Projective Geometry

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Preface

This document began as a record of a two-week course called *Geometriler* at the Nesin Mathematics Village, Şirince, Selçuk, Izmir, Turkey, September 12–25, 2016.

I gave a similar course the next summer, July 31–August 13, 2017, though I have for this course only a handwritten record, in a coil-bound yellow A4 notebook from the Village.

In December of 2017, I published a relevant article, “Thales and the Nine-point Conic” [33].

The first week of my Şirince course was on projective geometry, with Pappus [28] as a text; the second, hyperbolic, with Lobachevsky [23]. The present document covers only the former week.

Otherwise, I have spelled out many details in my course notes, sometimes after further consultation with Hilbert [20] or Coxeter [5]. I have mostly kept the original ordering of topics, and the chapters are still titled with the days of the week.

I now draw diagrams with the `pst-eucl`, which has commands for drawing parallels and for naming the intersections of straight lines that themselves pass through named points.

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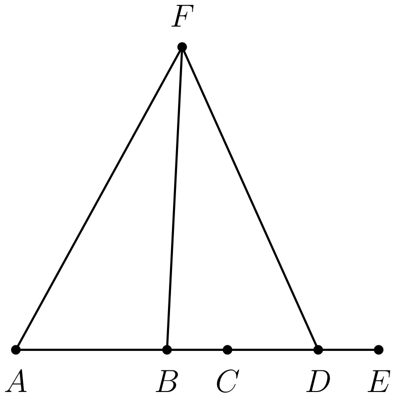
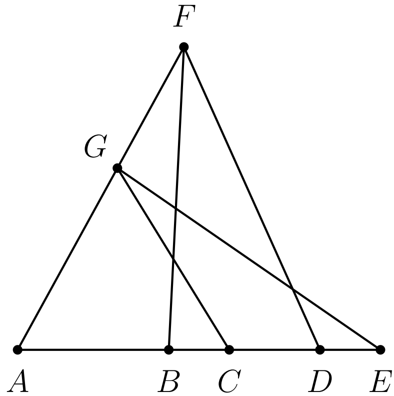
1. Monday

1.1. Quadrangle Theorem

Suppose five points, A through E, fall on a straight line, and F is a random point not on the straight line. Join FA, FB, and FD, as in Fig. 1a. Now let G be a random point on FA, and



(a) Point F is chosen



(b) Point G is chosen

Figure 1. Quadrangle Theorem set up

join GC and GE, as in Fig. 1b. Supposing these two straight lines cross FB and FD at H and K respectively, join HK as in Fig. 2. If this straight line crosses the original straight line AB at L, we shall show that L depends only on the original five points, not on F or G. Let us call this the **Quadrangle Theorem.** It is about how the straight line AB crosses the

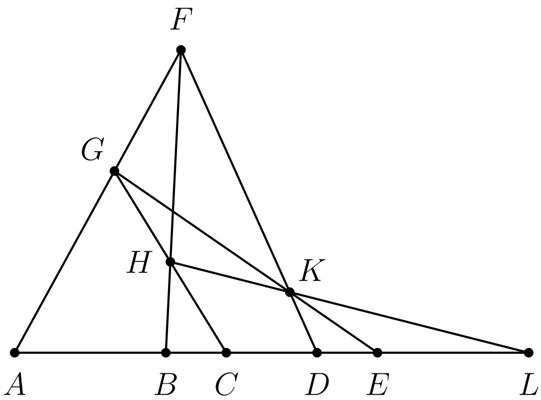
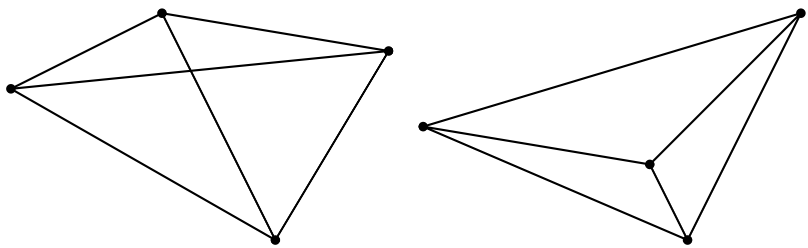


Figure 2. Quadrangle Theorem

six straight lines that pass through pairs of the four points F, G, H, and K. Any such collection of four points, no three of which are collinear, together with the six straight lines that they determine, as in Fig. 3, is called a **complete quadrangle**



(a) (b)

Figure 3. Complete quadrangles

(*tam dörtgen*). Similarly, any collection of four straight lines, no three passing through the same point, together with the six points at the intersections of pairs of these six straight lines, as in Fig. 4, is a **complete quadrilateral** (*tam dörtkenar*).

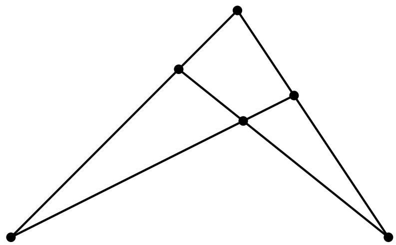


Figure 4. Complete quadrilateral

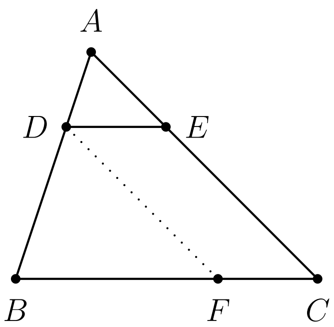
As stated, the Quadrangle Theorem is a consequence of what we shall call Lemma IV of Pappus. Pappus was the last great mathematician of antiquity, and Lemma IV is one of the lemmas in Book VII of his *Collection* [25, 26, 27, 28] that are intended for use with Euclid's now-lost three books of *Porisms.* We shall prove Lemma IV in §2.2 (p. 38). Lemmas I, II, V, VI, and VII (not proved in these notes) treat other cases, such as when HK in Fig. 2 is parallel to AE. We shall give a second proof of the Quadrangle Theorem in §6.5 (p. 95).

1.2. Thales's Theorem

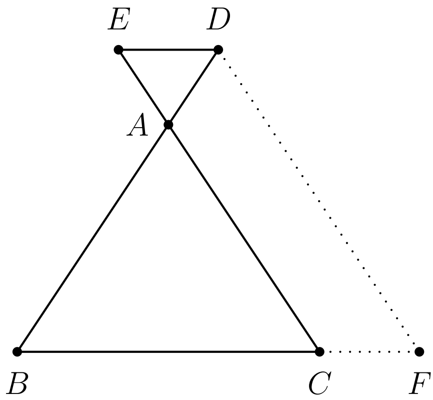
1.2.1. Proportion Theorem

Pappus's proofs of results such as Lemma IV rely heavily on what for now I shall call the **Proportion Theorem.** This is Proposition 2 of Book VI of Euclid's *Elements* [13, 14]:

> If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally, [and conversely].



(a) Internal division



(b) External division

Figure 5. Proportion Theorem

Symbolically, if the triangle is ABC as in Fig. 5, and D and E are on AB and AC respectively, or possibly on the extensions of these bounded straight lines, then, according to the Proportion Theorem,

$$DE \parallel BC \iff AD : DB :: AE : EC. \tag{1}$$

This result is known in some countries as *Thales's Theorem* [29], but for now I want to reserve this name for a related result, as follows.

1.2.2. Thales's Theorem

In (1) we can read the proportion

$$AD : DB :: AE : EC \tag{2}$$

in any of several ways:

1. AD, DB, AE, and EC are proportional.
2. AD is to DB as AE is to EC.
3. AD has to DB the same ratio that AE has to EC.

In particular, the proportion expresses not the *equality* of the ratios $AD : DB$ and $AE : EC$, but their **sameness.**

Having the same ratio is an *equivalence relation.* In particular, it is transitive. See §1.3 (p. 17). Thus, if (2) holds, and also

$$AE : EC :: AH : HG$$

as in Fig. 6, then

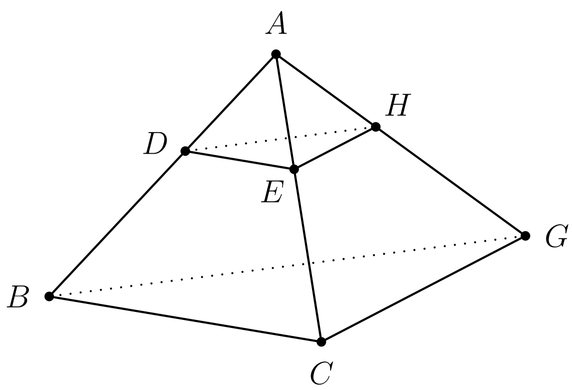


Figure 6. Thales's Theorem

$$AD : DB :: AH : HG.$$

By the Proportion Theorem,

$$DE \parallel BC \ \ \& \ \ EH \parallel CG \implies DH \parallel BG. \tag{3}$$

Let us call this **Thales's Theorem.** We can count (3) as true, even if G lies on AB, since then DH and BG lie on the same straight line.

Historical notes on Thales are in Appendix A.

1.2.3. Affine plane

Truth of Thales's Theorem in the sense just defined is a fundamental property of an *affine plane.* By definition, an **affine plane** is a collection of points and straight lines of which the following axioms are true.

1. There exist at least three points, not all on the same straight line.
2. Any two distinct points lie on a unique straight line.
3. To a given straight line, through a given point not on the line, there is a unique parallel straight line.
4. Thales's Theorem holds.

We can understand axiom 2 here as the first of Euclid's five postulates. In axiom 3,

- existence of the parallel is a consequence of Proposition 31 of Book I of the *Elements*;
- uniqueness, Propositions 27 and 29, the latter relying on the fifth postulate.

1.2.4. Ratios

In an affine plane, the relation of having the same ratio is indeed an equivalence relation, if we take the Proportion Theorem as a *definition* of proportion. We shall do this. Then for any triangle ABC and point D on AB, there is unique point E on AC such that (2) holds. The ratio $AD : DB$ is now the equivalence class consisting of all ordered triples (A, E, C) such that E lies on AC and

- if C is not on AB, then

$$DE \parallel BC;$$

- if C is on AB, then for some G not on AB and some H

on AG,

$$DH \parallel BG \ \ \& \ \ HE \parallel GC.$$

1.2.5. Product of ratios

We can now define the product of two ratios. In Fig. 7, with

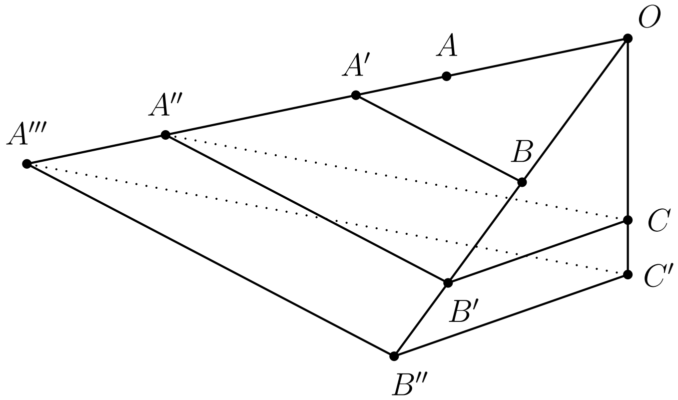


Figure 7. Multiplication of ratios

$$A'B \parallel A''B', \tag{4}$$

we let

$$(OA : AA')(OB : BB') :: OA : AA''. \tag{5}$$

Since ratios are equivalence classes, we have to confirm that this is a valid definition; but it is, by Thales's Theorem. As a special case, from which we can now derive (5), we have

$$(OA : AA')(OA' : A'A'') :: OA : AA''. \tag{6}$$

1.2.6. Associativity of multiplication

Thales's Theorem gives us also associativity of multiplication, since, still in Fig. 7, if, in addition to (4), also

$$A''B' \parallel A'''B'', \qquad\qquad B'C \parallel B''C',$$

then

$$\begin{aligned}(OB : BB')(OC : CC') &:: OB : BB'',\\ (OA : AA')(OB : BB'') &:: OA : AA'''.\end{aligned}$$

By this and (6),

$$\begin{aligned}&(OA : AA')\big((OB : BB')(OC : CC')\big)\\ &\qquad :: \big((OA : AA')(OB : BB')\big)(OC : CC')\\ &\qquad\qquad \iff A'''C' \parallel A''C;\end{aligned}$$

but the parallelism holds by Thales's Theorem.

The commutativity of multiplication of ratios will need Pappus's Lemma VIII, which is one case of Pappus's Theorem, to be defined in §1.5 (p. 29).

1.3. Equivalence relations

1.3.1. Equality

In Euclid, two bounded straight lines may be *equal* without being the same. For example, in an isosceles triangle, two of the sides are equal. Equality of bounded straight lines is an **equivalence relation.** This means equality is *transitive, symmetric,* and *reflexive.*

1. Equality is transitive, by the first of the Common Notions in Euclid's *Elements*:

   > Equals to the same thing are equal to one another.

2. That equality is symmetric is implicit in the same common notion, as well as in one of the definitions at the head of the *Elements* that we have just alluded to:

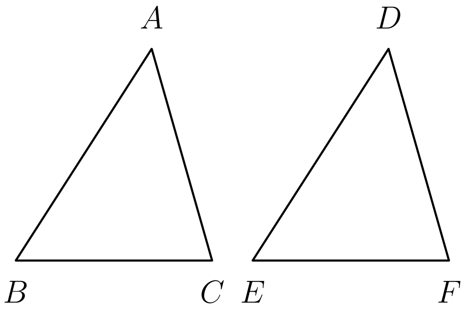
   > Of trilateral figures,
   > - an **equilateral triangle** is that which has its three sides equal;
   > - an **isosceles triangle,** that which has two of its sides alone equal; and
   > - a **scalene triangle,** that which has its three sides unequal.

   If the sides AB and AC of a triangle are equal, we can write this indifferently as $AB = AC$ or $AC = AB$.

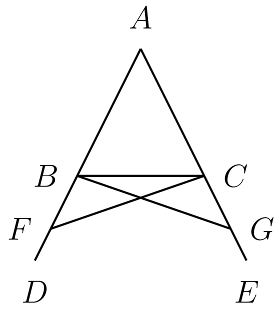
3. That equality is reflexive in the *Elements* is seen in how Proposition 4 of Book I is applied. This proposition is the theorem about triangles that we now call Side-Angle-Side, or SAS. In triangles ABC and DEF of Fig. 8a,

$$\left.\begin{gathered} BA = ED \\ \angle BAC = \angle EDF \\ AC = DF \end{gathered}\right\} \implies \left\{\begin{gathered} \angle ABC = \angle DEF \\ AC = DF \\ \angle ACB = \angle DFE \\ \triangle ABC = \triangle ABC. \end{gathered}\right. \tag{7}$$

   The equality of triangles here is in the sense to be discussed in §1.6 (p. 32). The point now is that, when (7) is applied in Proposition 5, to prove that the base angles of isosceles triangle ABC in Fig. 8b are equal, we have $AB = AC$, and we make also $AF = AG$, and therefore angles ABG and ACF are equal, since also angles BAG and CAF are equal, being the same angle.



(a) I.4



(b) I.5

Figure 8. *Elements* Propositions I.4 and 5

Equality being an equivalence relation, we may say that equal straight lines have the same **length.** Length is an abstraction, which we cannot draw in a diagram. We can define the length of a line AB as the equivalence class, denoted by

$$|AB|,$$

consisting of all of the straight lines XY such that $AB = XY$.

Equality has a criterion in the fourth of Euclid's Common Notions that his editor Heiberg [12] accepts as genuine:

> Things congruent with one another
> are equal to one another.

I discuss this in "On Commensurability and Symmetry" [36]. Two equal straight lines can have different directions and end-points: this is seen in

- Euclid's third postulate, that a circle can be drawn with any center and passing through any other point;
- Proposition I.3, whereby, from any straight line, we can cut off a part that is equal to any shorter straight line.

We shall be interested in seeing how far we can go, treating only opposite sides of a parallelogram as equal. This is the only kind of equality that we can talk about in an affine plane. It is also the equality of which our sign = of equality is an icon; see the paper [36] just mentioned.

1.3.2. Sameness of ratio

Being an equivalence relation is even more fundamental to *sameness* than to equality. In ancient Greek mathematics at least, any definition of proportion should make it obvious that sameness of ratio is indeed an equivalence relation. There are two theories of proportion in the *Elements*:

1) for *magnitudes,* such as bounded straight lines, in Books V and VI;
2) for *numbers,* in Books VII, VIII, and IX.

By the definition at the head of Book VII,

> [Four] numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

If we take seriously the use of the word "same" here, then, in the context of the whole of Book VII of the *Elements,* the definition of proportion of numbers must mean that, for counting numbers k, ℓ, m, and n, we have $k : \ell :: m : n$ precisely when the Euclidean algorithm has the same steps, whether applied to (k, ℓ) or (m, n). Thus for example $32 : 14 :: 48 : 21$, because of the common sequence $(2, 3, 2)$ of multipliers in the computations

$$\begin{aligned} 32 &= 14 \cdot 2 + 4, & 48 &= 21 \cdot 2 + 6, \\ 14 &= 4 \cdot 3 + 2, , & 21 &= 6 \cdot 3 + 3, \\ 4 &= 2 \cdot 2, & 6 &= 3 \cdot 2. \end{aligned}$$

I have written about this elsewhere [35].

We shall look at Euclid's definition (and Hilbert's definition) of proportion of bounded straight lines in §3.2 (p. 45).

As we cannot draw lengths as such in a diagram, so can we not draw ratios.

1.3.3. Parallelism

Parallelism is transitive by Proposition I.30 of the *Elements.* Since it is obviously symmetric, it is an equivalence relation, provided we understand a straight line to be parallel to itself.

That parallelism is transitive is also a theorem about affine planes. If $AB \parallel CD$ as in Fig. 9, but a third line ED meets

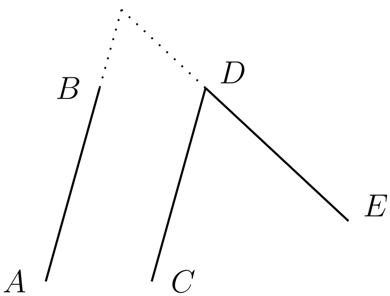


Figure 9. Parallelism in an affine plane

CD at D, then, CD being the only parallel to AB that passes through C, ED must meet AB somewhere.

1.3.4. Sameness of direction and length

Since parallelism is an equivalence relation, we may say that parallel straight lines have the same **direction.** Hence having the same direction and length is an equivalence relation.

In Euclid and Pappus, expressions such as AB and BA for bounded straight lines are interchangeable. We may however distinguish them, considering AB as the ordered pair (A, B). As in §1.2 (p. 12) we assigned to an ordered triple (A, D, B) the ratio $AD : DB$, so now we assign to AB, considered as (A, B), the **directed length,** or **vector,** denoted by

$$\overrightarrow{AB}.$$

This is the equivalence class consisting of all ordered pairs (C, D) of points such that

$$AB \parallel CD, \qquad\qquad AB = CD,$$

and A is on the same side of B that C is of D. However, this last condition is imprecise.

Propositions 33 and 34 of Book I of Euclid's *Elements* constitute what we shall call the **Equality Theorem:** two bounded straight lines that are not part of one straight line, but are parallel, are equal if and only if they are the sides of a parallelogram. Now we can say that $\overrightarrow{AB}$ consists of those (C, D) such that

- if A and B are the same, then so are C and D;
- if C does not lie on AB, then $ABDC$ is a parallelogram;
- if C does lie on AB, then for some E and F, both $ABEF$ and $EFCD$ are parallelograms.

If (A, B) and (C, D) represent the same vector, we may write

$$\overrightarrow{AB} :: \overrightarrow{CD}.$$

If $\overrightarrow{AB} :: \overrightarrow{AC}$, then B and C must be the same point. The Side-Angle-Side Theorem, discussed in §1.3 (p. 17), now takes the form that, in two triangles ABC and DEF,

$$\overrightarrow{AB} :: \overrightarrow{DE} \ \& \ \overrightarrow{BC} :: \overrightarrow{EF} \implies \overrightarrow{AC} :: \overrightarrow{DF}.$$

Let us call this the **Prism Theorem,** even though a prism is normally a solid figure, and we are working in a plane.

More precisely, if, as in Figure 13a, ABC is a triangle, $ABED$ is a parallelogram, and AD does not lie along AC, but DF is drawn parallel to AC, the Prism Theorem is

$$AD \parallel CF \implies BC \parallel EF. \tag{8}$$

This is an easy consequence of the Equality Theorem. For suppose now, in Fig. 13a, in addition to the conditions already stated,

$$AD \parallel CF. \tag{9}$$

Then

$$BE = AD, \qquad\qquad AD = CF,$$

by the Equality Theorem. Since equality is transitive,

$$BE = CF.$$

Again by the Equality Theorem,

$$BC \parallel EF. \tag{10}$$

This gives (8).

Euclid proves the Equality Theorem, already having equality as an equivalence relation, in the sense discussed above. We can use the Theorem as a *definition* of equality of parallel bounded straight lines, provided we know that equality so defined will be transitive; but then the Prism Theorem guarantees this transitivity, just as Thales's Theorem guarantees that sameness of ratio, as given by the Proportion Theorem, is transitive.

1.4. Desargues's Theorem

The Prism Theorem and Thales's Theorem are specials case of **Desargues's Theorem.** We shall use this result for the second proof of the Quadrangle Theorem, mentioned at the end of §1.1 (p. 10). Desargues was a contemporary of Descartes, and the theorem named for him concerns two triangles. If these are ABC and DEF, we assume that the lines AD, BE, and CF that connect corresponding vertices either

- have a common point G, or
- are parallel to one another.

There are only the following three possibilities for the pairs $\{BC, EF\}$, $\{AC, DF\}$, and $\{AB, DE\}$ of corresponding sides.

Parallelism: each pair are parallel.

Intersecting: each pair intersect, and the three intersection points lie along a common straight line.

Mixing: Two pairs intersect, and the line that the two intersection points determine is parallel to each line in the third pair.

That is Desargues's Theorem, and there are six cases in all. The two cases of parallelism are, again, The other cases are shown in Fig. 10 and Fig. 11.

1.4.1. Parallel cases in an affine plane

By definition, in every affine plane, Thales's Theorem is true. The Prism Theorem is also true in every affine plane. To prove this, we first establish a converse of Thales's Theorem. In Fig. 12, assuming that DE is drawn parallel to AB in the triangle GAB, and the sides AC and DF of the triangles ABC and DEF are also parallel, we have

$$BC \parallel EF \implies F \text{ lies on } GC.$$

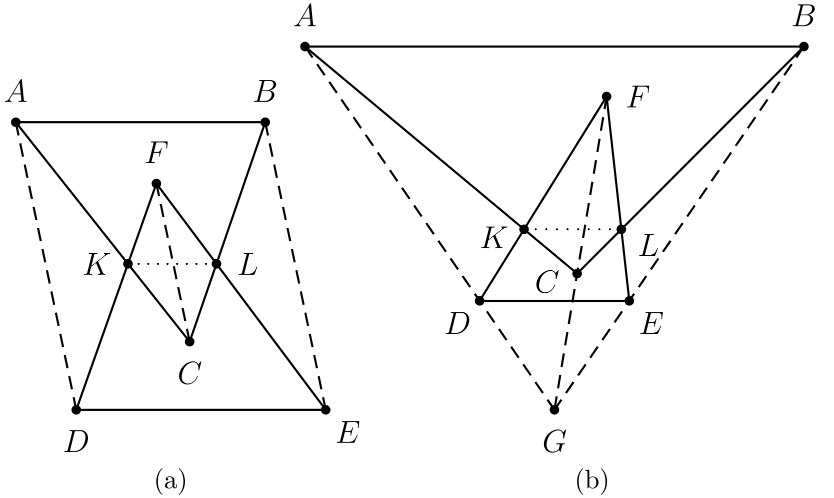


Figure 10. Desargues's Theorem, mixed cases

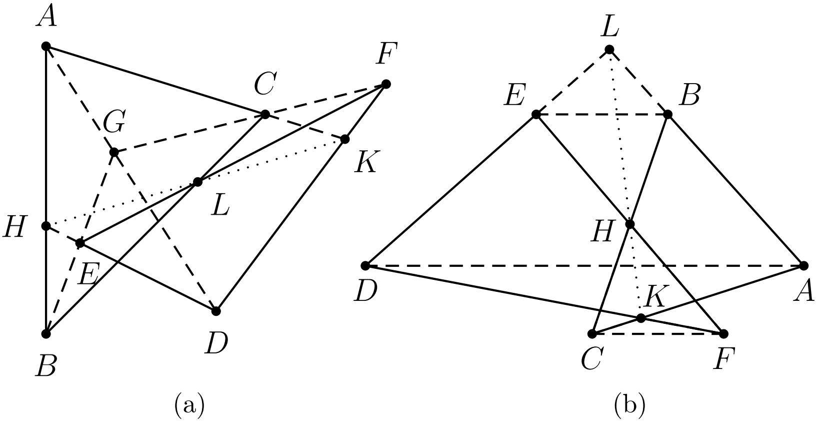


Figure 11. Desargues's Theorem, intersecting cases

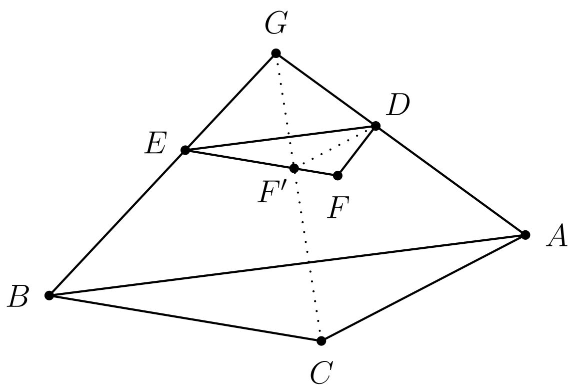


Figure 12. Converse of Thales's Theorem

For, if $BC \parallel EF$, but F is not on GC, this intersects EF at some other point F'. Then Thales's Theorem applies, yielding $DF' \parallel AC$. Thus F' lies on the straight line through D that is parallel to AC. This line being DF, F' must lie on this, as well as on EF. Only one point can do this, and that point is F. So F' and F must be the same point after all, and F lies on GC.

Next we establish a converse of the Prism Theorem. In Fig. 13b, $ABED$ is a parallelogram and $AC \parallel DF$. If $CF \nparallel AD$, then they meet at a point G. Since E does not lie on BG, we conclude by the converse of Thales's Theorem that $BC \nparallel EF$. By contraposition, the converse of (8) holds.

Finally we prove the Prism Theorem itself in an affine plane. In Fig. 13c, $ABED$ is a parallelogram and $AC \parallel DF$. If $BC \nparallel EF$, then $BC \parallel EF'$ for some F' on DF, and therefore $AD \parallel CF'$ by what we have just proved, so $CF \nparallel AD$.

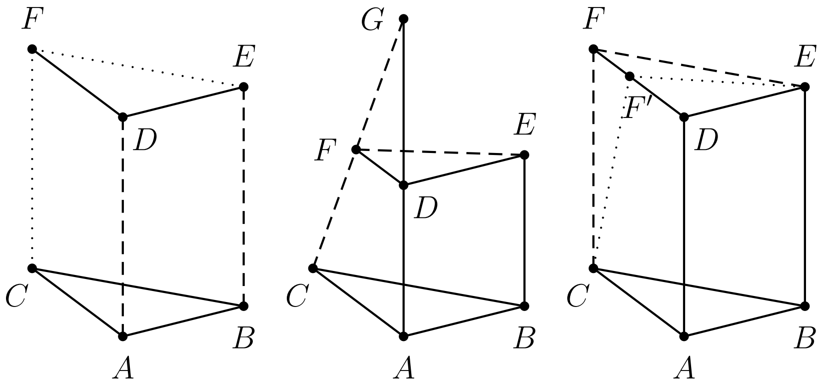


Figure 13. Prism Theorem

1.4.2. Equality in proportions

In an affine plane, if some bounded straight line appears in a proportion, we may now replace it with a bounded straight line representing the same vector. For if, as in Figure 14,

$$AC : CB :: AH : HG, \tag{11}$$

so that $HC \parallel GB$, and if all of the straight lines XX' are parallel to one another, and also

$$AB \parallel A'B', \qquad\qquad AG \parallel A'G',$$

so that the XX' all represent the same vector, then

$$H'C' \parallel HC \parallel GB \parallel G'B',$$

so

$$A'C' : C'B' :: A'H' : H'G'.$$

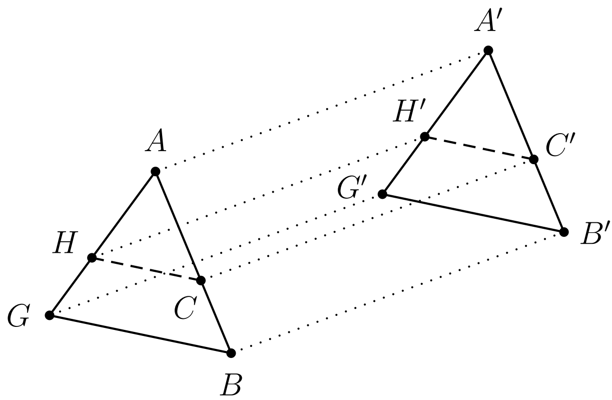


Figure 14. Proportions from the Prism Theorem

Therefore, in (11), we can replace any particular XY with $X'Y'$. Thus, back in Fig. 5, if $DF \parallel AC$, then

$$BD : DA :: BF : FC :: BF : DE.$$

We can conclude from this, as an alternative form of the Proportion Theorem, still in Fig. 5,

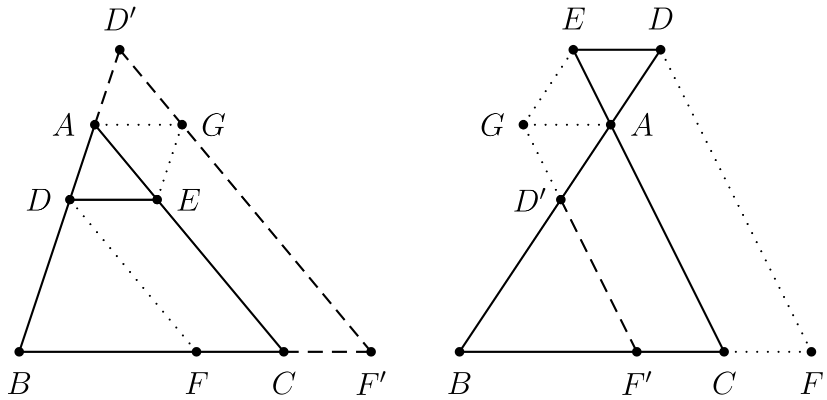
$$BA : DA :: BC : DE.$$

For, if we augment Fig. 5 as in Fig. 15, where now $ADEG$ is a parallelogram, and the straight line through G parallel to AC cuts AB and AC at D' and F' respectively, then

$$BA : AD' :: BC : CF',$$

but also

$$AD' = DA, \qquad CF' = FC = DE.$$



(a) Composition (b) Separation

Figure 15. Composition and separation of ratios

We can therefore write the rule (6) for multiplication of ratios as

$$(OA : OA')(OA' : OA'') :: OA : OA''.$$

This becomes more succinct when we write a for OA, and so forth:

$$(a : a')(a' : a'') :: a : a''.$$

1.5. Pappus's Theorem

We shall prove Desargues's Theorem in general in §6.3 (p. 90), by means of **Pappus's Hexagon Theorem.** This concerns a hexagon, such as $ABCDEF$, whose vertices lie alternately on two straight lines. Thus AC contains E, and BD contains F, and the two straight lines either

- have a common point G, or
- are parallel to one another.

The following are the only three possibilities for the pairs $\{AB, DE\}$, $\{BC, EF\}$, and $\{CD, FA\}$ of opposite sides of the hexagon.

Parallelism: each pair are parallel.

Intersecting: each pair intersect, and the three intersection points lie along a common straight line.

Mixing: Two pairs intersect, and the line that the two intersection points determine is parallel to each line in the third pair.

That is Pappus's Theorem, and just as for Desargues's Theorem, there are six cases in all.

Pascal generalized the Hexagon Theorem, though without proof, to allow the vertices of the hexagon to lie on an arbitrary conic section [4, 40].

Pappus himself proves three cases of the Hexagon Theorem:

1) the parallel case with vertices on intersecting lines, as Lemma VIII;
2) the intersecting case with vertices on parallel lines, as Lemma XII;
3) the intersecting case with vertices on intersecting lines, as Lemma XIII.

The proofs of the last two lemmas will use Lemmas III, X, and XI, concerning the *cross ratio* of four straight lines. We shall take up

- Lemma VIII, and the other parallel case, in §2.1 (p. 36);
- Lemmas III and X in §3.4 (p. 53);
- Lemmas XI, XII, and XIII in §4.3 (p. 68);
- the mixed cases in §4.4 (p. 72).

Lemma VIII is illustrated in Figure 16, where

$$BC \parallel EF \ \& \ CD \parallel FA \implies AB \parallel DE.$$

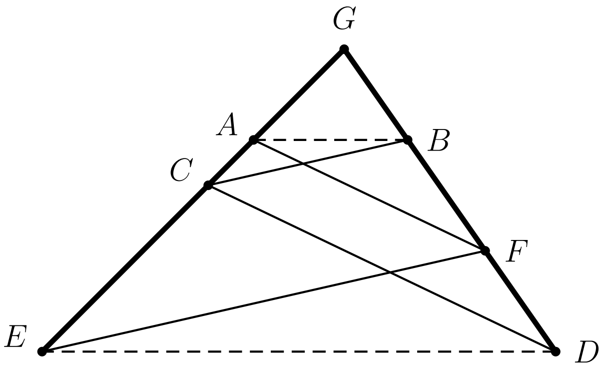


Figure 16. Commutativity of multiplication of ratios

This just means multiplication of ratios is commutative, since we have now

$$\begin{aligned}(GA : AC)(GB : BF) &:: GA : AE\\ &:: GB : BD :: (GB : BF)(GA : AC).\end{aligned}$$

Pappus's proof of Lemma VIII uses only Propositions I.37 and 39 of Euclid's *Elements,* whereby, under the hypothesis that two triangles have the same base, as in Fig. 17, the

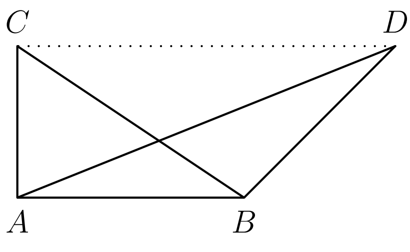


Figure 17. Triangles on the same base

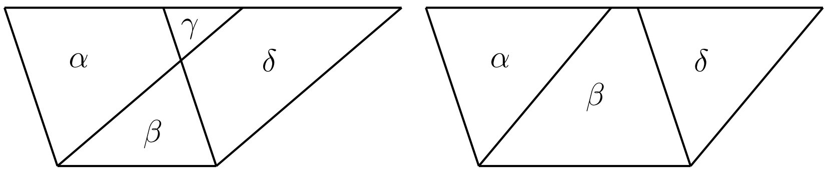
straight line joining the apices of the triangles is parallel to the common base just in case the triangles are *equal* to one another.

1.6. Equality of polygons

Equality of triangles means not congruence of the triangles, but sameness of their areas. We have seen in §1.2 (p. 12) that congruence is *one* way to establish this sameness. Euclid's proof of his Proposition I.35, that *parallelograms* on the same base and in the same parallels are equal, is by

1) adding congruent pieces to the parallelograms, then
2) dividing the resulting polygons into congruent pieces.

Thus in Fig. 18a,



(a) Euclid's case

(b) Simpler case

Figure 18. Euclid's Proposition I.35

$$\alpha + \gamma = \gamma + \delta$$

because the two sums are congruent triangles. The third of the Common Notions of Euclid is, "If equals be subtracted from equals, the remainders are equal"; thus

$$\alpha = \delta.$$

The second of the Common Notions is, "If equals be added to equals, the wholes are equal," and so

$$\alpha + \beta = \beta + \delta,$$

which is the desired equation of parallelograms.

There is a simpler case, in Fig. 18b, where the parallelograms themselves are composed of congruent parts. The same is actually true in Fig. 18a, where we can analyze the parallelograms further as in Fig. 19. Here, by congruence,

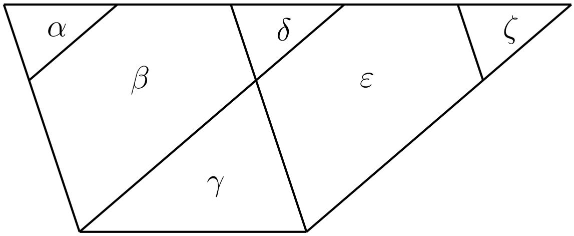


Figure 19. Alternative proof of I.35

$$a = \delta = \zeta, \qquad \beta = \varepsilon,$$

and therefore

$$\alpha + \beta + \gamma = \gamma + \varepsilon + \zeta.$$

However, as Fig. 20 suggests, there is no bound on the number of congruent parts that we may have to analyze the parallelograms into, if we want to avoid *adding* congruent parts.

1.7. Projective plane

The first four books of Euclid's *Elements* concern a **Euclidean plane.** Provided we can, as we shall in §3.3 (p. 51), prove Thales's Theorem using only those books, and not Book VI, which we mentioned in §1.2 (p. 12), and which we shall use in §3.1 (p. 44), a Euclidean plane is a special case of an affine plane, as defined in §1.2 (p. 12). Books XI–XIII of Euclid concern Euclidean space.

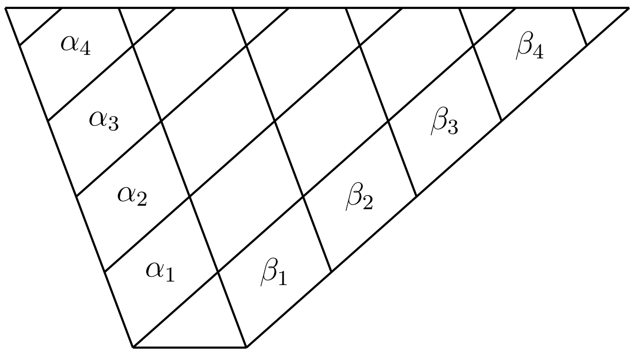


Figure 20. Complications of I.35

We can understand Pappus's geometry as concerning a *projective plane.* Here, for either of Desargues's Theorem and Pappus's Theorem, the six cases can be given a single expression, because formerly parallel straight lines are now allowed to intersect "at infinity." In Chapter 5, we shall obtain a projective plane from an affine plane by adding

1) new points, called **points at infinity,** one for each family of parallel straight lines, to which the point is considered to be common; and
2) a new straight line, the **straight line at infinity,** which is common to all of the points at infinity.

Meanwhile, by definition, a **projective plane** satisfies the following axioms.

1. Any two distinct points lie on a single straight line.
2. Any two distinct straight lines intersect at a single point.
3. There is a complete quadrangle, in the sense of §1.1 (p. 10).

In not every projective plane is Pappus's Theorem true. A **Pappian plane** is a projective plane in which Pappus's Theorem *is* true, in the sense illustrated below by Fig. 46 in §4.3

(p. 68): when the vertices of a hexagon lie alternately on two straight lines, which now necessarily intersect, then the intersection points, which now always exist, of opposite sides lie on a straight line. This is Pappus's Lemma XIII, which Pappus proves for a Euclidean plane.

A Pappian plane has no specified line at infinity. When we remove any straight line and its points, what remains is an affine plane, for which the line removed may be conceived as a line at infinity.

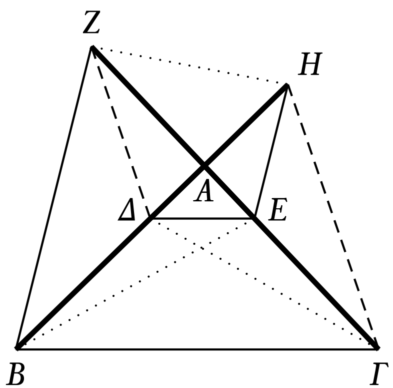
We shall show in §6.3 (p. 90) that Desargues's Theorem, in the projective sense illustrated by Fig. 11a, is true in every Pappian plane.

2. Tuesday

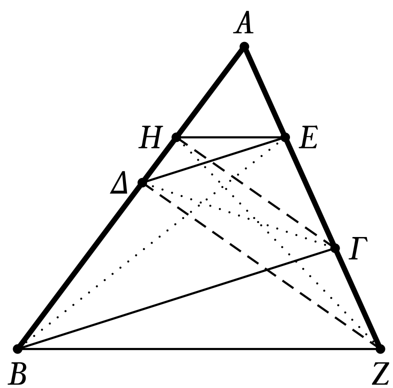
2.1. Pappus's Theorem, parallel cases

In a Euclidean plane, we prove now the parallel cases of Pappus's Theorem, stated in §1.5 (p. 29). One of them, Lemma VIII, will give us commutativity of multiplication of ratios in an affine plane, as we said.

In Lemma VIII, two pairs of opposite sides of the hexagon are parallel, and the two bounding lines intersect. In particular, letting the hexagon be $B\Gamma HE\Delta Z$ in Fig. 21a (which is close



(a) Pappus's figure



(b) Alternative figure

Figure 21. Lemma VIII

to the figure in Hultsch's text of Pappus [25], except that the lines there all look the same), we suppose

$$B\Gamma \parallel \Delta E, \qquad\qquad HE \parallel ZB.$$

We prove $\Gamma H \parallel \Delta Z$ using several equations of triangles:

1)	$\Delta BE = \Delta\Gamma E$,	[*Elements* I.37, since $B\Gamma \parallel \Delta E$]
2)	$ABE = \Gamma\Delta A$,	[add ΔAE]
3)	$BZE = BZH$,	[*Elements* I.37, since $BZ \parallel EH$]
4)	$ABE = AHZ$,	[subtract ABZ]
5)	$A\Gamma\Delta = AHZ$,	[steps 2 and 4]
6)	$\Gamma\Delta H = \Gamma ZH$,	[add $A\Gamma H$]
7)	$\Gamma H \parallel \Delta Z$.	[*Elements* I.39]

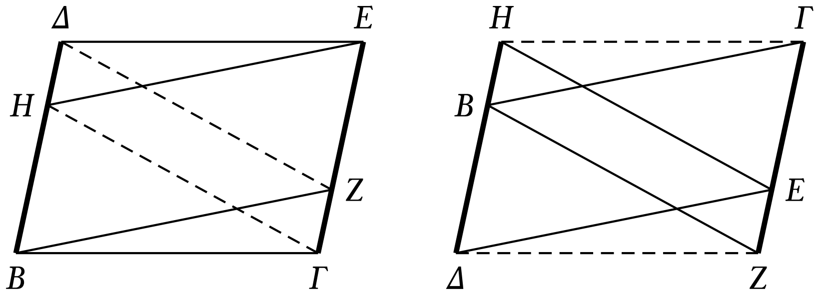
If the diagram is as in Fig. 21b, then we must adjust the proof by *subtracting* BZE and BZH from ABZ in step 4, and subtracting $A\Gamma H$ from $A\Gamma\Delta$ and AHZ in step 6. Note then that the proof does not make sense in an abstract affine plane, where there is no ordering of points on a straight line.

Possibly the intersection point A does not exist, because $B\Delta \parallel \Gamma E$. This is the situation of Fig. 22, where, by the Equality Theorem of §1.4 (p. 24), being opposite sides of parallelograms,

$$B\Delta = \Gamma E, \qquad\qquad BH = ZE.$$

Therefore

- the differences are equal in Fig. 22a, by the third common notion, mentioned in §1.6 (p. 32);
- the sums are equal in Fig. 22b, by the second common notion.



(a) (b)

Figure 22. Parallel Theorem

That is,

$$\Delta H = \Gamma Z.$$

These too must be the opposite sides of a parallelogram, by the Equality Theorem again; in particular,

$$.\Gamma H \parallel Z\Delta. \tag{12}$$

Let us call this case of Pappus's Theorem the **Parallel Theorem.** We shall use it in §4.1 (p. 58). Again the proof is in a Euclidean plane; but now there is a proof for an affine plane. In Fig. 23, the triangles $B\Gamma Z$ and $E\Delta H$ having corresponding sides parallel, the straight lines BE, $\Gamma\Delta$, and ZH must have a common point A, by the converse of Thales's Theorem. Applied then to the triangles $B\Gamma H$ and $E\Delta Z$, Thales's Theorem yields (12).

2.2. Quadrangle Theorem proved by Pappus

Pappus's **Lemma IV** is that, in Fig. 24, where the solid lines

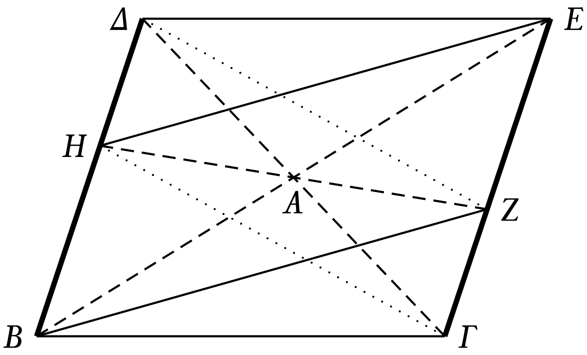


Figure 23. Parallel Theorem in an affine plane

are straight, if the proportion

$$(AZ : AB)(B\Gamma : \Gamma Z) :: (AZ : A\Delta)(\Delta E : EZ) \tag{13}$$

holds for the points A, B, Γ, Δ, E, and Z one one of the straight lines, then Θ, H, and Z are in a straight line. Proving this will involve various manipulations. Pappus writes the products of ratios in (13) as ratios of products:

$$AZ \cdot B\Gamma : AB \cdot \Gamma Z :: AZ \cdot \Delta E : A\Delta \cdot EZ. \tag{14}$$

Now we apply **alternation,** which is the rule

$$a : b :: c : d \implies a : c :: b : d. \tag{15}$$

We prove this using commutativity of multiplication. From the hypothesis $a : b :: c : d$, we compute

$$a : c :: (a : b)(b : c) :: (c : d)(b : c) :: (b : c)(c : d) :: b : d.$$

Now (14) is equivalent to

$$AZ \cdot B\Gamma : AZ \cdot \Delta E :: AB \cdot \Gamma Z : A\Delta \cdot EZ. \tag{16}$$

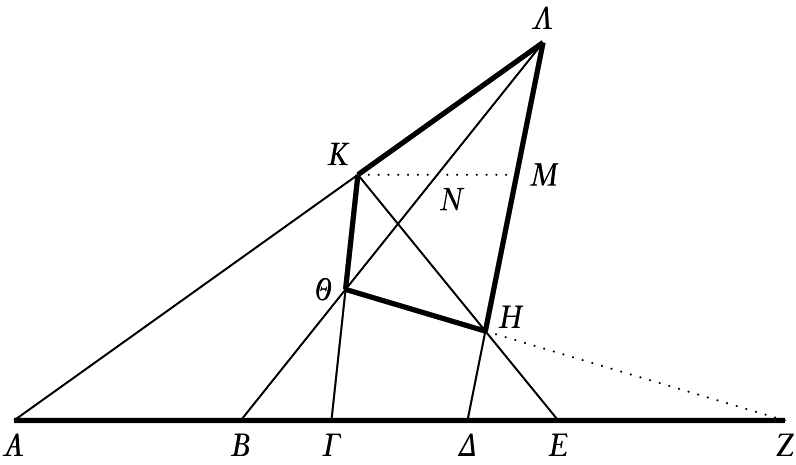


Figure 24. Lemma IV

The left-hand member simplifies, and then we expand it as a product:

$$\begin{aligned} AZ \cdot B\Gamma : AZ \cdot \Delta E :: B\Gamma : \Delta E \\ :: (B\Gamma : NK)(NK : KM)(KM : \Delta E). \quad (17) \end{aligned}$$

Pappus has KN for NK here, and similar variants elsewhere. He analyzes the right-hand member of (16) as a product of ratios:

$$AB \cdot \Gamma Z : A\Delta \cdot EZ :: (BA : A\Delta)(\Gamma Z : ZE). \qquad (18)$$

Assuming KM is drawn parallel to AZ, by Thales's Theorem we have

$$NK : KM :: BA : A\Delta.$$

Eliminating this common ratio from the members of (16) given in (17) and (18), then reversing the order of the new members, we obtain

$$\Gamma Z : ZE :: (B\Gamma : NK)(KM : \Delta E),$$

and therefore, by Thales's Theorem applied to each ratio in the compound,

$$\mathit{\Gamma Z} : \mathit{ZE} :: (\mathit{\Theta\Gamma} : \mathit{\Theta K})(\mathit{KH} : \mathit{HE}). \tag{19}$$

Pappus says now that $\mathit{\Theta HZ}$ is indeed straight. Although he provides a reminder, he may expect his readers to know, as some students today know from high school, the generalization of Thales's Theorem known as **Menelaus's Theorem** whose diagram is in Fig. 25.[1] Rewriting (19), we have the hypothesis

$$\mathit{\Gamma Z} : \mathit{ZE} :: (\mathit{\Gamma\Theta} : \mathit{\Theta K})(\mathit{KH} : \mathit{HE}). \tag{20}$$

We extend $\mathit{\Theta H}$ and let it be met at $\mathit{\Xi}$ by the parallel to $\mathit{\Gamma K}$ through E. By Thales's Theorem then, from (20) we have

$$\mathit{\Gamma Z} : \mathit{ZE} :: (\mathit{\Theta\Gamma} : \mathit{K\Theta})(\mathit{K\Theta} : \mathit{E\Xi}) :: \mathit{\Theta\Gamma} : \mathit{E\Xi}.$$

By the same theorem in the other direction, the points $\mathit{\Theta}$, $\mathit{\Xi}$, and Z must be collinear, and therefore the same is true for $\mathit{\Theta}$, H, and Z. This completes the proofs of

[1] Menelaus's *Sphaerica* survives in Arabic translation [17, p. 261]; but we also have Menelaus's Theorem in Ptolemy, where I read it as a student at St John's College, just before Toomer's 1984 translation [39] came out; we used the translation that Taliaferro had made for the College [38, I.13, p. 26]. Thomas also puts Menelaus's Theorem in his anthology [41, pp. 445 ff.]. In the commentary for their translation of Desargues, Field and Gray remark that Pappus's Lemma IV is proved by "chasing ratios much in the fashion Desargues was later to use. In this case collinearity could have been established by appealing to the converse of Menelaus' theorem, but when Pappus reached that point he missed that trick and continued to chase ratios until the conclusion was established—in effect, proving the converse of Menelaus' theorem without saying so" [15, pp. 10–1]. I would add that the similarity of "fashion" in Pappus and Desargues is probably due to the latter's having studied the former. Moreover, Pappus seems *not* to have "missed the trick," since he asserts the desired collinearity at a point when it can be recognized only by somebody who knows Menelaus's Theorem.

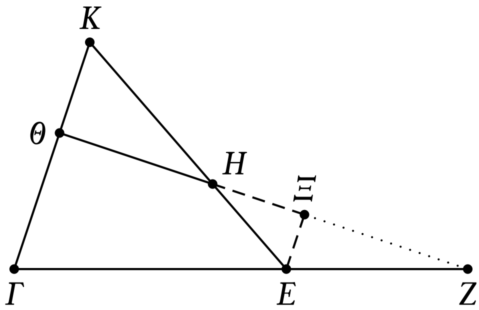


Figure 25. Menelaus's Theorem

- one direction of Menelaus's Theorem,
- Lemma VIII.

The steps of the proof are reversible. Thus, if we are given the complete quadrangle ***HΘKΛ*** of Fig. 24 and the points ***A***, ***B***, ***Γ***, ***Δ***, ***E***, and ***Z*** where its sides cross a given straight line, the proportion (13) must be satisfied. Therefore if five sides of another complete quadrangle, as ***ΠPΣT*** in Fig. 26, should pass through the points ***A***, ***B***, ***Γ***, ***Δ***, and ***E***, then the sixth side would pass through ***Z***. This is the **Quadrangle Theorem.**

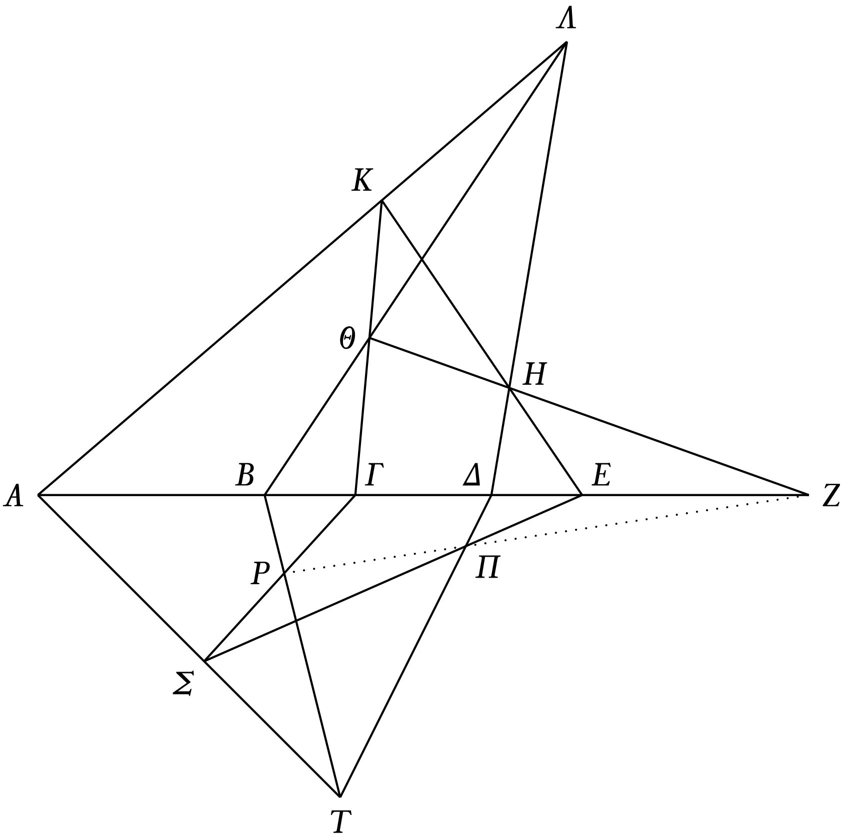


Figure 26. Quadrangle Theorem

3. Wednesday

3.1. Thales's Theorem proved by Euclid

Proposition VI.1 of Euclid's *Elements* is that triangles and parallelograms under the same height are to one another in the ratio of their bases. Thus, in Fig. 27,

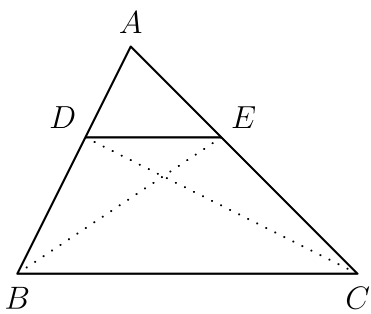


Figure 27. Thales's Theorem

$$\left.\begin{aligned} AD : DB &:: ADE : DBE, \\ AE : EC &:: ADE : EDC. \end{aligned}\right\} \tag{21}$$

By V.7 and 8, the ratios of ADE to DBE and EDC are the same, just in case these two triangles are equal in the sense of §1.6 (p. 32); symbolically,

$$ADE : DBE :: ADE : EDC \iff DBE = EDC. \tag{22}$$

Finally, by I.35 and 37, as discussed in §1.6 (p. 32),

$$DBE = EDC \iff DE \parallel BC. \tag{23}$$

Combining (21), (22), and (23), using the transitivity of sameness of ratio, we conclude

$$AD : DB :: AE : EC \iff DBE = EDC.$$

That is Euclid's proof of Thales's Theorem, or more precisely the Proportion Theorem, which is Proposition VI.2 of the *Elements*, as discusses in §1.2 (p. 12). The proof takes place in a Euclidean plane, in the sense of §1.7 (p. 33), but also using a theory of proportion. We have developed such a theory only in an affine plane, in the sense of §1.2 (p. 12).

3.2. Proportion in a Euclidean plane

For Euclid, a proportion is a relation of four **magnitudes.** Magnitudes come in three kinds: lines, surfaces, or solids (all bounded). The magnitudes in the proportion are considered in two pairs, the magnitudes in each pair **having a ratio** to one another; and these ratios composing the proportion are the same. Only magnitudes of the same kind can have a ratio.

In any proportion, we expect to be able to replace a magnitude by an equal magnitude in the sense of §1.6 (p. 32). We may then confuse a magnitude with its **size**—its length, area, or volume—, understood as the class of magnitudes equal to the original one.

Given three lengths a, b, and c, we can form the products

$$ab, \qquad\qquad abc,$$

which are respectively

- the area of a rectangle having dimensions a and b,
- the volume of a rectangular parallelepiped having dimensions a, b, and c.

The order in which dimensions are given is irrelevant. Suppose we have

$$ay = bx, \qquad cy = dx. \tag{24}$$

Then

$$bcx = acy = adx. \tag{25}$$

The fifth of Euclid's Common Notions is, "the whole is greater than the part." If $bc \neq ad$, then we may assume $bc < ad$, and thus bc is the area of a part of a surface that has area ad. In this case, bcx is the volume of a part of a solid having volume adx; in particular, (25) fails. Thus from (25), and therefore from (24), we conclude

$$bc = ad. \tag{26}$$

As a consequence, the relation of ratios of segments given by the rule

$$AB : CD :: EF : GH \iff |AB| \cdot |GH| = |CD| \cdot |EF| \tag{27}$$

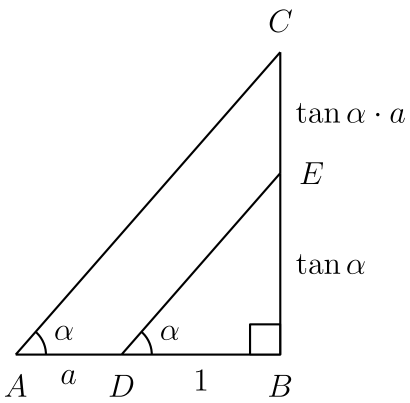
is transitive. It now makes some sense to take (27) for a definition of proportion of lengths. A convenience of this definition is that alternation, as in (15), is immediate. There remain two problems, one more serious than the other.

1. As observed in §1.2 (p. 12), one way to read the proportion in (27) is as "AB has the same ratio to CD that EF has to GH"; and as argued in §1.3 (p. 17), transitivity of sameness of ratio ought to be obvious, not needing such a proof as we have given.

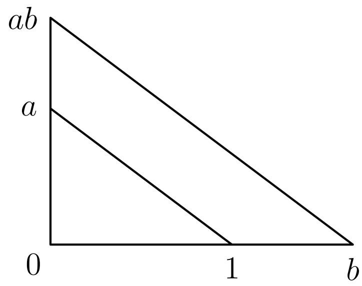
2. Euclid's proof of Thales's Theorem uses also ratios of *areas.*

3.2.1. Hilbert's definition

We can avoid these problems by using the method of Descartes [8, 10] and fixing a unit length, so that we can obtain products of lengths as lengths. However, Descartes assumes Euclid's theory of proportion to begin with. Hilbert does not, but works only in a Euclidean plane. Following Hilbert's idea [19, pp. 29–32], in Fig. 28a, we suppose DB has unit length. If the



(a)



(b) Hilbert's diagram

Figure 28. Hilbert's multiplication

size of angle EDB is α, then by definition

$$|BE| = \tan\alpha.$$

Thus every length is the tangent of the size of *some* angle. If $AC \parallel DE$, we define

$$|BC| = \tan\alpha \cdot |AB|.$$

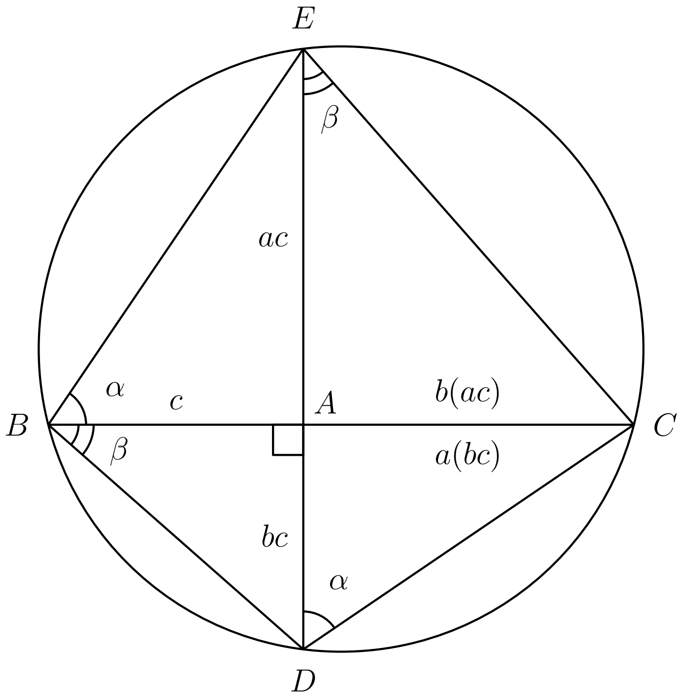


Figure 29. Associativity and commutativity

Hence also

$$|EC| = \tan\alpha \cdot |AD|.$$

Hilbert does not mention tangents of angles, but just defines multiplication using a figure like 28b.

To prove that this multiplication is associative and commutative, Hartshorne, in *Geometry: Euclid and Beyond* [16, pp. 168–72], presents the streamlined method found in later editions of Hilbert [20, pp. 203–6] and attributed to Enriques. In Fig. 29, using that, “In a circle, the angles in the same segment

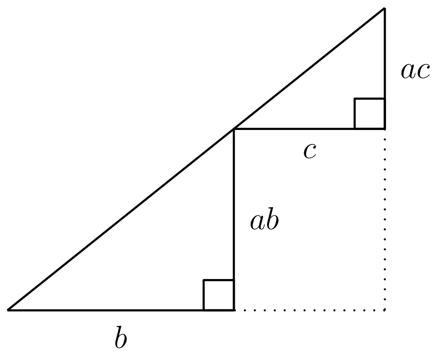


Figure 30. Distributivity

are equal to one another” (*Elements* III.21), if

$$\tan\alpha = a, \qquad\qquad \tan\beta = b,$$

then AC has the two lengths indicated, so these are the same; that is,

$$a(bc) = b(ac). \tag{28}$$

Letting $c = 1$ gives commutativity; then this with (28) gives

$$a(cb) = (ac)b$$

and thus associativity. It is clear how to add lengths, but see also §4.1 (p. 58). Distributivity of multiplication over addition follows from Fig. 30, where

$$ab + ac = a(b + c).$$

Every product of lengths is now a length. Moreover, with respect to this multiplication, every length has an inverse: in Fig. 28b, we can let $ab = 1$. Thus multiplication admits cancellation:

$$ax = bx \implies a = b.$$

Finally,

$$a < b \implies ax < bx.$$

In algebraic terms, lengths are now the positive elements of an ordered field $\mathbb{F}$. We obtain the whole ordered field $\mathbb{F}$ by selecting points O and U on an infinite straight line that are the unit distance apart; then the points on the line correspond to all of the elements of $\mathbb{F}$, with O as 0 and U as 1. We can now define the ratio $AB : CD$ to be the quotient

$$\frac{|AB|}{|CD|},$$

in $\mathbb{F}$, of the lengths of AB and CD. This makes

- sameness of ratio transitive;
- alternation, (15), an easy theorem;
- (27) also an easy theorem.

Now we can *state* the Proportion Theorem, which implies Thales's Theorem.

3.2.2. Euclid's definition

It remains to *prove* the Proportion Theorem, and thus Thales's Theorem, under Hilbert's definition of ratios. Meanwhile, let us note that, for Euclid, the ratio $a : b$ is effectively what we now call a **Dedekind cut,** because of its use in Dedekind's development of the real numbers [7]. We can understand a Dedekind cut as a partition of the positive rational numbers into two nonempty sets A and B, where every element of A is less than every element of B. In the cut corresponding to $a : b$,

$$A = \left\{ \frac{x}{y} : bx \leqslant ay \right\}, \qquad B = \left\{ \frac{x}{y} : bx > ay \right\}.$$

Here x and y are counting numbers. We can replace $\leqslant$ with $<$, and $>$ with $\geqslant$: this moves at most one element from A to B. We assume that A and B are indeed both nonempty, when a and b are magnitudes of the same kind: this is the **Archimedean Axiom.** In this case, the ordered pair (a, b) defines the given cut; Dedekind's innovation was to recognize that he could define cuts without reference to magnitudes.

3.3. Thales's Theorem proved by Hilbert

As Hilbert shows, we can prove the Proportion Theorem, under the definition of ratios as quotients of lengths, without using the Archimedean Axiom. Given triangles ABC and DEF that are **similar,** in the sense that the angles at A and B are respectively equal to the angles at D and E, and therefore also the angles at C and F are equal by Euclid's I.32, we want to show

$$AB : DE :: AC : DF. \tag{29}$$

If the angles at B and E are right angles, then (29) follows from our definition of multiplication of lengths. In the general case, we can arrange the triangles as in Fig. 31, so that their corresponding sides are parallel and their incenters coincide at a point I. Thus AI and BI bisect the angles at A and B, and therefore CI does the same for the angle at C. Then because of the right angles,

$$ar' = a'r, \qquad br' = b'r, \qquad cr' = c'r,$$

so that

$$(a + b)r' = ar' + br' = a'r + b'r = (a' + b')r,$$

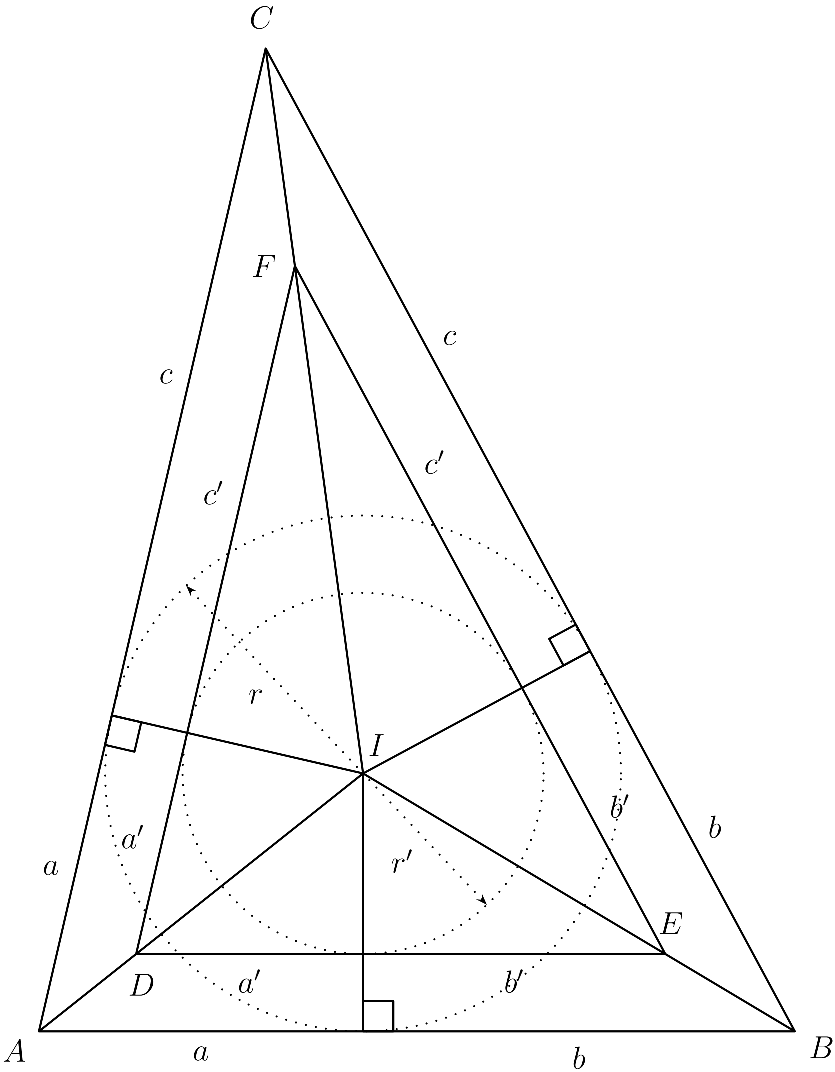


Figure 31. Proof of Thales's Theorem

and likewise

$$(a+c)r' = (a'+c')r.$$

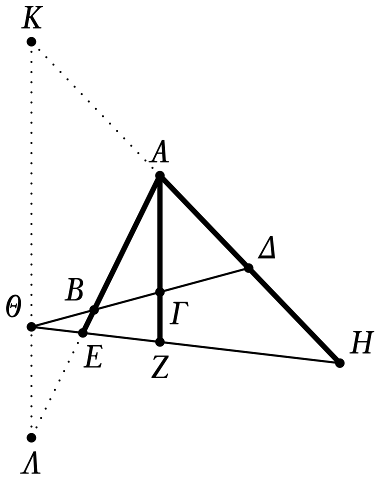
Therefore

$$\frac{a+b}{a'+b'} = \frac{r}{r'} = \frac{a+c}{a'+c'},$$

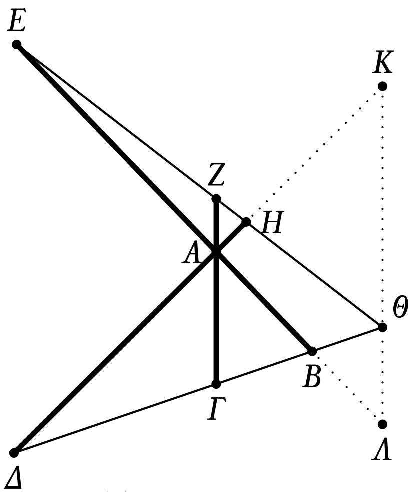
which yields (29).

3.4. Cross ratio

In Pappus's **Lemma III,** straight lines $\Theta\Delta$ and ΘH cut the straight lines AB, $A\Gamma$, and $A\Delta$ as in Fig. 32. We are going to



(a) Pappus's figure



(b) alternative figure

Figure 32. Lemma III

show

$$(\Theta E : EZ)(ZH : H\Theta) :: (\Theta B : B\Gamma)(\Gamma\Delta : \Delta\Theta).$$

We complete the diagram by making

$$K\Lambda \parallel AZ, \qquad \Lambda M \parallel AH.$$

By Thales,

$$(\Theta E : EZ)(ZH : H\Theta) :: (\Theta\Lambda : AZ)(AZ : \Theta K) :: \Theta\Lambda : \Theta K. \tag{30}$$

Since the last ratio is independent of the choice of H along $A\Delta$, we are done.

In (30), the first product of ratios, written also as

$$\Theta E \cdot ZH : EZ \cdot H\Theta,$$

and which we shall denote by

$$[\Theta, E, Z, H],$$

is the **cross ratio** (*çapraz oran*) of the ordered quadruple (Θ, E, Z, H). Some permutations of the points do not change the cross ratio. For example, we can reverse their order:

$$\begin{aligned}[H, Z, E, \Theta] &:: HZ \cdot E\Theta : ZE \cdot \Theta H \\ &:: \Theta E \cdot ZH : EZ \cdot H\Theta :: [\Theta, E, Z, H].\end{aligned}$$

Thus, by Lemma III, if four straight lines in a plane intersect at a point, then the cross ratio of the four points where some other straight line crosses the lines is always the same; we can see this using Fig. 33.

Lemma X is a converse to Lemma III. The hypothesis is that, in Fig. 34,

$$[\Theta, H, Z, E] :: [\Theta, \Delta, \Gamma, B]. \tag{31}$$

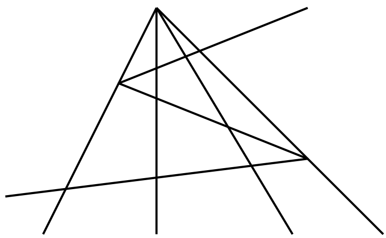


Figure 33. Invariance of cross ratio

We shall show that $\mathit{\Gamma}$, A, and Z are collinear. Through $\mathit{\Theta}$ we draw a line parallel to $\mathit{\Gamma}A$, and this line is cut at $\mathit{\Lambda}$ and K by the extensions of EB and $\mathit{\Delta}H$. We let $\mathit{\Lambda}M$ be parallel to $\mathit{\Delta}H$ and meet the extension of $E\mathit{\Theta}$ at M. We compute

$$\begin{aligned}[\mathit{\Theta},\mathit{\Delta},\mathit{\Gamma},B] &:: \mathit{\Theta}K : \mathit{\Theta}\mathit{\Lambda} && \text{[Lemma III proof]}\\ &:: \mathit{\Theta}H : \mathit{\Theta}M && \text{[Thales]}\\ &:: \mathit{\Theta}H \cdot EZ : \mathit{\Theta}M \cdot EZ.\end{aligned}$$

By definition,

$$[\mathit{\Theta},H,Z,E] :: \mathit{\Theta}H \cdot ZE : HZ \cdot E\mathit{\Theta}.$$

Our hypothesis (31) then yields

$$\begin{aligned} ZH \cdot E\mathit{\Theta} &= \mathit{\Theta}M \cdot EZ, &&\\ E\mathit{\Theta} : \mathit{\Theta}M &:: EZ : ZH, &&\\ E\mathit{\Theta} : EM &:: EZ : EH, && [\textit{componendo}]\\ E\mathit{\Theta} : EZ &:: EM : EH && \text{[alternation]}\\ &:: E\mathit{\Lambda} : EA. && \text{[Thales]}\end{aligned}$$

By the converse of Thales's Theorem,

$$AZ \parallel \mathit{\Lambda}\mathit{\Theta},$$

and this yields the claim.

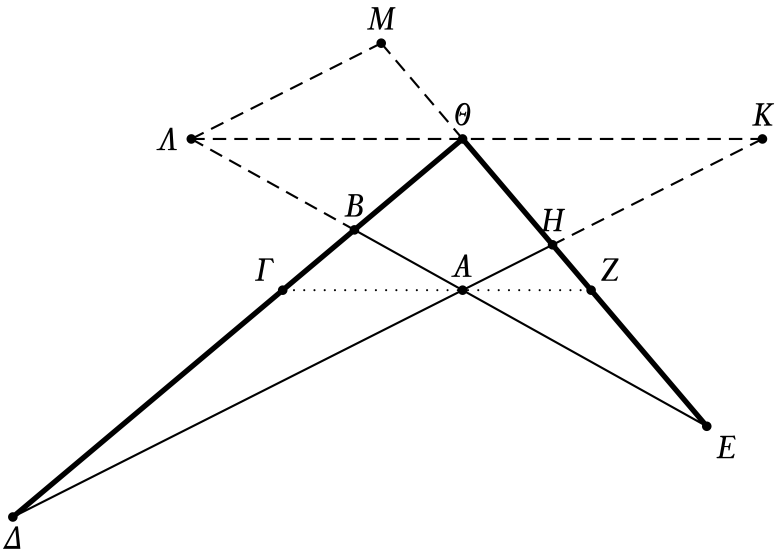


Figure 34. Lemma X

4. Friday

In Euclid, a **bounded straight line** (*sınırlanmış doğru çizgi*) is called more simply a **straight line** (*doğru çizgi*), and more simply still, a **"straight"** (*doğru*). In English, this is usually called a **line segment** (*doğru parçası*), although for Euclid, and sometimes in English too, a **line** (*çizgi*) may be curved. For example, a circle is a certain kind of line. For us though, henceforth lines will always be straight and unbounded.

In §3.2 (p. 45), we saw how to understand the points of a line in a Euclidean plane as the elements of a commutative field. Now we are going to do the same thing in an arbitrary affine plane, except that the field may not be commutative: it may be a **skew field,** usually called today a non-commutative division ring. To ensure that the field of ratios *is* commutative, to the axioms for an affine plane we can add one axiom:

5. Pappus's Lemma VIII, proved in §2.1 (p. 36) on the basis of Book I of the *Elements.*

The commutative field in this case still need not be ordered.

Here is where we are going:

- Today, after obtaining the field of ratios, we shall independently prove the remaining cases of Pappus's Theorem, stated in §1.5 (p. 29), of which Lemma VIII is a special case.
- Tomorrow we shall look at geometry over fields, possibly skew.
- The day after that, we shall prove Desargues's Theorem in general, in §6.3 (p. 90).

4.1. Arithmetic in an affine plane

4.1.1. Multiplication

We have seen in §1.2 (p. 12) and §1.4 (p. 24) that we can define ratios in an affine plane to meet the following conditions.

1. In Fig. 35a,

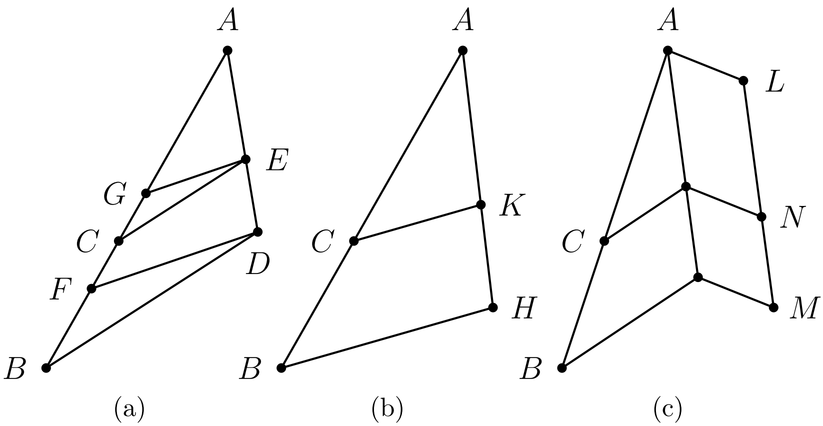


Figure 35. Ratios

a) if C is on AB, and E is on AD, but D is not on AB, then

$$AC : AB :: AE : AD \iff CE \parallel BD;$$

b) if also F and G lie on AB, then

$$AC : AB :: AE : AD \ \& \ AE : AD :: AG : AF \\ \implies AC : AB :: AE : AD.$$

2. For any ratio $AC{:}AB$ and any point H, there is a unique point K such that, as in Fig. 35b,

$$AK : AH :: AC : AB.$$

3. For any ratio $AC : AB$ and any points L and M, there is a unique point N such that, as in Fig. 35c,

$$LN : LM :: AC : AB.$$

Given a ratio r and two points A and B, we can now define

$$r \cdot_A B = C,$$

where C lies on AB and

$$AC : AB :: r.$$

If s is another ratio, we have defined the product sr; associativity of the multiplication here gives us

$$(sr) \cdot_A B = s \cdot_A (r \cdot_A B).$$

4.1.2. Addition

With respect to a point A, we can form the sum of two points B and C by completing the parallelogram, if there is one. That is, assuming C does not lie on AB, we define

$$B +_A C = D,$$

where $ABDC$ is a parallelogram as in Fig. 36. This means

$$AB \parallel CD, \qquad\qquad AC \parallel BD.$$

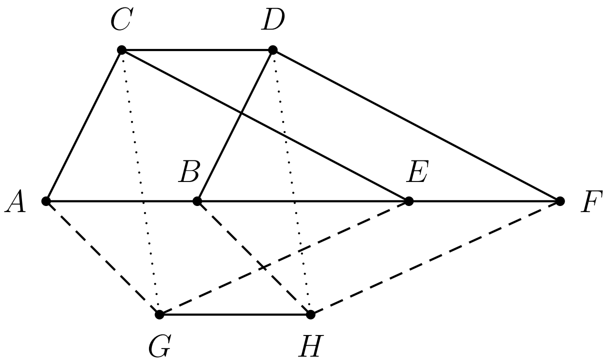


Figure 36. Addition

This definition of the sum is symmetric in B and C. If however E lies on AB, then, using the point C, which does not lie on AB, and using the sum $B +_A C$, which is D, we define

$$E +_A B = F,$$

where, as in Fig. 36 again,

$$F = E +_C D.$$

Thus

$$E +_A B = E +_C (B +_A C).$$

This definition is independent of the choice of C, by the Prism Theorem. Indeed, if also G does not lie on AB, and

$$B +_A G = H,$$

so that, as in Fig. 36,

$$CD \parallel AB \parallel GH, \qquad AC \parallel BD, \qquad AG \parallel BH,$$

then

$$CG \parallel DH.$$

Likewise, since also $CE \parallel DF$, we have $GE \parallel HF$.

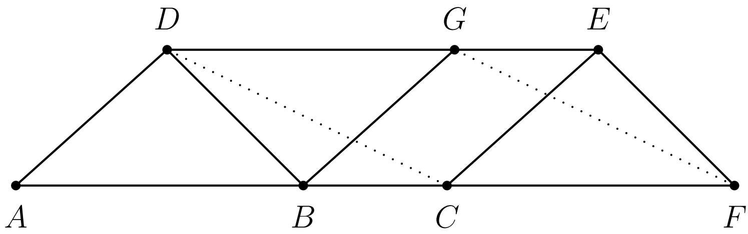


Figure 37. Commutativity of addition

4.1.2.1. Commutativity

By the symmetry of the definition, as we have noted, addition of points not collinear with the reference point is commutative. For points that *are* collinear with the reference point, the definition of their sum is not symmetric, but commutativity is equivalent to the case of Pappus's Theorem called the Parallel Theorem, and proved for affine planes, in §2.1 (p. 36). In Fig. 37 now,

$$\begin{gathered} B +_A C = B +_D (D +_A C) = B +_D E = F, \\ C +_A B = C +_D (D +_A B) = C +_D G, \end{gathered}$$

and therefore

$$B +_A C = C +_A B \iff DC \parallel GF.$$

We have the last parallelism by the Parallel Theorem, applied to the hexagon $DBGFEC$, since

$$BF \parallel DE, \qquad DB \parallel FE, \qquad BG \parallel EC.$$

4.1.2.2. Associativity

To prove associativity of addition, we have three cases to consider.

1. When A is not collinear with any two of B, C, and D, as in Fig. 38, then

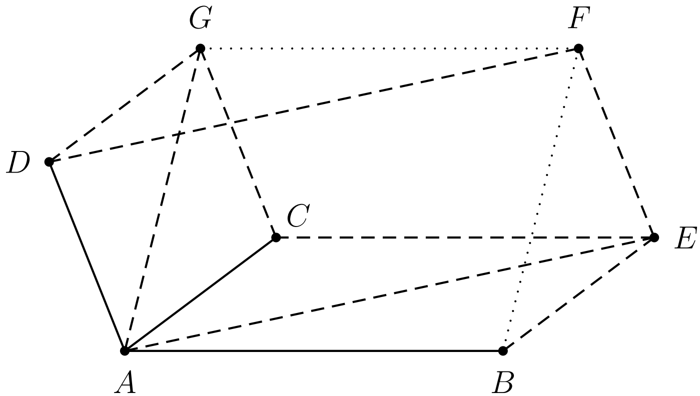


Figure 38. Associativity of addition: easy case

$$B +_A C = E, \qquad E +_A D = F, \qquad C +_A D = G,$$

and then

$$B +_A G = F \iff AG \parallel BF \ \& \ GF \parallel AB.$$

They *are* parallel, by the Prism Theorem. Indeed,

$$CG \parallel AD \parallel EF, \qquad AC \parallel DG, \qquad AE \parallel DF,$$

and therefore $CE \parallel GF$; but also $AB \parallel CE$, so

$$AB \parallel GF.$$

Now we can derive $AG \parallel BF$ similarly. Thus

$$(B +_A C) +_A D = B +_A (C +_A D). \tag{32}$$

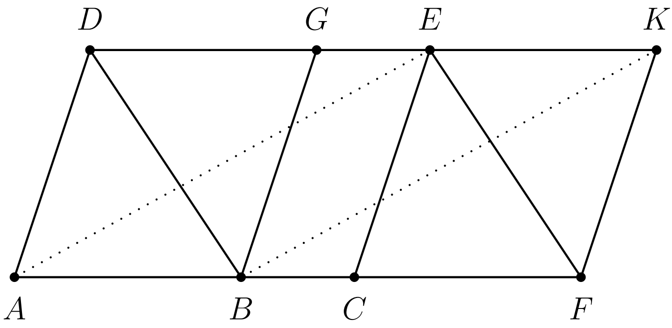


Figure 39. Associativity of addition: less easy case

2. When AB contains C, but not D, then (32) still holds, since in Fig. 39,

$$(B +_A C) = F, \qquad F +_A D = K, \qquad C +_A D = E,$$

so that

$$K = B +_A E \iff AE \parallel BK.$$

Now apply the Parallel Theorem to $ADBKFE$.

3. Finally, when AB contains both C and D, then, making use of commutativity, in Fig. 40 we have

$$C +_A B = G, \qquad G +_A D = K, \qquad C +_A D = L,$$

and

$$B +_A L = K \iff EB \parallel MK.$$

By the Parallel Theorem applied to $BFGMLH$,

$$BH \parallel GM;$$

then, applied to $KHBEGM$, $BE \parallel MK$.

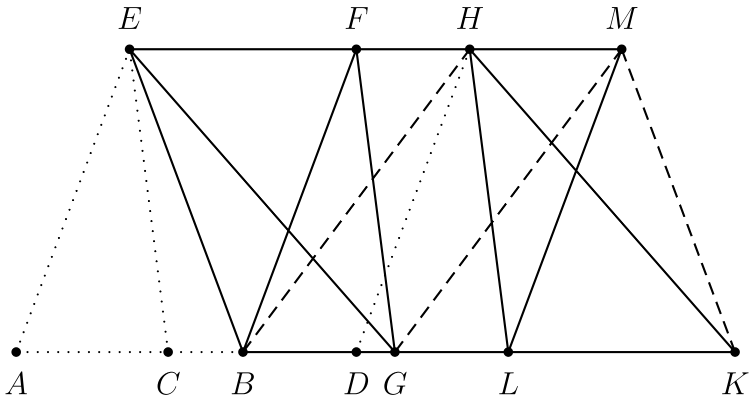


Figure 40. Associativity of addition: hardest case

4.1.2.3. Negatives

Having defined vectors as in §1.3 (p. 17), we can define

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AE},$$

where, as in Fig. 41,

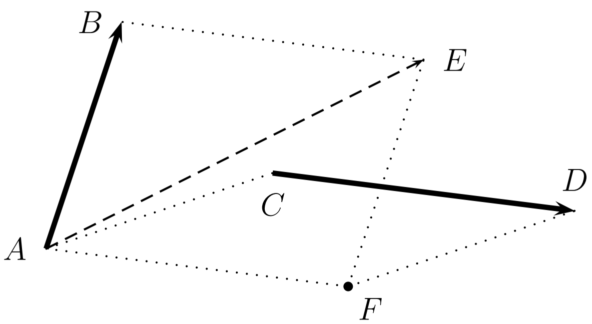


Figure 41. Addition of vectors

$$E = B +_A F, \qquad\qquad F = (A +_C D)$$

By what we have shown, the vectors compose an abelian group with respect to this addition. In particular, the 0 of the group is $\overrightarrow{AA}$, and

$$-\overrightarrow{AB} = \overrightarrow{BA}.$$

4.1.3. Distributivity

By Thales's Theorem,

$$r \cdot_A (B +_A C) = r \cdot_A B +_A r \cdot_A C.$$

For any additional ratio s, there is a ratio t such that, for all A and B,

$$r \cdot_A B + s \cdot_A B = t \cdot_A B.$$

Then we can define

$$r + s = t.$$

We can also define

$$r \cdot \overrightarrow{AB} = r \cdot_A B.$$

We are not writing out all details of the proofs here. Ratios now compose a field, possibly a skew field, and the vectors themselves compose a vector space over this field.

4.2. A skew field

An *example* of a skew field is the field $\mathbb{H}$ of **quaternions,** discovered by Hamilton. We can obtain this field from the field $\mathbb{C}$ of complex numbers as we can obtain $\mathbb{C}$ from $\mathbb{R}$.

4.2.1. Complex numbers

We define
$$\mathbb{C} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : (x, y) \in \mathbb{R}^2 \right\}.$$
This is a subspace of the space of all 2×2 matrices over $\mathbb{R}$. Using the abbreviations
$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = x, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{i},$$
we have
$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} = x + y\mathbf{i}.$$
In particular, $(1, \mathbf{i})$ is a basis of $\mathbb{C}$ over $\mathbb{R}$. Moreover, since
$$\mathbf{i}^2 = -1,$$
$\mathbb{C}$ is closed under multiplication, and multiplication is commutative on $\mathbb{C}$. Finally,
$$(x + y\mathbf{i})(x - y\mathbf{i}) = x^2 + y^2,$$
which is in $\mathbb{R}$, and therefore $\mathbb{C}$ is a field. We define
$$\overline{x + y\mathbf{i}} = x - y\mathbf{i}.$$

4.2.2. Quaternions

By analogy, we define
$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : (z, w) \in \mathbb{C}^2 \right\}.$$

This is *not* a subspace of the space of 2×2 matrices over $\mathbb{C}$. It is however a *real* subspace. Moreover, using the abbreviations

$$\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} = z, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{j},$$

we have

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} + \begin{pmatrix} 0 & w \\ -\overline{w} & 0 \end{pmatrix} = z + w\mathbf{j},$$

but also

$$\overline{w}\mathbf{j} = \mathbf{j}w, \qquad \mathbf{j}^2 = -1.$$

Thus $\mathbb{H}$ is a sub-ring of the ring of 2×2 matrices over $\mathbb{C}$, and moreover, $\mathbb{H}$ in turn has the sub-ring

$$\left\{\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} : z \in \mathbb{C}\right\},$$

which is isomorphic to $\mathbb{C}$, and $\mathbb{H}$ is both a *left* and a *right* vector-space over this, with basis $(1, \mathbf{j})$ in each case. Also

$$(z + w\mathbf{j})(\overline{z} - w\mathbf{j}) = z\overline{z} + w\mathbf{j}\overline{z} - zw\mathbf{j} - w\mathbf{j}w\mathbf{j} = z\overline{z} + w\overline{w},$$

which is in $\mathbb{R}$, and therefore $\mathbb{H}$ is a field, albeit a skew field.

$\mathbb{H}$ is also a two-sided real vector space, with basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$, where

$$\mathbf{k} = \mathbf{ij}.$$

In analytic geometry, in $\mathbb{R}^2$, we define lines by equations

$$ax + by = c,$$

where not both of a and b are 0. We can do the same in $\mathbb{H}^2$, or else we can use equations

$$xa + yb = c,$$

but we cannot do both. For example, the solution sets of

$$\mathbf{i}x + y = 0, \qquad\qquad x\mathbf{i} + y = 0$$

have $(0, 0)$ and $(\mathbf{i}, 1)$ in common, but are not identical: $(\mathbf{j}, -\mathbf{k})$ solves the former equation, not the latter; $(\mathbf{j}, \mathbf{k})$, the latter, not the former. We shall continue with these ideas tomorrow.

4.3. Pappus's Theorem, intersecting cases

Pappus's **Lemma XI** is that, in Fig. 42a or 42b (Pappus only alludes to the latter),

$$\mathit{\Delta E} \cdot \mathit{ZH} : \mathit{EZ} \cdot \mathit{H\Delta} :: \mathit{\Gamma B} : \mathit{BE}.$$

This is, in the notation for cross ratios of §3.4 (p. 53), with a correction of the ordering of points,

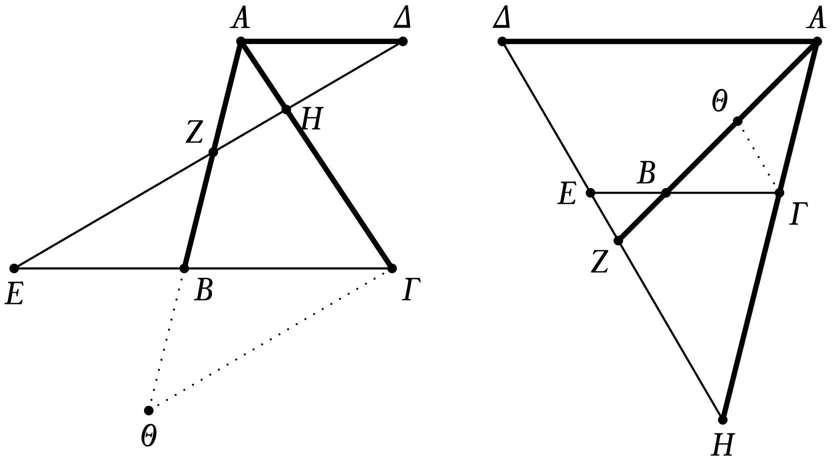
$$[\mathit{\Delta}, E, Z, H] :: \mathit{\Gamma B} : \mathit{EB}. \tag{33}$$

By Lemma III, if $\mathit{A\Delta}$ and $\mathit{E\Gamma}$ met at a point K, then

$$[\mathit{\Delta}, E, Z, H] :: [K, E, B, \mathit{\Gamma}] :: (\mathit{EK} : \mathit{\Gamma K})(\mathit{\Gamma B} : \mathit{EB}).$$

This yields (33), if $\mathit{EK} : \mathit{\Gamma K}$ becomes identity when K is at infinity. Pappus does not argue this way, but, drawing $\mathit{\Gamma\Theta}$ parallel to $\mathit{\Delta E}$, he has by Thales

$$\mathit{\Gamma\Theta} : \mathit{ZH} :: \mathit{\Gamma A} : \mathit{AH} :: \mathit{E\Delta} : \mathit{\Delta H},$$



(a) (b)

Figure 42. Lemma XI

$$
\begin{aligned}
E\Delta \cdot ZH &= \Gamma\Theta \cdot \Delta H, \\
E\Delta \cdot ZH : \Delta H \cdot EZ &:: \Gamma\Theta \cdot \Delta H : \Delta H \cdot EZ \\
&:: \Gamma\Theta : EZ \\
&:: \Gamma B : EB,
\end{aligned}
$$

which is (33). Alternatively, from the proof of Lemma III, we know

$$[E, H, \Delta, Z] :: E\Gamma : EB.$$

From this we obtain (33) in modern notation as follows.

$$
\begin{aligned}
[\Delta, E, Z, H] = \frac{\Delta E \cdot ZH}{EZ \cdot H\Delta} &= \frac{(\Delta H + HE) \cdot ZH}{EZ \cdot H\Delta} \\
&= \frac{\Delta H \cdot ZH + HE \cdot ZH}{EZ \cdot H\Delta}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mathit{\Delta H} \cdot (ZE + EH) + HE \cdot ZH}{EZ \cdot H\mathit{\Delta}} \\
&= 1 + \frac{\mathit{\Delta H} \cdot EH + HE \cdot ZH}{EZ \cdot H\mathit{\Delta}} \\
&= 1 - \frac{H\mathit{\Delta} \cdot EH + EH \cdot ZH}{EZ \cdot H\mathit{\Delta}} \\
&= 1 - \frac{Z\mathit{\Delta} \cdot EH}{EZ \cdot H\mathit{\Delta}} \\
&= 1 - [E, H, \mathit{\Delta}, Z] = 1 - \frac{E\mathit{\Gamma}}{EB} = \frac{\mathit{\Gamma}B}{EB}.
\end{aligned}
$$

Lemma XII is that, in Fig. 43 and Fig. 44, where $AB \parallel \mathit{\Gamma\Delta}$,

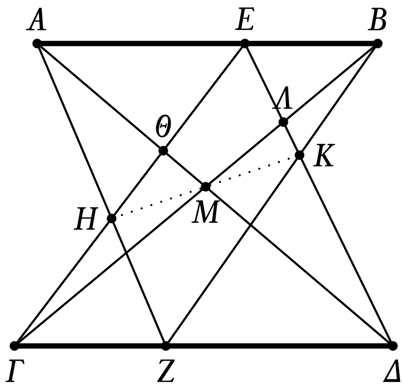
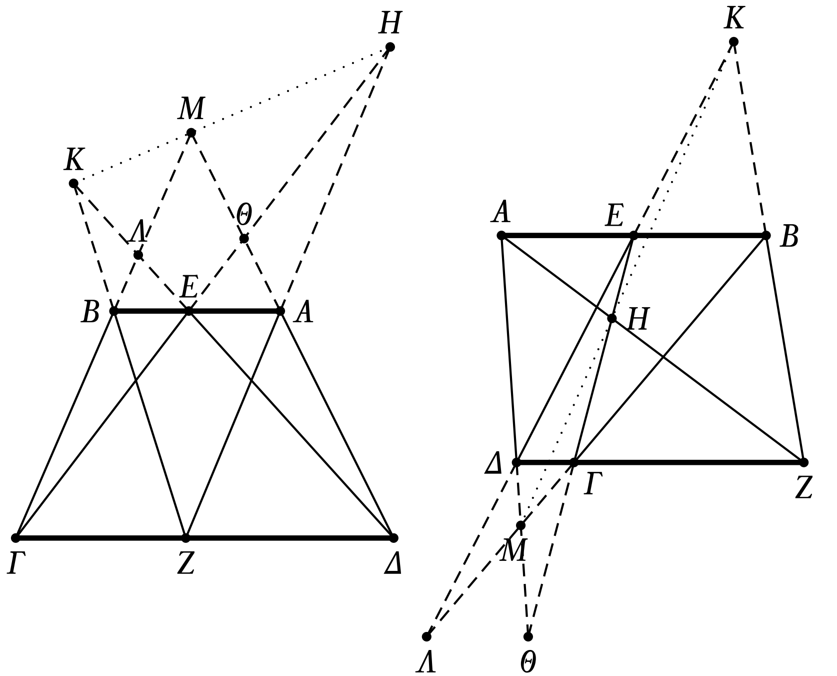


Figure 43. Lemma XII

the points H, M, and K are on a straight line. The proof considers the parts of the diagram shown in Fig. 45. Applying Lemma XI to the first two parts yields

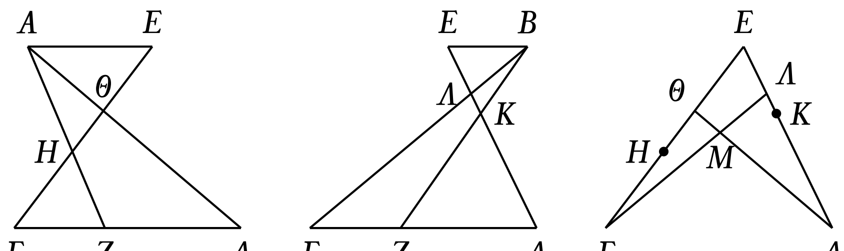
$$[E, \mathit{\Gamma}, H, \mathit{\Theta}] :: \mathit{\Delta}Z : \mathit{\Gamma}Z :: [E, \mathit{\Lambda}, K, \mathit{\Delta}]$$

By Lemma X then, HMK is straight.



(a) (b)

Figure 44. Lemma XII, alternative figures



(a) (b) (c)

Figure 45. Steps of Lemma XII

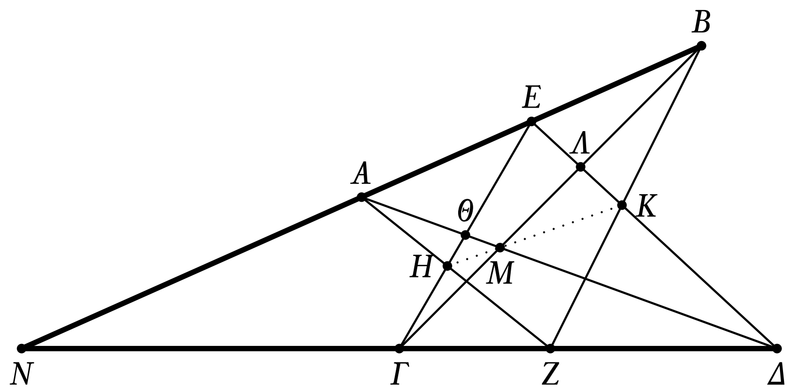


Figure 46. Lemma XIII: Pappus's figure

Lemma XIII is the same, except that AB and $\Gamma\Delta$ meet at a point N, as in Fig. 46 and Fig. 47, so that Lemma III is used in place of Lemma XI. This gives us the intersecting cases of Pappus's Theorem.

4.4. Pappus's Theorem by projection

The mixed cases of Pappus's Theorem are as in Fig. 48, where

- the hexagon is $B\Gamma HE\Delta Z$,
- $B\Gamma$ and $E\Delta$ meet at K,
- ΓH and ΔZ meet at Λ.

The theorem is,

$$BZ \parallel HE \implies HE \parallel K\Lambda.$$

We can prove this, and all other cases of Pappus's Theorem, except for the Parallel Theorem, using only Lemma VIII and *projection.*

If a diagram is drawn on a transparent notebook cover, and the cover is raised at an angle to the first page, and a shadow of

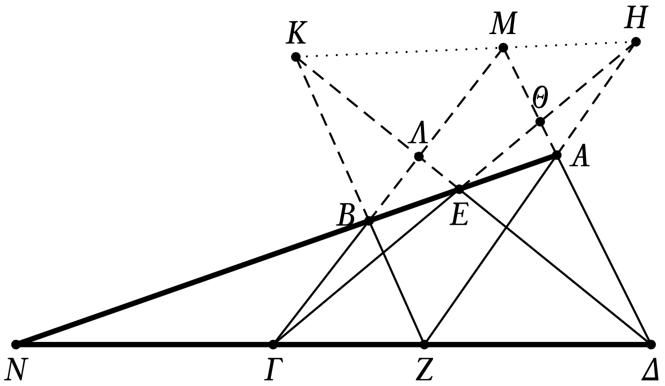
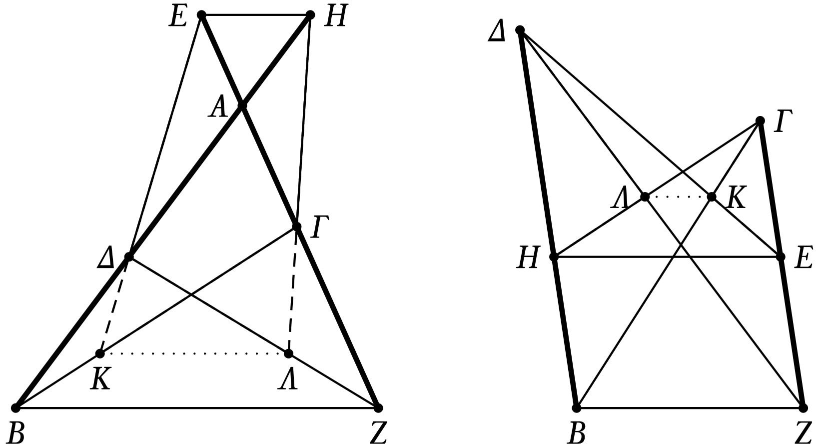


Figure 47. Lemma XIII: alternative figure



(a) Intersecting bounding lines (b) Parallel bounding lines

Figure 48. Pappus's Theorem: mixed cases

the diagram is cast on that page, all straight lines will remain straight, but some parallel lines will cease to be so, and some intersecting lines will become parallel.

For example, adding to Fig. 21a, we draw MN through A, parallel to BZ, and we conceive of the diagram as lying in a vertical plane in Fig. 49 where MN is horizontal. We let I lie not on MN, but in a horizontal plane that contains MN, and we project the diagram from I onto another horizontal plane, so that B becomes B', and Γ becomes Γ', and so on. All straight lines, such as BZ and HE, that were parallel to MN remain parallel to one another in the new diagram. Lines such as $B\Gamma$ and ΔE that were parallel to one another, but not to MN, now intersect; but their intersection points all lie on a single line, $K\Lambda$, which represents the line at infinity of the old diagram. Lines such as AB and AZ that intersected on MN become parallel in the new diagram. Thus we obtain the mixed cases of Pappus's Theorem with parallel bounding lines (Fig. 48b). Similarly,

- if MN is parallel to BZ, but does not contain A, we obtain the mixed case with intersecting bounding lines (Fig. 48a);
- if MN is not parallel to any other lines of the figure, and
  - does not contain A, we obtain Lemma XIII (Fig. 46);
  - but does contain A, we obtain Lemma XII (Fig. 43).

Tomorrow we shall obtain a third proof of Pappus's Theorem by coordinatizing a projective plane.

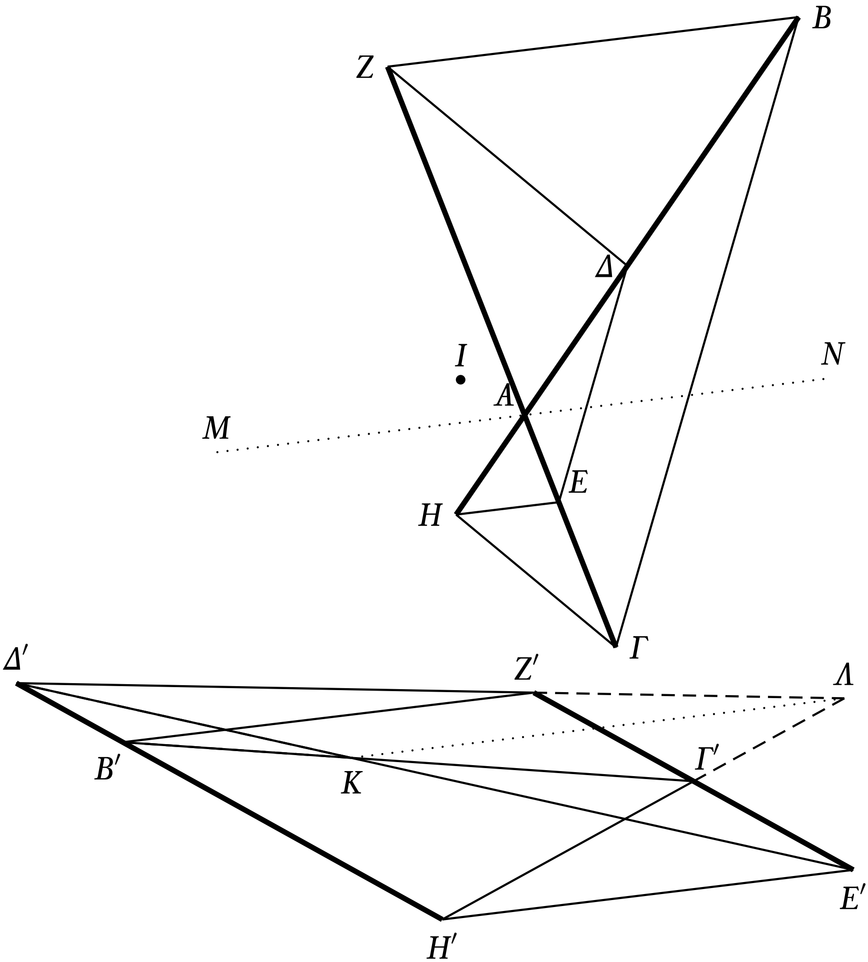


Figure 49. Mixed case by projection

5. Saturday

5.1. Cartesian coordinates

In §4.1 (p. 58), we have obtained a field $\mathbb{K}$ of ratios in an affine plane. Given a triangle ABC in such a plane, we can consider the plane as a left vector space over $\mathbb{K}$ with neutral point C and basis (A, B). In particular, for any point M of the plane, there is a unique ordered pair (s, t) of ratios, meaning $(s, t) \in \mathbb{K}^2$, such that

$$M = s \cdot_C A +_C t \cdot_C B,$$

or equivalently

$$\overrightarrow{CM} = s \cdot \overrightarrow{CA} + t \cdot \overrightarrow{CB}. \tag{34}$$

Here (s, t) is the ordered pair of **Cartesian coordinates** of M with respect to ABC. Conversely, every element of $\mathbb{K}^2$ corresponds to a point of the plane in this way.

5.2. Barycentric coordinates

Since $\overrightarrow{CC}$ is the neutral or zero vector, we can rewrite (34) as

$$\overrightarrow{CM} = s \cdot \overrightarrow{CA} + t \cdot \overrightarrow{CB} + (1 - s - t) \cdot \overrightarrow{CC}. \tag{35}$$

The point of doing this is that now the coefficients on the right add up to 1. Since, for all points X and D,

$$\overrightarrow{CX} = \overrightarrow{CD} + \overrightarrow{DX},$$

substitution into (35) yields

$$\overrightarrow{DM} = s \cdot \overrightarrow{DA} + t \cdot \overrightarrow{DB} + (1 - s - t) \cdot \overrightarrow{DC},$$

or equivalently

$$M = s \cdot_D A +_D t \cdot_D B +_D (1 - s - t) \cdot_D C.$$

Since D is arbitrary, we may write simply

$$M = sA + tB + (1 - s - t)C.$$

Conversely, if p, q, and r are three ratios (elements of $\mathbb{K}$) for which

$$p + q + r = 1, \tag{36}$$

then the linear combination

$$pA + qB + rC$$

is unambiguous, defining the point

$$p \cdot_C A +_C q \cdot_C B.$$

We may write the same point as

$$(p : q : r)$$

when we consider ABC as fixed. But now we can allow

$$(p : q : r) = (pt : qt : rt) \tag{37}$$

for any nonzero t in $\mathbb{K}$. It will be important that the multiplier t is on the right. Given arbitrary ratios p, q, and r for which the equation

$$p + q + r = 0 \tag{38}$$

fails, we have

$$(p : q : r) = ptA + qtB + rtC = pt \cdot_C A + qt \cdot_C B, \qquad (39)$$

where

$$t = (p + q + r)^{-1}.$$

The point $(p : q : r)$ has the **barycentric coordinates** p, q, and r, but each must be considered together with the sum $p + q + r$. The idea is that the point is the center of gravity (the barycenter, from βαρύς, -εῖα, -ύ "heavy") of the system with weights p, q, and r at A, B, and C respectively.

5.3. Ceva's Theorem

Given (p, q, r) satisfying (36), if we define

$$D = (q + r)^{-1}qB + (q + r)^{-1}rC = (q + r)^{-1}q \cdot_C B, \qquad (40)$$

then D is a point on BC, and since t in (39) is 1 we have

$$(p : q : r) = pA + (q + r)D.$$

Thus $(p : q : r)$ is a point on AD. Similarly, when we define

$$E = (p + r)^{-1}pA + (p + r)^{-1}rC,$$
$$F = (p + q)^{-1}pA + (p + q)^{-1}qC,$$

these points are on AC and AB respectively, so $(p : q : r)$ is on BE and CF. We have for example

$$BD : DC :: (BD : BC)(BC : DC),$$

which from (40) is

$$(q + r)^{-1}qr^{-1}(q + r).$$

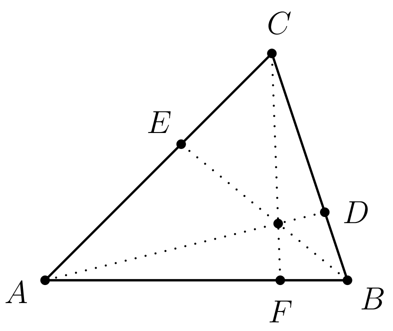


Figure 50. Ceva's Theorem

This ratio is just qr^{-1}, if $\mathbb{K}$ is commutative, and in this case we have **Ceva's Theorem:** in Fig. 50, the lines AD, BE, and CF have a common point if and only if

$$BD : DC \;\&\; CE : EA \;\&\; AF : FB = 1.$$

5.4. Projective coordinates

We have given geometric meaning to $(p : q : r)$ whenever the equation (38) fails. We can still understand $(p : q : r)$ to be the equivalence class defined by (37), even when (38) holds. In this case, we shall give geometric meaning to $(p : q : r)$, if at least one of p, q, and r is not 0.

For every straight line in the plane, there are ratios a, b, and c, where at least one of a and b is not 0, such that the straight line consists of the points such that, if their Cartesian coordinates are (s, t), then

$$as + bt + c = 0.$$

Here it will be important that the coefficients are on the left. If the same point (s, t) has barycentric coordinates $(p : q : r)$, where (36) holds, then $(s, t) = (p, q)$, and so

$$\begin{aligned} 0 &= ap + bq + c(p + q + r) \\ &= (a + c)p + (b + c)q + cr. \end{aligned}$$

Thus the same line is given by

$$ax + by + c = 0$$

in Cartesian coordinates and

$$(a + c)x + (b + c)y + cz = 0 \tag{41}$$

in barycentric coordinates. The straight lines parallel to this one are obtained by changing c alone. Since at least one of a and b is not 0, the coefficients in (41) are not all the same. We obtain all parallel lines by adding the same ratio to each coefficient.

Relabelling, we now have that every straight line is given by an equation

$$ax + by + cz = 0 \tag{42}$$

in barycentric coordinates, where one of the coefficients a, b, and c is different from the others. As $(p : q : r)$ and $(pt : qt : rt)$ are the same point if $t \neq 0$, so then (42) and

$$tax + tby + tcz = 0 \tag{43}$$

define the same line. If $(p : q : r)$ satisfies (42) and therefore (43), and also (38) holds, although $(p, q, r) \neq (0, 0, 0)$, then $(p : q : r)$ satisfies the equation of every straight line parallel to the one defined by (42), and no other straight line. Thus we can understand $(p : q : r)$ as the point at infinity of the straight lines parallel to (42). The line at infinity is then defined by

$$x + y + z = 0.$$

A projective plane now consists of points $(p : q : r)$, where $(p, q, r) \neq (0, 0, 0)$. The expression $(p : q : r)$ consists of **projective coordinates** for the point. The definition in §1.7 (p. 33) of a projective plane is indeed satisfied by such points and the lines defined by the equations (42). For, given two such equations, the coefficients in one not being the same multiples of the coefficients of the other, by Gaussian elimination we can find a nonzero solution, unique up to scaling. (If $\mathbb{K}$ is not commuting, we cannot use Cramer's Rule.) In the same way, two distinct points determine uniquely, again up to scaling, the coefficients in the equation of the line that contains them. Finally, no three of the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$ and $(1 : 1 : 1)$ are collinear.

5.5. Change of coordinates

Any triangle ABC determines a system of barycentric coordinates for the points of the affine plane of the triangle, hence a system of projective coordinates for the projective plane. However, suppose a fourth point D in this plane does not lie on any of the three sides of ABC. Then D has projective coordinates $(\mu : \nu : \rho)$, with $\mu\nu\rho \neq 0$. There is now a bijection

$$(x : y : z) \mapsto (\mu^{-1}x : \nu^{-1}y : \rho^{-1}z)$$

from the set of points of the projective plane to itself. This bijection preserves linearity; in particular, it takes the line given by (42) to the line given by

$$a\mu x + b\nu y + c\rho z = 0.$$

The bijection fixes A, B, and C, which are $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$ respectively, but takes D to $(1 : 1 : 1)$, the barycenter of ABC. In particular, the points that used to be on the line at infinity, defined by (38), are sent to the line given by

$$\mu x + \nu y + \rho z = 0, \tag{44}$$

while of course the points now at infinity satisfy (38).

If μ, ν, and ρ are not all equal to one another, that is, D is not the barycenter of ABC, then, as noted in the previous section, there is a unique point satisfying both (44) and (38).

5.6. Pappus's Theorem, third proof

We show that Pappus's Hexagon Theorem holds a projective plane if and only if the field $\mathbb{K}$ of ratios is commutative. For convenience, let us write

$$\mathrm{Z}(ax + by + cz)$$

for the line given by (42).

We have a hexagon $ABCDEF$, vertices lying alternately on two lines, opposite sides intersecting at G, H, and K respectively, as in Fig. 51. The three lines through A pass respectively through B, E, and K. Assuming these last three points are not collinear, we may let

$$\begin{aligned} A &= (1:1:1), & B &= (1:0:0), \\ E &= (0:1:0), & K &= (0:0:1). \end{aligned}$$

In particular, K is the barycenter of ABC. For the lines through A we have equations as follows:

$$AB = \mathrm{Z}(y - z), \qquad AE = \mathrm{Z}(x - z), \qquad AK = \mathrm{Z}(x - y).$$

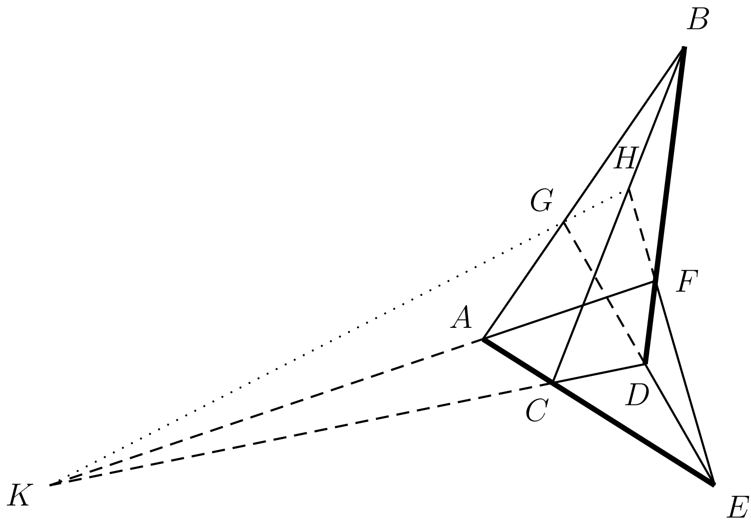


Figure 51. Hexagon Theorem in projective coordinates

For some p, q, and r then,

$$G = (p : 1 : 1), \qquad C = (1 : q : 1), \qquad F = (1 : 1 : r).$$

Consequently,

$$BF = \mathrm{Z}(z - ry), \quad EG = \mathrm{Z}(x - pz), \quad KC = \mathrm{Z}(y - qx).$$

Since these three lines have a common point, namely D, this is each of

$$(1 : q : rq), \qquad (pr : 1 : r), \qquad (p : qp : 1).$$

Hence for example

$$(1, q, rq) = (qpr, q, qr),$$

and therefore

$$1 = qpr. \tag{45}$$

In the same way, the products prq and rqp are 1. But we have also

$$BC = \mathrm{Z}(y - qz), \quad EF = \mathrm{Z}(z - rx), \quad KG = \mathrm{Z}(x - py).$$

The intersection of the first two of these is H, so

$$H = (1 : qr : r).$$

This lies on KG if and only if $pqr = 1$. Comparison with (45) yields the claim.

6. Sunday, September 18, 2016

6.1. Projective plane

Given an affine plane, we have obtained a field $\mathbb{K}$ of ratios, possibly not commutative. Using this, we have extended the affine plane to a projective plane. This plane has points and lines.

6.1.1. Points

We have obtained the set

$$\big\{(x:y:z)\colon (x,y,z) \in \mathbb{K}^3 \smallsetminus \{(0,0,0)\}\big\}$$

of points of the projective plane, where

$$(p:q:r) = \big\{(pt, qt, rt)\colon t \in \mathbb{K} \smallsetminus \{0\}\big\}.$$

Thus the set of points of the projective plane is the quotient of $\mathbb{K}^3 \smallsetminus \{(0,0,0)\}$ by the equivalence relation L given by

$$(p,q,r)\ R\ (p',q',r') \iff (p:q:r) = (p':q':r').$$

6.1.2. Lines

We have shown that the set of lines in the projective plane is

$$\big\{\mathrm{Z}(ax+by+cz)\colon (a,b,c) \in \mathbb{K}^3 \smallsetminus \{(0,0,0\}\big\},$$

whose elements are well-defined by the rule

$$\mathrm{Z}(ax+by+cz) = \{(p:q:r) : ap+bq+cr=0\}.$$

There is an equivalence relation L on $\mathbb{K}^3 \smallsetminus \{(0,0,0)\}$ such that

$$\begin{aligned}(a,b,c)\ L\ (a',b',c')& \\ \iff \mathrm{Z}(ax+by+cz) &= \mathrm{Z}(a'x+b'y+c'z).\end{aligned}$$

The equivalence class of (a,b,c) with respect to L is

$$\bigl\{(ta,tb,tc)\colon t \in \mathbb{K} \smallsetminus \{0\}\bigr\}.$$

6.1.3. Duality

There is no particular reason not to let $\mathrm{Z}(ax+by+cz)$ simply *be* the L-class of (a,b,c). There is then no reason why points should be R-classes, and lines, L-classes, and not the other way around. It just depends on the side we want to write coefficients on. We have chosen the left in the equation (42) for a line.

The field $\mathbb{K}$ gives us the sets

$$\mathrm{M}_1^3(\mathbb{K}), \qquad\qquad \mathrm{M}_3^1(\mathbb{K}),$$

of 3×1 matrices, or column vectors, and of 1×3 matrices, or row vectors, respectively. The sets are isomorphic an abelian groups with respect to addition. If we write column vectors as $\boldsymbol{x}$, we can write row vectors as $\boldsymbol{x}^\top$. We are interested in three matrix multiplications.

- With respect to
$$(\boldsymbol{x},t) \mapsto \boldsymbol{x}t$$
from $\mathrm{M}_1^3(\mathbb{K}) \times \mathbb{K}$ to $\mathrm{M}_1^3(\mathbb{K})$, the latter is a right vector space over $\mathbb{K}$.

- With respect to
$$(t, \boldsymbol{x}^\top) \mapsto t\boldsymbol{x}^\top$$
from $\mathbb{K} \times \mathrm{M}_3^1(\mathbb{K})$ to $\mathrm{M}_1^3(\mathbb{K})$, the latter is a left vector space over $\mathbb{K}$.
- Using
$$(\boldsymbol{x}^\top, \boldsymbol{y}) \mapsto \boldsymbol{x}^\top \boldsymbol{y}$$
from $\mathrm{M}_3^1(\mathbb{K}) \times \mathrm{M}_1^3(\mathbb{K})$ to $\mathbb{K}$, we shall define the relation of a line to a point, whereby the point is on the line, or the line contains or passes through the point.

These multiplications are associative, in the sense that
$$(t\boldsymbol{x}^\top)\boldsymbol{y} = t(\boldsymbol{x}^\top\boldsymbol{y}), \qquad (\boldsymbol{x}^\top\boldsymbol{y})t = \boldsymbol{x}^\top(\boldsymbol{y}t).$$
We have also
$$t\boldsymbol{x}^\top = (\boldsymbol{x}t)^\top.$$
For all $\boldsymbol{a}$ and $\boldsymbol{b}$, the condition
$$\boldsymbol{a}^\top\boldsymbol{b} = 0$$
is equivalent to either of the following:

(i) for all nonzero t in $\mathbb{K}$,
$$\boldsymbol{a}^\top(\boldsymbol{b}t) = 0;$$
(ii) for all nonzero t in $\mathbb{K}$,
$$(t\boldsymbol{a}^\top)\boldsymbol{b} = 0.$$
We can understand the relations R and L defined above as being on $\mathrm{M}_1^3(\mathbb{K}) \smallsetminus \{0\}$ and $\mathrm{M}_3^1(\mathbb{K}) \smallsetminus \{0^\top\}$ respectively. Thus, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero, the two conditions
$$\boldsymbol{a} \mathrel{R} \boldsymbol{b}, \qquad \boldsymbol{a}^\top \mathrel{L} \boldsymbol{b}^\top$$

are equivalent to one another and to the existence of a nonzero t such that

$$\boldsymbol{a}t = \boldsymbol{b}.$$

Therefore, for all nonzero $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$,

$$\boldsymbol{a}^\top\boldsymbol{b} = 0 \ \ \& \ \ \boldsymbol{b}\ R\ \boldsymbol{c} \implies \boldsymbol{a}^\top\boldsymbol{c} = 0,$$

and equivalently

$$\boldsymbol{b}^\top\boldsymbol{a} = 0 \ \ \& \ \ \boldsymbol{b}^\top\ L\ \boldsymbol{c}^\top \implies \boldsymbol{c}^\top\boldsymbol{a} = 0.$$

If the R-class of $\boldsymbol{a}$ is $[\boldsymbol{a}]$, and the L-class of $\boldsymbol{b}^\top$ is $[\boldsymbol{b}^\top]$, we can understand $[\boldsymbol{a}]$ as a point, and $[\boldsymbol{b}^\top]$ as a line. The point is on the line if and only if

$$\boldsymbol{b}^\top\boldsymbol{a} = 0.$$

6.1.4. Plane

Given a field $\mathbb{K}$, we can denote by

$$\mathbb{P}^2(\mathbb{K})$$

the projective plane that we have just described. The points and lines compose the quotients

$$(\mathrm{M}_1^3(\mathbb{K}) \smallsetminus \{0\})/R, \qquad\qquad (\mathrm{M}_3^1(\mathbb{K}) \smallsetminus \{0^\top\})/L$$

respectively. Typical elements are respectively

$$\bigl\{\boldsymbol{a}t \colon t \in \mathbb{K} \smallsetminus \{0\}\bigr\}, \qquad\qquad \bigl\{t\boldsymbol{a}^\top \colon t \in \mathbb{K} \smallsetminus \{0\}\bigr\},$$

denoted by

$$[\boldsymbol{a}], \qquad\qquad [\boldsymbol{a}^\top].$$

We already conceive of $\mathbb{K}^3$ as consisting of points in a three-dimensional space. Then $\mathrm{M}_1^3(\mathbb{K})$ and $\mathrm{M}_3^1(\mathbb{K})$ are somehow the same space, and each of $[\boldsymbol{a}]$ and $[\boldsymbol{a}^\top]$ consists of the nonzero points on a line that passes through zero and the same nonzero point. However, when $\mathbb{K}$ is non-commutative, the lines may be different, as we noted at the end of §4.2 (p. 65). Returning to our earlier notation, we have the one-to-one correspondence

$$(p : q : r) \leftrightsquigarrow \mathrm{Z}(p^*x + q^*y + r^*z)$$

between $(\mathrm{M}_1^3(\mathbb{K}) \smallsetminus \{0\})/R$ and $(\mathrm{M}_3^1(\mathbb{K}) \smallsetminus \{0^\top\})/L$, where

$$t^* = \begin{cases} t^{-1}, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

6.2. Fano Plane

For the simplest example of a projective plane, we may let $\mathbb{K}$ be the two-element field $\mathbb{F}_2$, thus obtaining the **Fano Plane.** The four points of ${\mathbb{F}_2}^2$ in barycentric coordinates are

$$(1:0:0), \qquad (0:1:0), \qquad (0:0:1), \qquad (1:1:1).$$

There are three points at infinity in $\mathbb{P}^2(\mathbb{K})$:

$$(1:1:0), \qquad (1:0:1), \qquad (0:1:1).$$

If we call these A, B, C, D, E, F, and G respectively, then the seven lines are as follows:

$$\begin{aligned} \mathrm{Z}(x) &= BCG, & \mathrm{Z}(y-z) &= ADG, \\ \mathrm{Z}(y) &= ACF, & \mathrm{Z}(x-z) &= BDF, \\ \mathrm{Z}(z) &= ABE, & \mathrm{Z}(x-y) &= CDE, \end{aligned}$$

and finally,

$$Z(x + y + z) = EFG,$$

the line at infinity. All of this can be depicted as in Fig. 52.

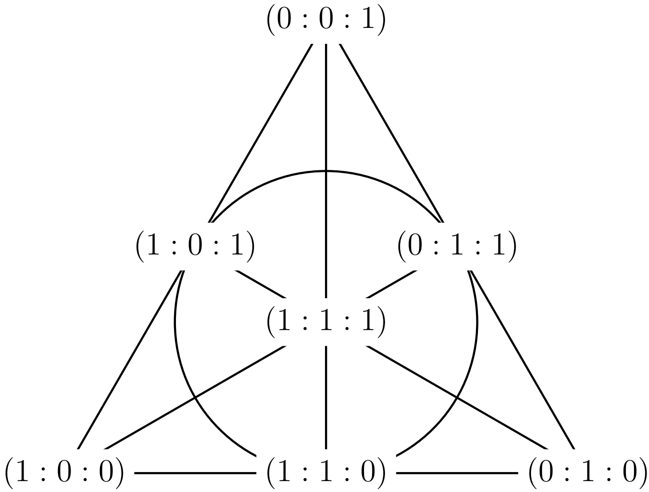


Figure 52. Fano Plane

6.3. Desargues's Theorem proved

6.3.1. In the projective plane over a field

We now prove Desargues's Theorem in $\mathbb{P}^2(\mathbb{K})$ for arbitrary $\mathbb{K}$. First we prove it in $\mathbb{K}^3$. We suppose triangles ABC and DEF lie in two different planes, and

- AB and DE intersect at H,
- BC and EF intersect at K,
- CA and FD intersect at L.

In particular, the two lines in each of the three respective pairs are not identical. There are two conclusions.

1. The two lines in each of the three respective pairs lie in a common plane, and the three resulting planes have a common point G (possibly at infinity), which is where AD, BE, and CF intersect.
2. Each of H, K, and L lies in both the plane of ABC and the plane of DEF, and therefore in the intersection of those planes, which is a straight line.

If the planes of ABC and DEF are parallel, then they meet at their common line at infinity.

If now ABC and DEF lie in the same plane, and AD, BE, and CF intersect at G, the triangles are projections of triangles in different planes meeting the conditions above. Thus Desargues's Theorem holds in $\mathbb{P}^2(\mathbb{K})$.

6.3.2. In a Pappian plane

If $\mathbb{K}$ is not commutative, then Pappus's Theorem does not hold in $\mathbb{P}^2(\mathbb{K})$. However, in a Pappian plane, as defined in §1.7 (p. 33), we can prove Desargues's Theorem as follows. There will be two cases. First we assume

- C is not on DE,
- D is not on BC,

as in Fig. 53. This is the general case proved by Hessenberg in 1905, in a paper [18] using diagrams.

1. Let BC and DE intersect at M.
2. Because ABC and DEF are proper triangles, not lines, M cannot be A or F.
3. Because BC and EF are not the same line, so that BC does not contain G, also M cannot be G.
4. Because of our additional assumptions for the present case, M cannot be any of the points B, C, D, or E.

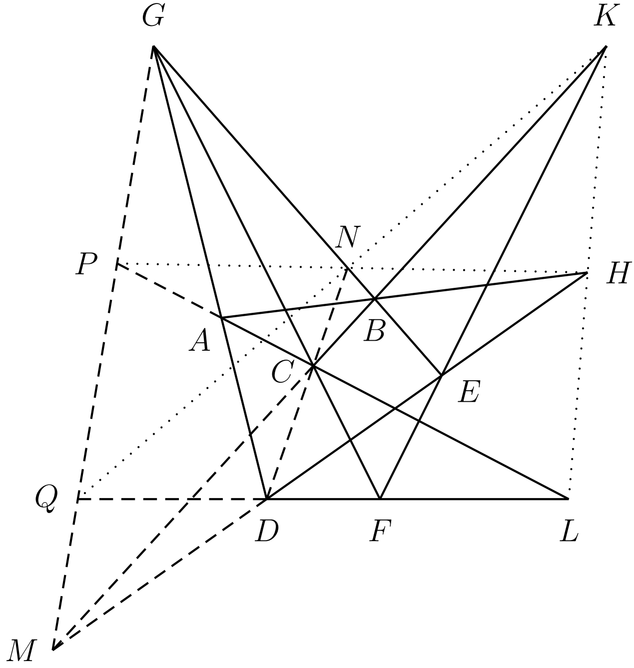


Figure 53. Hessenberg's proof

5. Thus the vertices of the hexagon $ABGMDC$ are distinct.
6. Let the intersection of
   - DC and BE (which is BG) be N,
   - CA and GM be P.
7. Since the vertices of hexagon $ABGMDC$ lie alternately on two distinct straight lines, by Pappus's Theorem the intersections H, N, and P of the pairs of opposite sides are collinear.
8. Let DF and GM intersect at Q.
9. Since the vertices of hexagon $CDFEGM$ are distinct

and lie alternately on two distinct straight lines, the intersections K, N, and Q are collinear.

10. Since CD does not contain B, it does not contain M either.
11. Since AD is not identical with CF, the point P, which lies on AC, can lie also on CD only by being the point C; but then MC would contain both B and G, so BC would contain G, and it doesn't. Thus CD does not contain P.
12. Likewise, Q can lie on CD only by being D, but then DE would contain G; so CD does not contain Q.
13. Now Pappus applies to the hexagon $CMDQNP$, and the points H, K, and L are collinear.

In the other case, as in Fig. 54,

- C lies on DE,
- B lies on DF,
- A lies on EF.

By applying Pappus's Theorem in turn to hexagons $GCELBA$, $GAELBC$, and $SRDCAF$, we have that FHS, DKR and finally LKH are straight. This proof is by Cronheim, in a 1953 paper [6] that uses no diagrams.

6.4. Duality

The **dual** of a statement about a projective plane is obtained by interchanging points and lines. Thus the dual of Pappus's Theorem is that if the sides of a hexagon alternately contain two points, then the straight lines containing pairs of opposite vertices have a common point. So, in the hexagon $ABCDEF$, let AB, CD, and EF intersect at G, and let BC, DE, and FA intersect at H, as in Fig. 55. If the diagonals AD and

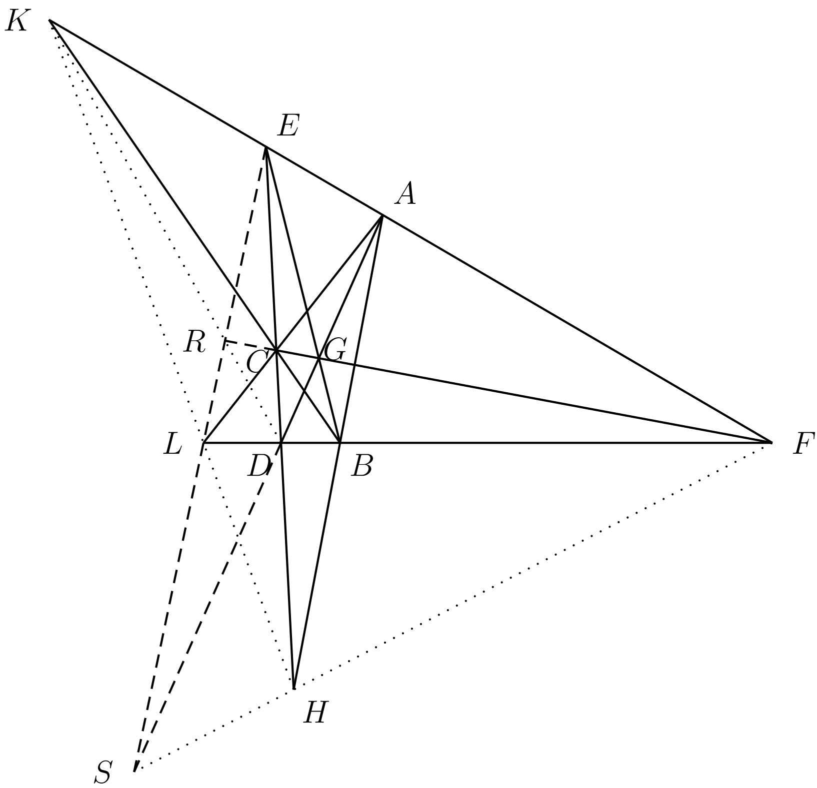


Figure 54. Cronheim's proof

BE meet at K, then the diagonal CF also passes through K. For we can apply Pappus's Theorem itself to the hexagon $ADGEBH$, since AGB and DEH are straight. Since AD and EB intersect at K, and DG and BH at C, and GE and HA at F, it follows that KCF is straight.

It now follows that the dual of Desargues's Theorem is true in a Pappian plane. But the dual is precisely the converse.

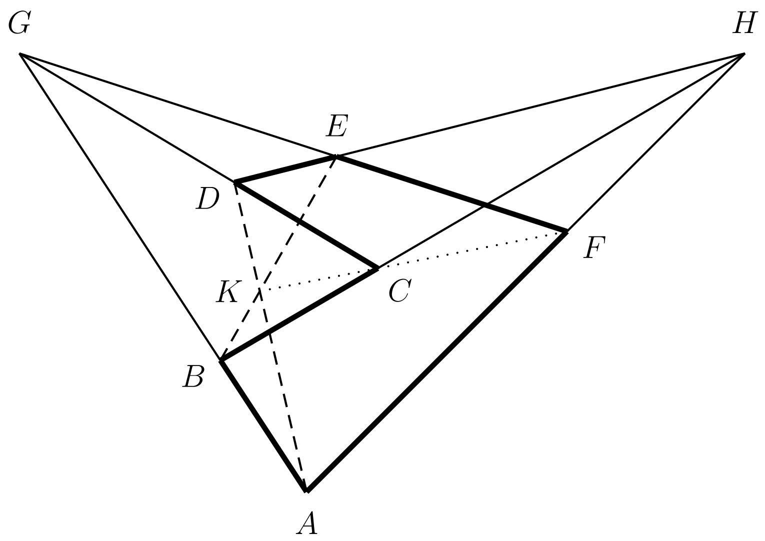


Figure 55. Dual of Pappus's Theorem

6.5. Quadrangle Theorem proved by Desargues

We now use the converse of Desargues's Theorem twice, and the original Theorem once, to prove the Quadrangle Theorem. We shall show that, in Fig. 56, the line PQ passes through F.

1. By the converse of Desargues's Theorem applied to triangles GHL and MNQ, since
   - GH and MN meet at A,
   - GL and MQ meet at D, and
   - HL and NQ meet at E,

   and ADE is straight, it follows that GM, HN, and LQ intersect at a common point R.
2. Likewise, in triangles GHK and MNP, since
   - GH and MN meet at A,

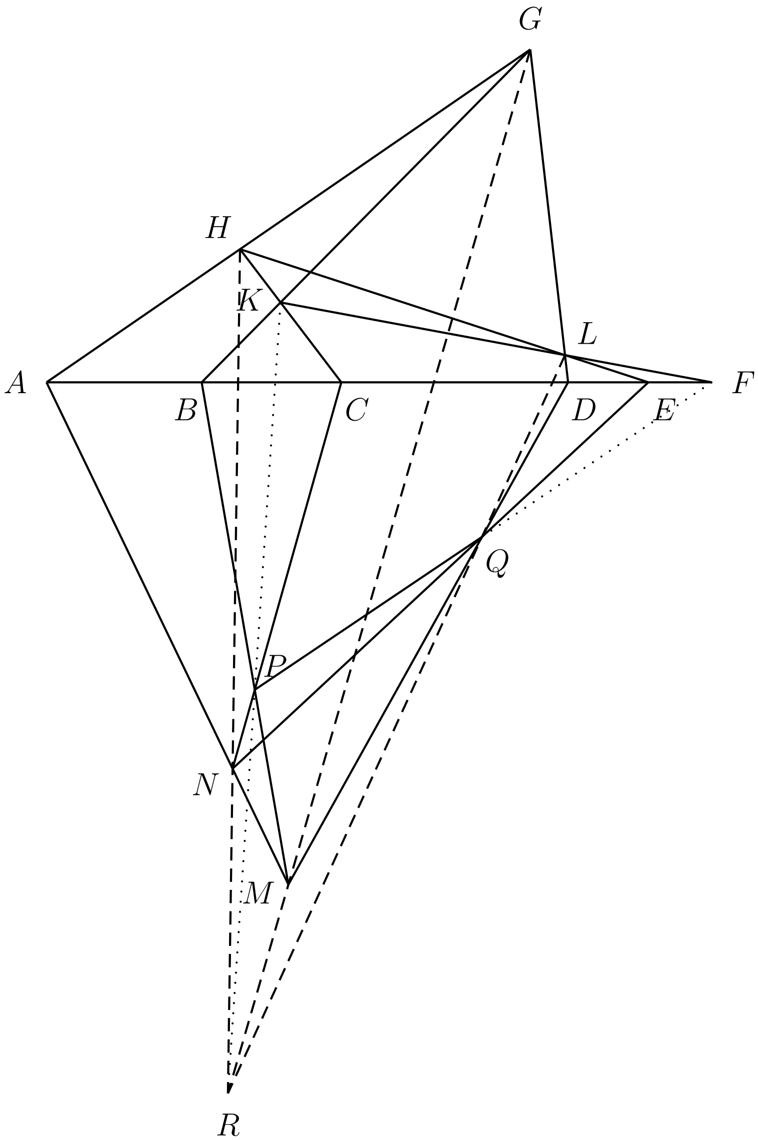


Figure 56. Quadrangle Theorem from Desargues

- GK and MP meet at B, and
- HK and NP meet at C,

and ABC is straight, it follows that KP passes through the intersection point of GM and HN, which is R.

3. Since now HN, KP, and LQ intersect at R, the respective sides of triangles HKL and NPQ intersect along a straight line, by Desargues's Theorem. But
   - HK and NP meet at C,
   - HL and NQ meet at E, and
   - KL and CE meet at F;

   therefore PQ must also meet CE at F.

A. Thales himself

There is little evidence that Thales knew, in full generality, the theorem named for him. I learned this while preparing for the Thales Meeting held on Saturday, September 24, 2016, in Thales's home town of Miletus [34]. See also my "Thales and the Nine-point Conic" [33]. Thales supposedly measured the heights of the Pyramids by considering their shadows; but he may have done this just when his own shadow was as long as a person is tall, since in this case the height of the pyramid would be the same as the length of its own shadow (as measured from the center of the base).

Thales may have recognized that two triangles are congruent if they have two angles equal respectively to two angles and the common sides equal. (This is the so-called Angle-Side-Angle or ASA Theorem.) According to the commentary by Proclus on Book I of Euclid's *Elements* [37], Thales also knew the following three theorems found in that book:

1) the diameter of a circle divides the circle into two equal parts;
2) vertical angles formed by intersecting straight lines are equal to one another;
3) the base angles of an isosceles triangle are equal to one another.

According to Diogenes Laërtius [11, i.24–5], Thales also knew that

4) the angle inscribed in a semicircle is right.

Dates of activity for the persons we have named are roughly as in Fig. 57. All four of the listed theorems can be understood

585 B.C.E.	Thales
300	Euclid
250 C.E.	Diogenes Laërtius
340	Pappus
450	Proclus

Figure 57. Dates of some ancient writers and thinkers

to be true by *symmetry.* For example, the equation

$$\angle ABC = \angle CBA$$

basically establishes the equality of vertical angles. Also, suppose we complete the diagram of an angle inscribed in a semicircle as in Fig. 58. Here the quadrilateral $BCDE$ has four

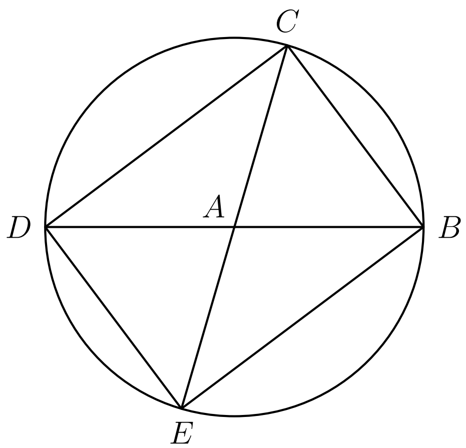


Figure 58. Angle in a semicircle

equal angles. If it follows that those angles must be right, then the theorem of the semicircle is proved.

Those four equal angles are right in Euclidean geometry. Here, by Euclid's fifth postulate, if the angles at DCB and CBE are together less then two right angles, then CD and BE must intersect when extended. In that case, for the same reason, they intersect when extended in the other direction; but this would be absurd.

B. Origins

The origin of this course is my interest in the origins of mathematics. This interest goes back at least to a tenth-grade geometry class in 1980–1. We students were taught to write proofs in the two-column, statement–reason format. I understood the purpose of the class, not as learning geometry as such, but as learning proof. That was good, but I did not much care for our textbook, by Weeks and Adkins. Their example of congruence was a machine in a photograph, stamping out foil trays for TV dinners [42, p. 13].

Weeks and Adkins confuse equality with sameness, as I mention in "On Commensurability and Symmetry" [36]. A geometrical equation like $AB = CD$ means not that the segments AB and CD are the same, but that their *lengths* are the same. Length is an abstraction from a segment, as ratio is an abstraction from two segments. This is why Euclid uses "equal" to describe two equal segments, but "same" to describe the ratios of segments in a proportion. One can maintain the distinction symbolically by writing a proportion as $A:B :: C:D$, rather than as $A/B = C/D$. I noticed the distinction many years after high school; but even in tenth grade I thought we should read Euclid. I went on to read him at St John's College [31], along with Homer, Aeschylus, and Plato, and Apollonius, Ptolemy, Newton, and Lobachevski.

In 2008, the first course I taught at the Nesin Mathematics Village was an opportunity to review some of that reading. Called "Conic Sections *à la* Apollonius of Perga," my course reviewed the propositions of Book I of the *Conics* [1, 2] that pertained to the parabola. I shall say more about this later; for now, while the course was great for me, I don't think it meant much for the students who sat and watched me at the board. One has to engage with the mathematics for oneself, especially when it is something so unusual as Apollonius. A good way to do this is to have to go to the board and present the mathematics, as at St John's.

In 2010 at METU in Ankara, I taught the course called History of Mathematical Concepts in the manner of St John's. We studied Euclid, Apollonius, and (briefly) Archimedes in the first semester; Al-Khwārizmī, Thābit ibn Qurra, Omar Khayyám, Cardano, Viète, Descartes, and Newton in the second [30].

At METU I loved the content of the course called Fundamentals of Mathematics, required of all first-year students. I even wrote a text for the course, a rigorous text that might overwhelm students, but whose contents I thought at least teachers should know. In the end I didn't think it was right to try to teach equivalence relations and proofs to beginning students, independently from a course of traditional mathematics. When I oved to Mimar Sinan in 2011, my colleagues and I were able to develop a course in which first-year students read and presented the proofs that taught mathematics to practically all mathematicians until the twentieth century. Among other things, students would learn the non-trivial (because non-identical) equivalence relation of congruence. I did not actually recognize this opportunity until I had seen the way students tended to confuse equality of line segments with sameness.

Our first-semester Euclid course is followed by an analytic geometry course. Pondering the transition from the one course to the other led to some of the ideas about ratio and proportion that are worked out in the present course. My study of Pappus's Theorem arose in this context, and I was disappointed to find that the *Wikipedia* article called "Pappus's Hexagon Theorem" did not provide a precise reference to its namesake. I rectified this condition on May 13, 2013, when I added to the article a section called "Origins," giving Pappus's proof.

In order to track down that proof, I had relied on Heath, who in *A History of Greek Mathematics* summarizes most of Pappus's lemmas for Euclid's lost *Porisms* [17, p. 419–24]. In this summary, Heath *may* give the serial numbers of the lemmas as such: these are the numbers given here as Roman numerals. Heath *always* gives the numbers of the lemmas as propositions within Book VII of Pappus's *Collection,* according to the enumeration of Hultsch [25]. Apparently this enumeration was made originally in the 16th century by Commandino in his Latin translation [26, pp. 62–3, 77].[1]

According to Heath, Pappus's Lemmas XII, XIII, XV, and XVII for the *Porisms,* or Propositions 138, 139, 141, and 143 of Book VII, establish the Hexagon Theorem. The latter two propositions can be considered as converses of the former two, which consider the hexagon lying respectively between parallel and intersecting straight lines.

In *Mathematical Thought from Ancient to Modern Times,*

[1]Searching for Commandino's name in Jones's book reveals an interesting tidbit on page 4: Book III of the *Collection* is addressed to an otherwise-unknown teacher of mathematics called Pandrosion. She must be a woman, since she is given the feminine form of the adjective κράτιστος, -η, -ον ("most excellent"); but "in Commandino's Latin translation her name vanishes, leaving the absurdity of the polite epithet κρατίστη being treated as a name, 'Cratiste'; while for no good reason Hultsch alters the text to make the name masculine."

Kline cites only Proposition 139 as giving Pappus's Theorem [22, p. 128]. This proposition, Lemma XIII, follows from Lemmas III and X, as XII follows from XI and X. For Pappus's Theorem in the most general sense, one should cite also Proposition 134, Lemma VIII, which is the case where two pairs of opposite sides of the hexagon are parallel; the conclusion is then that the third pair are also parallel. Heath's summary does not seem to mention this lemma at all. The omission must be a simple oversight. For Hilbert, the lemma is Pascal's Theorem; he never mentions Pappus [20].

In the catalogue of my home department at Mimar Sinan, there is an elective course called Geometries, meeting two hours a week. I offered it in the fall of 2015; it had last been taught in the fall of 2010. For use in the first half of the course, from Hultsch's text [25] I translated the first 19 of Pappus's 38 lemmas for Euclid's *Porisms* [28]; in the second half, we used the existing English translation of Lobachevski [23]. I was able to go over the same material in the following summer at the Nesin Mathematics Village in Şirince.

I had not thought there was an English version of the Pappus; but at the end of my work, I found Jones's. This helped me to parse a few confusing words. What I found first, on *Library Genesis,* was the first volume of Jones's work [26]; Professor Jones himself supplied me with Volume II, the one with the commentary and diagrams [27].

Book VII of Pappus's *Collection* is an account of the so-called *Treasury of Analysis* (ἀναλυόμενος τόπος). This *Treasury* consisted of works by Euclid, Apollonius, Aristaeus, and Eratosthenes, most of them now lost. As a reminder of the wealth of knowledge that is no longer ours, I ultimately wrote out, on the back of my Pappus translation, a table of the contents of the *Treasury.* Pappus's list of the contents is included by Thomas in his *Selections Illustrating the History of Greek Mathematics* in the Loeb series [41, pp. 598–601]. Thomas's anthology includes more selections from Pappus's *Collection,* but none involving the Hexagon Theorem. He does provide Proposition 130 of Book VII, that is, Lemma IV for the *Porisms* of Euclid, the lemma that I am calling the Quadrangle Theorem.

C. Involutions

Pappus's Lemma IV is that if the points $ABCC'B'A'$ in Fig. 59 satisfy the proportion

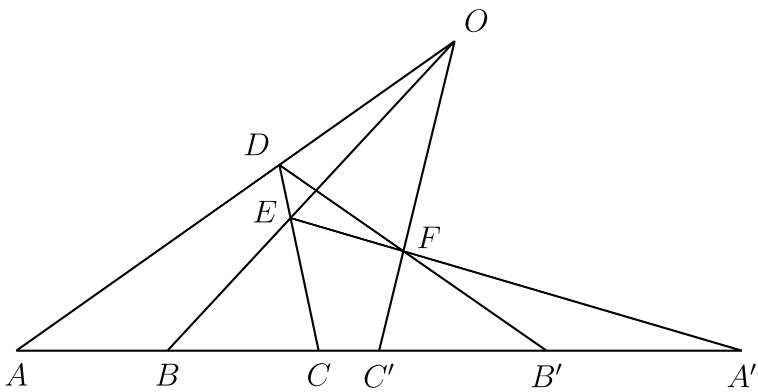


Figure 59. Lemma IV (Quadrangle Theorem)

$$(AB : BC)(CA' : A'A) :: (A'B' : B'C')(C'A : AA'), \qquad (46)$$

then EFA' is straight. Eliminating AA', Chasles [3, p. 102] rewrites (46) in the form

$$BC \cdot C'A \cdot A'B' = AB \cdot B'C' \cdot CA'.$$

We can write this as

$$BC \cdot C'A : AB \cdot C'B' :: CA' : B'A',$$

which makes it easier to see that, when $ABCC'B'$ are given, then some unique A' exists so as to satisfy (46). As A' varies, the ratio $CA' : B'A'$ takes on all possible values but unity. If $BC \cdot C'A = AB \cdot C'B'$, that is,

$$AB : BC :: C'A : C'B',$$

then $EF \parallel AB'$ by Lemma I. Otherwise, by Lemma IV, EF must pass through the A' that satisfies (46). Thus the converse of the lemma holds as well. We might speculate whether this converse was one of Euclid's original porisms. Chasles seems to think it was.

Thomas observes [41, pp. 612–3],

> [The converse of Lemma IV] is one of the ways of expressing the proposition enunciated by Desargues: *The three pairs of opposite sides of a complete quadrilateral are cut by any transversal in three pairs of conjugate points of an involution.*

Following Coxeter, I would call the "complete quadrilateral" here a *complete quadrangle,* as in Fig. 3 (p. 11). The proposition to which Thomas refers is apparently the **Involution Theorem,** which Desargues proves in his *Rough Draft of an Essay on the results of taking plane sections of a cone* [15, p. 54].

One way to understand the Involution Theorem is to observe that, in Fig. 59, if the points $BCC'B'$ are conceived of as fixed, then A determines A'. Moreover, A' determines A in the same way, as in Fig. 60. Thus we have an operation that *transposes* A and A', and so it is an *involution* of the straight line BB'.

Desargues proceeds towards the Involution Theorem by first observing that if seven points $OABCC'B'A'$ are arranged, as in Fig. 61, on a straight line so that

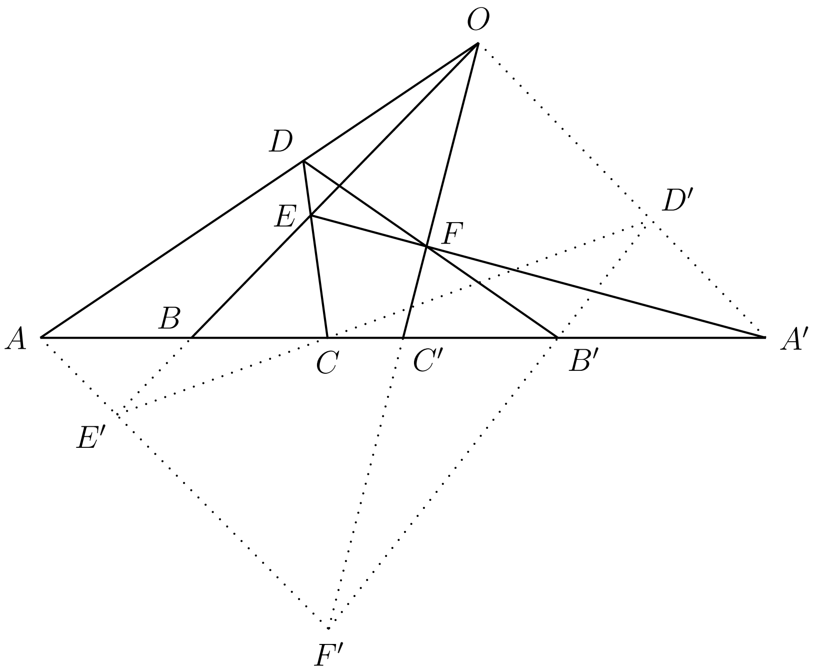


Figure 60. Involution

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC', \tag{47}$$

then, without reference to O, we have [15, p. 48]

$$CB' \cdot C'B' : BC \cdot BC' :: AB' \cdot B'A' : AB \cdot BA'. \tag{48}$$

To prove this, by second equation in (47), we have

$$OB' : OC' :: OC : OB, \tag{49}$$

so by subtracting the terms of the first ratio from the terms of the second, we obtain

$$OB' : OC' :: CB' : BC'. \tag{50}$$

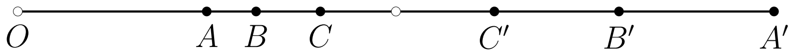


Figure 61. Points in involution

Similarly, after alternating (49), we obtain

$$OC' : OB :: OB' : OC,$$
$$OC' : OB :: C'B' : BC.$$

Composing the latter with (50) yields

$$OB' : OB :: CB' \cdot C'B' : BC \cdot BC'. \tag{51}$$

Replacing C with A and C' with A' in the second equation of (47) yields the first equation. Hence we can do the same in (51), obtaining

$$OB' : OB :: AB' \cdot B'A' : AB \cdot BA'.$$

Eliminating the common ratio from the last two proportions yields (48). For meeting this condition, the three pairs AA', BB' and CC' of points are said to be in **involution,** by Desargues's definition.

Now suppose instead that the pairs AA', BB' and CC' consist of the points where a transversal cuts the pairs of opposite sides of a complete quadrangle, as in Thomas's description and as in Fig. 59. Then the converse of Pappus's Lemma IV, expressed in (46), gives us now

$$AA' \cdot BC : AB \cdot CA' :: AA' \cdot C'B' : AC' \cdot B'A'. \tag{52}$$

This proportion is equivalent to (48). However, the best way to show this may not be obvious. One approach is to introduce the **cross ratio,** as we did in §3.4 (p. 53), though apparently Desargues does not do this [15, p. 52]. Despite the misgivings expressed on page 101, we turn to modern notation for ratios. If $ABCD$ are points on a straight line, we let

$$\frac{AB \cdot CD}{AD \cdot CB} = [A, B, C, D] \tag{53}$$

by definition. Note the pattern of repeated letters on the left. We could use a different pattern; we just have to be consistent. We take line segments to be directed, so that $CB = -BC$. Then (52) is equivalent to

$$[A, A', C, B] = [A, A', B', C'], \tag{54}$$

while, since (48) is, in modern notation,

$$\frac{CB' \cdot C'B'}{BC \cdot BC'} = \frac{AB' \cdot B'A'}{AB \cdot BA'},$$

we obtain from this

$$\frac{CB' \cdot AB}{BC \cdot AB'} = \frac{BC' \cdot B'A'}{C'B' \cdot BA'},$$

that is,

$$[C, B', A, B] = [B, C', B', A']. \tag{55}$$

But (54), obtained from Pappus, must still hold if we permute the pairs AA', BB', and CC'. Sending each pair to the next (and the last to the first), we obtain from (54) the equivalent equation

$$[B, B', A, C] = [B, B', C', A']. \tag{56}$$

We shall have that this is equivalent to (55), once we understand how cross ratios are affected by permutations of their entries.

The cross ratios that can be formed from $ABCD$ are permuted transitively by a group of order 24. Then there are at most 6 different cross-ratios, since

$$\begin{aligned}[C, D, A, B] &= [A, B, C, D],\\ [B, A, D, C] &= [A, B, C, D].\end{aligned}$$

Moreover, from (53) we can read off

$$[A, D, C, B] = \frac{1}{[A, B, C, D]},$$

while

$$\begin{aligned}[A, C, B, D] &= \frac{AC \cdot BD}{AD \cdot BC}\\ &= \frac{(AB + BC) \cdot (BC + CD)}{AD \cdot BC}\\ &= \frac{AC \cdot BC + BC \cdot CD}{AD \cdot BC} - [A, B, C, D]\\ &= 1 - [A, B, C, D].\end{aligned}$$

The involutions $x \mapsto 1/x$ and $x \mapsto 1 - x$ of the set of ratios generate a group of order 6. (Here we either exclude the ratios 0 and 1, or allow them along with ∞.) Now we have accounted for all permutations of points. We have

$$\begin{aligned}&[B, B', A, C] = [B, B', C', A']\\ &\qquad \iff [B, A, B', C] = [B, C', B', A']\\ &\qquad \iff [A, B, C, B'] = [B, C', B', A']\\ &\qquad \iff [C, B', A, B] = [B, C', B', A'],\end{aligned}$$

that is, (56) and (55) are equivalent, as desired.

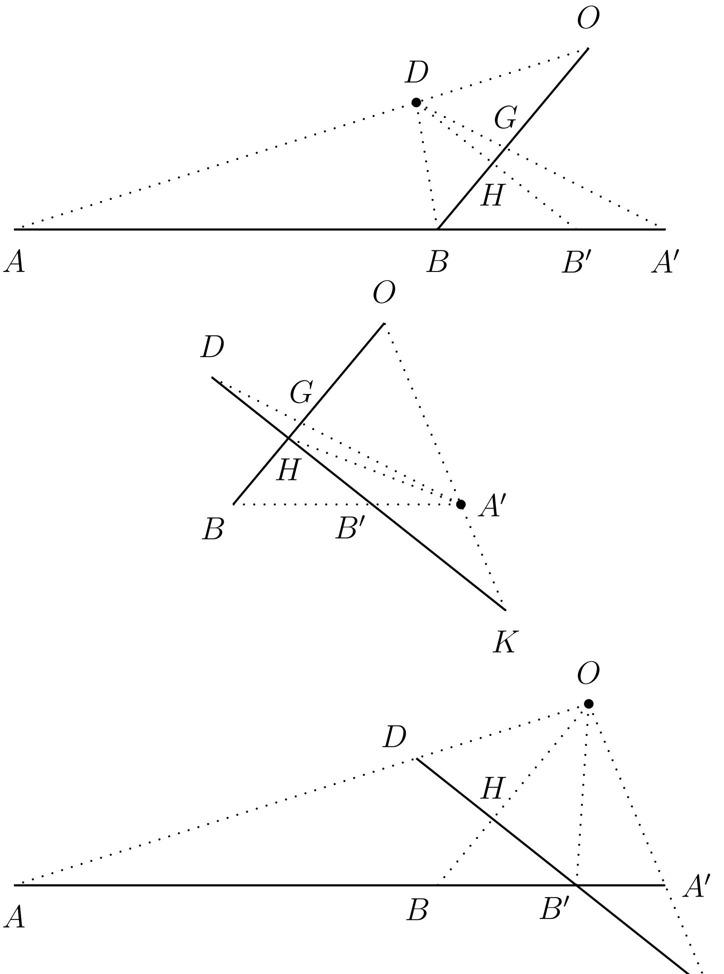


Figure 62. Involution

The involution of the straight line BB' in Fig. 59 that transposes A and A' will transpose B and B', and also C and C'. Following Coxeter [5, **14.56**, p. 246], we obtain this transposition as in Fig. 62, as a composition of three projections of one straight line onto another. In Coxeter's notation [5, p. 242], the projections are

$$ABB'A' \underset{\wedge}{\overset{D}{=}} OBHG,$$

$$OBHG \underset{\wedge}{\overset{A'}{=}} KB'HD,$$

$$KB'HD \underset{\wedge}{\overset{O}{=}} A'B'BA,$$

where for example the first expression means that

$$ADO, \qquad B'DH, \qquad A'DG, \qquad ABB'A', \qquad OBHG$$

are straight. It follows from Lemma IV that this transformation is an involution and is uniquely determined by the pairs AA' and BB'.

D. Locus problems

Thomas's anthology [41, 600–3] includes the account by Pappus of five- and six-line locus problems that Descartes quotes in the *Geometry* [8, pp. 18–21]. Pappus suggests no solution to such problems; but later in the *Geometry,* Descartes solves a special case of the five-line problem, where, as in Fig. 63, four of the straight lines—say ℓ_0, ℓ_1, ℓ_2, and ℓ_3—are parallel

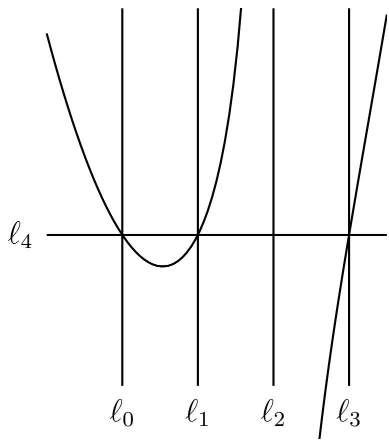


Figure 63. A five-line locus

to one another, each a distance a from the previous, while the fifth line—ℓ_4—is perpendicular to them. What is the locus of points such that the product of their distances to ℓ_0, ℓ_1, and ℓ_3 is equal to the product of a with the distances to ℓ_2 and ℓ_4? One can write down an *equation* for the locus, and Descartes does. Effectively letting the x- and y-axes be ℓ_2 and

ℓ_4, the positive direction of the latter being from ℓ_3 towards ℓ_0, Descartes obtains

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy.$$

This may allow us to plot points on the desired locus, as in Fig. 63; but we could already do that. The equation is thus not a solution to the locus problem, since it does not tell us what the locus *is.* But Descartes shows that the locus is traced by the intersection of a moving parabola with a straight line passing through one fixed point and one point that moves with the parabola. In Fig. 64, the parabola has axis sliding along

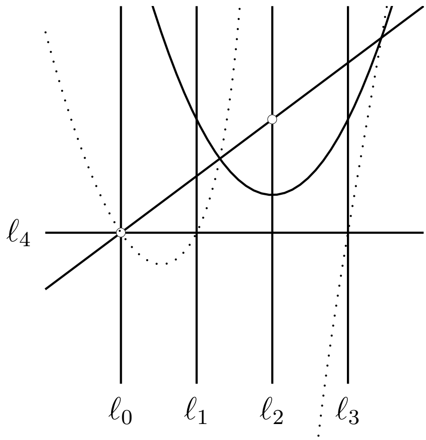


Figure 64. Solution of the five-line locus problem

ℓ_2, and a is its *latus rectum.*[1] The straight line passes through the intersection of ℓ_0 and ℓ_4 and through the point on the axis of the parabola whose distance from the vertex is a.

[1] See my article "Abscissas and Ordinates" [32] for more than you ever imagined wanting to know about the term *latus rectum.*

Descartes's solution is apparently one that Pappus would recognize as such. Thus Descartes's algebraic methods would seem to represent an advance, and not just a different way of doing mathematics.

E. Ancients and moderns

Modern mathematics is *literal* in the sense of relying on the letters that might appear in a diagram, rather than the diagram itself. The diagram is an integral part of ancient proofs. In some of his lemmas, as Jones observes [27, p. 456],

> Pappus does not take the trouble to define his figure (the reader with Euclid's *Porisms* before him perhaps would not have needed such help).

For example, in Lemma I of Pappus, there is no **enunciation** in the sense of Proclus [37, 203, p. 159], but the **exposition** reads,

Ἔστω καταγραφὴ	Let the diagram be
ἡ ΑΒΓΔΕΖΗ,	$AB\Gamma\Delta EZH$,
καὶ ἔστω	and let it be that
ὡς ἡ ΑΖ πρὸς τὴν ΖΗ,	as AZ is to ZH,
οὕτως ἡ ΑΔ πρὸς τὴν ΔΓ,	so is $A\Delta$ to $\Delta\Gamma$;
καὶ ἐπεζεύχθω ἡ ΘΚ.	and let ΘK have been joined.

Thus we are given

$$AZ : ZH :: A\Delta : \Delta\Gamma, \tag{57}$$

and we can infer that Z lies on AH, and Δ on $A\Gamma$. If H did not lie on $A\Gamma$, then we could immediately apply Thales's Theorem; so probably all of the points $A\Gamma\Delta ZH$ are on one straight line. Pappus mentions two more points, B and E; probably they are not on $A\Gamma\Delta ZH$. From the **specification,** we can infer that Θ and K do not lie on $A\Gamma$:

ὅτι	[I say] that
παράλληλός ἐστιν ἡ ΘΚ *τῇ* ΑΓ.	parallel is $\mathit{\Theta K}$ to $\mathit{A\Gamma}$.

Thus we aim to show

$$\mathit{\Theta K} \parallel \mathit{A\Gamma}. \tag{58}$$

The **construction** gives one more point:

Ἤχθω διὰ τοῦ Ζ	Let have been drawn through Z,
τῇ ΒΔ *παράλληλος ἡ* ΖΛ	to $\mathit{B\Delta}$ parallel, $\mathit{Z\Lambda}$.

We have now

$$\mathit{B\Delta} \parallel \mathit{Z\Lambda}. \tag{59}$$

Beginning as follows, the **demonstration** shows, in terms of B, where $\mathit{\Lambda}$ is along $\mathit{Z\Lambda}$:

ἐπεὶ οὖν	Since then
ἐστιν ὡς ἡ ΑΖ *πρὸς τὴν* ΖΗ,	as AZ is to ZH,
οὕτως ἡ ΑΔ *πρὸς τὴν* ΔΓ,	so $\mathit{A\Delta}$ to $\mathit{\Delta\Gamma}$,
ἀνάπαλιν	by inversion,
καὶ συνθέντι καὶ ἐναλλάξ	*componendo, alternando,*
ἐστιν ὡς ἡ ΔΑ *πρὸς τὴν* ΑΖ,	as $\mathit{\Delta A}$ is to AZ,
τουτέστιν ἐν παραλλήλῳ	that is, in parallel,
ὡς ἡ ΒΑ *πρὸς τὴν* ΑΛ,	as BA is to $\mathit{A\Lambda}$,
οὕτως ἡ ΓΑ *πρὸς τὴν* ΑΗ·	so $\mathit{\Gamma A}$ to AH;
παράλληλος ἄρα	parallel then
ἐστὶν ἡ ΛΗ *τῇ* ΒΓ·	is $\mathit{\Lambda H}$ to $\mathit{B\Gamma}$.

Thus we manipulate (57) to get

$$\mathit{\Delta A} : \mathit{AZ} :: \mathit{\Gamma A} : \mathit{AH}.$$

Also, by Thales's Theorem, apparently

$$\mathit{\Delta A} : \mathit{AZ} :: \mathit{BA} : \mathit{A\Lambda}.$$

This shows $\mathit{\Lambda}$ must have lain along AB. Now, with (59), the position of $\mathit{\Lambda}$ is determined. Combining the last two proportions yields

$$BA : A\mathit{\Lambda} :: \mathit{\Gamma}A : AH,$$

so by the converse of Thales,

$$B\mathit{\Gamma} \parallel \mathit{\Lambda}H.$$

What we know is in Fig. 65a. The demonstration continues:

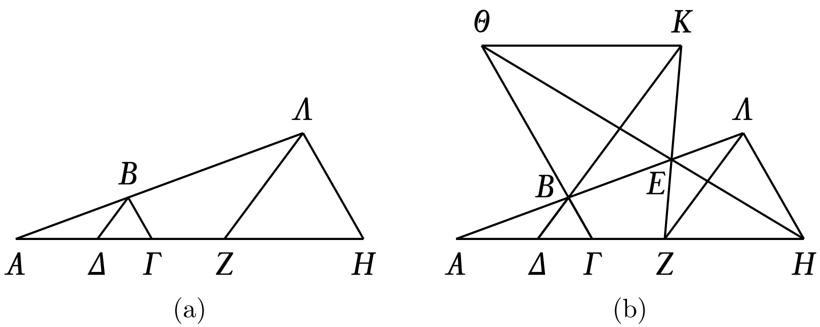


Figure 65. Pappus's Lemma I

ἔστιν ἄρα ὡς ἡ ΕΒ πρὸς τὴν ΒΛ,	hence as EB is to $B\mathit{\Lambda}$,
οὕτως ἐν παραλλήλῳ	so in parallel
ἡ ΕΚ πρὸς τὴν ΚΖ,	EK to KZ,
καὶ ἡ ΕΘ πρὸς τὴν ΘΗ·	and $E\mathit{\Theta}$ to $\mathit{\Theta}H$;
καὶ ὡς ἄρα ἡ ΕΚ πρὸς τὴν ΚΖ,	hence also as EK to KZ,
οὕτως ἐστὶν ἡ ΕΘ πρὸς τὴν ΘΗ·	so is $E\mathit{\Theta}$ to $\mathit{\Theta}H$;
παράλληλος ἄρα	parallel then
ἐστὶν ἡ ΘΚ τῇ ΑΓ.	is $\mathit{\Theta}K$ to $A\mathit{\Gamma}$.

By Thales's Theorem, apparently,

$$EB : B\mathit{\Lambda} :: \begin{cases} EK : KZ, \\ E\mathit{\Theta} : \mathit{\Theta}H. \end{cases}$$

We can infer

- from the ratios,
  - E lies on $B\Lambda$,
  - K lies on EZ,
  - Θ lies on EH;
- from the proportions,
  - K lies on ΔB,
  - Θ lies on ΓB.

Now we can complete the diagram as in Fig. 65b. Since, finally,

$$EK : KZ :: E\Theta : \Theta H,$$

we obtain (58) by the converse of Thales.

Netz observes that Greek mathematics *never* simply declares what letters in a diagram stand for [24, pp. 24–5]:

> Nowhere in Greek mathematics do we find a moment of specification *per se,* a moment whose purpose is to make sure that the attribution of letters in the diagram is fixed.

It may well be Descartes, he says, who first fixes such attributions. I note an early passage in *La Géométrie* [10, p. 2]:

> Mais souvent on n'a pas besoin de tracer ainsi ces lignes sur le papier, et il suffit de les désigner par quelques lettres, chacune par une seule. Comme pour ajouter la ligne BD à GH, je nomme l'une a et l'autre b, et écris $a + b$; et $a - b$ pour soustraire b de a; et ab pour les multiplier l'une par l'autre . . .

We have already seen how Descartes's abbreviations make solutions to ancient unsolved problems possible.

Nonetheless, it it possible to take cleverness with notation too far. The third proof of Pappus's Theorem, given in §5.6 (p. 82) is taken from Coxeter [5, p. 236], and it may appeal to modern Cartesian sensibilities; but it gives less sense of *why* the theorem is true than the second proof, in §4.4 (p. 72), where we pass to a third dimension and *project,* starting from the diagram of Lemma VIII. Coxeter's diagram for Pappus's Theorem [5, p. 232] is labelled as in Fig. 66; it allows

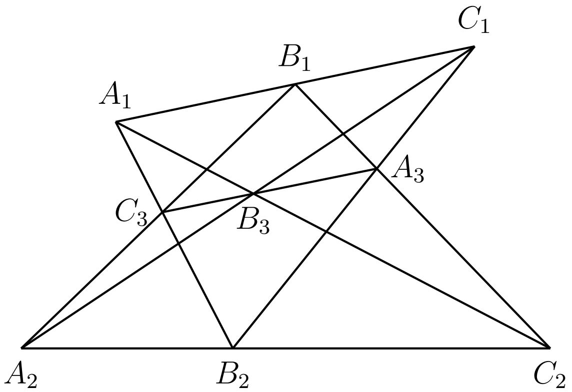


Figure 66. Pappus's Lemma XIII in modern notation

us to observe, once for all, that $A_iB_jC_k$ is straight whenever $\{i, j, k\} = \{1, 2, 3\}$. Coxeter can also express Pappus's Theorem with a matrix,

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}.$$

Here each of the first two rows consists of collinear points, as does each of the six diagonals $A_iB_{i\pm1}C_{i\pm2}$, indices considered *modulo* 3. The theorem then is that the bottom row consists of collinear points.

I do not know whether Pappus (or Euclid) recognized a single theorem lying behind Lemmas VIII, XII, and XIII. He may have recognized a similarity between the lemmas, but not a satisfactory way to prove the lemmas once for all. He may have thought it important to treat each case individually. By constrast, projective geometry collapses all cases into one.

Pappus knew, and probably Euclid before him knew [27, p. 594], that all of the conics sections can be given a single expression as the locus of points, whose distances from a given *focus* and a given *directrix* have a given ratio. In analytic geometry, we say that the conic sections are just the curves given by quadratic equations. However, when Omar Khayyām used conic sections to solve what we should call cubic equations, he considered several cases, depending on what we should call the signs of the coefficients of the equations [21, pp. 556–60]. For example, for the case where "a cube and sides are equal to squares and numbers," we can write the problem as the equation

$$x^3 + b^2x = cx^2 + b^2d,$$

which we manipulate into

$$\frac{x^2}{b^2} = \frac{d-x}{x-c}.$$

We can define y in terms of a solution so that

$$\frac{x}{b} = \frac{y}{x-c} = \frac{d-x}{y}.$$

Thus we solve the original equation by finding the intersection of the two conics given by

$$x^2 - cx = by, \qquad y^2 + (x-c)\cdot(x-d) = 0.$$

These are normally a parabola and a circle; however, if we have allowed negative coefficients, then we may have had to let b be imaginary. This matters if we want to construct real solutions.

In Rule Four of the posthumously published *Rules for the Direction of the Mind* [9, 373, p. 17], Descartes writes of a method that is

> so useful . . . that without it the pursuit of learning would, I think, be more harmful than profitable. Hence I can readily believe that the great minds of the past were to some extent aware of it, guided to it even by nature alone . . . This is our experience in the simplest of sciences, arithmetic and geometry: we are well aware that the geometers of antiquity employed a sort of analysis which they went on to apply to the solution of every problem, though they begrudged revealing it to posterity. At the present time a sort of arithmetic called "algebra" is flourishing, and this is achieving for numbers what the ancients did for figures.

We have already observed that Book VII of Pappus's *Collection* concerned the *Treasury of Analysis.* The term "treasury" is a modern flourish; but our word "analysis" is a transliteration of the Greek of writers like Pappus. It means freeing up, or dissolving. As Pappus describes it, mathematical analysis is assuming what you are trying to find, so that you can work backwards to see how to get there. We do this today by giving what we want to find a name x.

Today we think of conic sections as having *axes*: one for the parabola, and to each for the ellipse and hyperbola. The notion comes from Apollonius; but for him, an axis is just a special case of a *diameter.* A diameter of a conic section bisects the chords of the section that are parallel to a certain straight line. This straight line is the tangent drawn at a point where the diameter meets the section. Apollonius shows that *every* straight line through the center of an ellipse or hyperbola is a diameter in this sense; and every straight line parallel to the axis is a diameter of a parabola. Like Euclid's and Pappus's proofs, but unlike the proofs of analytic geometry, Apollonius's proofs rely on *areas.* There are areas of scalene triangles and non-rectangular parallelograms. In Appendix D we considered a locus problem in terms of *distances* from several given straight lines. For Pappus, what is involved is not distances as such, but the lengths of segments drawn to the given lines at given angles, which are not necessarily right angles. The apparently greater generality is trivial. This is why Descartes can solve a five-line problem using algebra. But if one is going to prove that a straight line parallel to the axis of a parabola is a diameter, one cannot just treat all angles as right. Apollonius may have had a secret weapon in coming up with his propositions about conic sections; but I don't think it was Cartesian analysis.

F. What happened in 2016

Of Pappus's lemmas for Euclid's *Porisms,* students presented six: VIII, IV, III, X, XI, and XII, in that order. Lemmas VIII, XII, and XIII are cases of what is now known as Pappus's Theorem, while Lemmas III, X, and XI are needed to prove XII and XIII. We skipped Lemma XIII in class, its proof being similar to that of XII. Lemma IV is effectively what I shall call the Quadrangle Theorem, although Coxeter gives it no name [5, **14.41,** p. 240]. There is a related theorem called Desargues's Involution Theorem by Field and Gray [15, p. 54]; Coxeter describes this as "the theorem of the quadrangular set" [5, **14.56,** p. 246].

Two students volunteered to present the first two (VIII and IV) of Pappus's lemmas above. For the next two lemmas (III and X), volunteers were not forthcoming, and so I picked two more students. When they had fulfilled their assignments, only two more students were still in class; I asked them to prepare the last two lemmas (XI and XII) for the next day. Class met at 8 A.M., a difficult time for many. Nonetheless, some absent students did return the next day.

Every presenter of a lemma came more or less prepared for the job, though sometimes needing help from the audience. Class was mostly in Turkish, except on the last day or two, when only I was speaking: with the remaining students' permission, I switched mostly to English.

In the following week, class was in the afternoon, as one returning student had begged for it to be. Most students in the second week were new. With a couple of notable exceptions, they did not prepare their presentations well. Some of them left the Village early, earlier than I did, without telling me, and having accepted assignments for the day (Friday) when they would be gone. I have doubts about how well even the remaining students understood Lobachevski's non-Euclidean conception of parallelism: they seemed to persist in their Euclidean notions.

On the first day of that second week, I reviewed the Euclidean geometry not requiring the Fifth Postulate that Lobachevski summarizes in his Theorems 1–15. This is the geometry of Propositions 1–28 of Book I of the *Elements.* There is also some solid geometry from Book XI, though I did not go into this. I do not know how much the review of Euclidean geometry meant to students who, unlike those at Mimar Sinan, had not read Euclid in the first place. I gave away the plot by describing the Poincaré half-plane model for Lobachevskian geometry; but given the quality of later student presentations, I have doubts that the model made much sense. If that part of the course is repeated, it should probably be coupled with a reading of Euclid; and then it would need a full week, if not two.

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