intersection, in contradiction to our construction.

If we admit the possibility, previously excluded, that 1′(2345) may coincide with 6′ or l(2346) with 5′, the proof is not essentially changed. In this case, too, we can conclude that the hyperboloid defined by 2, 3, and 4 would have to be a part of F_3. But the limiting process by which this case is derived from the general case can not be justified without the use of algebraic methods.

In the proof of the last incidence relation (1′6) of the double-six we used the fact, interesting in itself, that there is always a third-order surface F_3 that contains this configuration. It is easy to supplement the configuration with several additional straight lines which also lie on F_3. Consider, for instance, the plane spanned by the intersecting straight lines 1 and 2′ and the plane spanned by 1′ and 2 and let (12) denote the line in which the two planes intersect. Then (12) meets the four straight lines 1, 1′, 2, and 2′, all of which lie on F_3; hence (12) also lies on F_3. In all there are fifteen straight lines that bear the same relation to the double-six as (12) and therefore lie on F_3 as well. For, fifteen different pairs can be chosen from the numbers from 1 to 6. We have thus found $2 \times 6 + 15 = 27$ straight lines all lying on F_3.

Among the straight lines of the enlarged configuration that we have obtained in this way there are further incidence relations. In fact, it may be shown that all those pairs of the straight lines denoted by two numbers whose symbols have no number in common, and those only, will have a point of intersection. The proof can be based on the same idea as our proof that 1′ and 6 intersect, and we shall only give an indication of it. For reasons of symmetry it suffices to show that (12) meets (34). To this end, we consider the three straight lines 1, 2, (34), and note that 3′ and 4′ intersect them. If (12) did not intersect (34), there would be a straight line a that would meet the four lines 1, 2, 1′, and (34), and a straight line b that would meet 1, 2, 2′, and (34). b would necessarily be distinct from a, for if they were one and the same straight line, this would meet the four lines 1, 2, 1′, 2′, and would therefore be identical with (12) and yet meet (34), whereas we are assuming for the time being that (34) does not meet (12). Similarly a and b would have to be distinct from 3′ and 4′; for if, say, a coincided with 3′, then 3′ would intersect 1′, in contradiction to our construction. Now a and b, like 3′ and 4′, would have to lie on F_3, and because all of them meet the triple 1, 2, (34), the four straight lines would be hyperboloidal. But we have already seen that it is impossible for F_3 to contain a set of four straight lines in the hyperboloidal position. It follows that (12) does meet (34). For the same reasons it must meet (35), (36), (45), (46), and (56). Since (12) also meets 1, 2, 1′, and 2′, it follows that (12) intersects ten straight lines of the enlarged configuration, and does every one of the straight lines we denoted by two numbers. The same is true for the straight lines of the double-six itself; 1, for example, intersects the five lines 2′ to 6′ and the five lines (12), (13), (14), (15), (16). Accordingly, the configuration consisting of the 27 straight lines on F_3 together with their points of intersection has the symbol $(135_2 27_{10})$. The fact that there are exactly 135 points follows from the equation $135 \times 2 = 27 \times 10$. It can be shown, moreover, that the configuration is regular, and that many different double-sixes can therefore be found in it. Considering in addition the planes spanned by intersecting pairs of lines of the configuration, we can verify by referring to the incidence table that every such plane contains a third line of the configuration. This can also be seen by the following simple algebraic argument. Every plane necessarily intersects F_3 in a third-order curve. If the plane contains two straight lines of the configuration, this curve is bound to contain them, and it can be deduced algebraically that the curve must then consist of these two straight lines and a third straight line. It is easy to check by counting that five of our planes pass through each of the twenty-seven straight lines and that the planes number forty-five in all. Thus we see that the configuration is not self-dual, although the double-six, being built up on the self-dual relation of the incidence of two straight lines, is self-dual. The double-six can easily be extended to a configuration that is the dual of the configuration we have just constructed. To this end, we need to add a different set of straight lines $[ik]$ instead of the straight lines (ik), where, for example, $[12]$ passes through the points at which 1 intersects 2′ and 1′ intersects 2. The configuration obtained in this way has the symbol $(45_3 27_5)$.

Let us return to the original configuration of twenty-seven straight lines. We shall show by enumerative methods that there is such a configuration K on every third-order surface F_3. Here, as in all enumerative considerations, the cases where K is partly imaginary or degenerate must also be taken into account. The proof begins with the enumeration of the family of all double-sixes. According to our construction, the choice of the straight line 1 is completely free, and thus involves four parameters; the points where 1 intersects the straight lines 2′ to 6′ depend on another five parameters, and each of the lines 2′ to 6′ can assume ∞^2 positions once its point of intersection with 1 is fixed (thus accounting for ten more parameters). Since the straight lines 1, 2′, 3′, 4′, 5′, and 6′, uniquely define the double-six, we see that there are ∞^{19} double-sixes $(19 = 4 + 5 + 10)$. The family of configurations K has the same number of parameters; for, each configuration of this type is defined by one of the double-sixes in it, and obviously there is only a finite number of double-sixes in any one configuration K. Now we have given a construction for passing an F_3 through any given K; it follows either that the family of the surfaces F_3 constructed in this way comprises ∞^{19} surfaces or, should there be fewer surfaces, that at least ∞^1 configurations K lie on the same F_3, i.e. that F_3 would have to be a ruled surface of the third order. It can be shown, however, that there are less than ∞^{18} ruled surfaces of the third order; hence the F_3 we constructed would have to contain at least ∞^2 double-sixes. But since it was already demonstrated that the F_3 do not contain a hyperboloid and since any ruled surface of order higher than the second contains only one family of straight lines, such an F_3 cannot possibly carry ∞^2 double-sixes. Therefore our surfaces cannot in general be ruled surfaces, and it follows that our construction accounts for not less than ∞^{19} surfaces. On the other hand, as we have mentioned in the last section, there are only ∞^{19} third-order surfaces. From this, the algebraic nature of the figures under consideration being borne in mind, the truth of our assertion that every third-order surface contains a configuration of the type K can be rigorously deduced.