İstanbul Model Theory Seminar Notes 2012

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We are studying Hrushovski's 'Stable group theory and approximate subgroups' [5]. Secondary sources include

- Notes by Lou van den Dries [9],
- Terence Tao's blog [8].

These notes were first prepared for my talk on March 29 and then revised afterwards. I consulted my notes from earlier talks by Piotr Kowalski (February 16) and Gönenç Onay (February 23 and March 1; the three sessions between then and March 29 were devoted to talks by Bruno Poizat and Cédric Milliet on other matters).

The appendix contains notes that I wrote soon after March 1, in an attempt to justify the trouble van den Dries [9] takes to establish notation for many-sorted structures.

I expect to speak again on April 5 and then to add to these notes (and perhaps edit what is already here).

Contents

1. Setting

We fix a complete theory T. The signature of T may be *many-sorted*; this means there are variables for each sort, and function-symbols and predicates 'know' which sorts their arguments can come from.

We let letters like x and y denote (finite) tuples of (distinct) variables. In particular, x is of the form $(x_i : i < n)$, where each x_i belongs to a sort $s(i)$.

Say $\mathfrak{M} \models T$. We denote by M_x the set of instantiations of x in $\mathfrak{M}$; that is,

$$M_x = \prod_{i<n} M_{s(i)}.$$

Let A be a set of parameters from $\mathfrak{M}$. On the set of formulas in x over A we have the interpretation map $\varphi \mapsto \varphi^{\mathfrak{M}}$, where

$$\varphi^{\mathfrak{M}} = \{a \in M_x : \mathfrak{M} \models \varphi(a)\};$$

the range of the map is called

$$\mathrm{Def}_A(M_x),$$

the set of subsets of M_x that are definable over A. We identify formulas that have the same interpretation:

$$\varphi = \psi \iff \varphi^{\mathfrak{M}} = \psi^{\mathfrak{M}}.$$

Since T is complete, this identification is independent of choice of $\mathfrak{M}$. Considered under this identification, the set of formulas in x over A is

$$\mathrm{L}_x(A),$$

and this is a *Boolean algebra,* the **Lindenbaum algebra** in x over A with respect to $\mathfrak{M}$. This algebra is isomorphic to $\mathrm{Def}_A(M_x)$ under $\varphi \mapsto \varphi^{\mathfrak{M}}$:

$$\begin{aligned} (\varphi \vee \psi)^{\mathfrak{M}} &= \varphi^{\mathfrak{M}} \cup \psi^{\mathfrak{M}}, & \top^{\mathfrak{M}} &= M_x, \\ (\varphi \wedge \psi)^{\mathfrak{M}} &= \varphi^{\mathfrak{M}} \cap \psi^{\mathfrak{M}}, & \bot^{\mathfrak{M}} &= \varnothing. \end{aligned}$$

Here $\top$ stands for $\bigwedge_{i<n} x_i = x_i$, and $\bot$ for its negation.

Like every Boolean algebra, $\mathrm{L}_x(A)$ has a **Stone space,**

$$\mathrm{St}_x(A),$$

the set of *ultrafilters* of the algebra. First, a **filter** of $\mathrm{L}_x(A)$ is a nonempty subset F such that

- F is closed under $\land$;[1]
- if $\varphi \in F$, then $\varphi \lor \psi \in F$.

Then an **ultrafilter** is a maximal proper filter, equivalently a filter p such that

- $\varphi \notin p$ if and only if $\neg\varphi \in p$.

One way to prove this equivalence is to note that $\mathrm{L}_x(A)$ is also an *associative ring* with addition $(\varphi, \psi) \mapsto \neg(\varphi \leftrightarrow \psi)$ and multiplication $\land$; it is in particular a *Boolean ring*[2] because

$$\varphi \land \varphi = \varphi.$$

Then filters are duals of ideals: F is a filter if and only if $\{\neg\varphi \colon \varphi \in F\}$ is an ideal. Quotients of commutative rings by maximal ideals are fields, and the only Boolean field is the two-element field. This gives the characterization of ultrafilters.

There is an embedding $\varphi \mapsto [\varphi]$ of the Boolean algebra $\mathrm{L}_x(A)$ in the algebra $\mathscr{P}(\mathrm{St}_x(A))$, where

$$[\varphi] = \{p \colon \varphi \in p\}.$$

In particular

$$[\varphi \land \psi] = [\varphi] \cap [\psi], \qquad [\top] = \mathrm{St}_x(A), \qquad [\neg\varphi] = \mathrm{St}_x(A) \smallsetminus [\varphi].$$

[1] We can formulate this condition as the closure of F under $(\varphi_i \colon i < n) \mapsto \bigwedge_{i<n} \varphi_i$ for all n in ω. But in case $n = 0$, this nullary operation can be understood as the formula $\top$ (since this formula is true if only if φ_i is true for each i in $\varnothing$. Thus this formulation implies that F is nonempty.

[2] Boolean rings are commutative and have characteristic 2: if squaring is the identity, then $x + y = (x + y)^2 = x + xy + yx + y$, so $0 = xy + yx$.

Thus $\{[\varphi]\colon \varphi \in \mathrm{L}_x(A)\}$ is a basis of open sets for a topology on $\mathrm{St}_x(A)$, and these basic open sets are also closed. The topology is compact.[3]

Similarly, $\{\varphi^{\mathfrak{M}}\colon \varphi \in \mathrm{L}_x(A)\}$ is a basis for the ***A*-topology** on M_x. Closed sets in this topology can be called ***A*-closed;** open sets, ***A*-open.** Thus,

- the A-closed sets are are the **$\bigwedge$-definable,** or **type-definable,** sets over A;
- the A-open sets are the **$\bigvee$-definable** sets over A.

From M_x to $\mathrm{St}_x(A)$ there is a map $a \mapsto \mathrm{tp}(a/A)$, where

$$\mathrm{tp}(a/A) = \{\varphi\colon a \in \varphi^{\mathfrak{M}}\}.$$

This map is continuous with respect to the A-topology, because under the map the inverse image of $[\varphi]$ is $\varphi^{\mathfrak{M}}$. If the image of $\varphi^{\mathfrak{M}}$ is $[\varphi]$—that is, if the map is surjective—, then the map is also closed and open.

We can make the map surjective, and we can ensure that the inverse image of a singleton is one orbit under $\mathrm{Aut}(\mathfrak{M}/A)$. We do this by replacing $\mathfrak{M}$ with a *monster model* or **universal domain,** $\mathbb{U}$.

- $\mathbb{U}$ is $|\mathfrak{M}|^+$-saturated (for all $\mathfrak{M}$ that we shall consider), so we may assume
$$\mathfrak{M} \prec \mathbb{U}.$$
- $\mathbb{U}$ is $|\mathfrak{M}|^+$-homogeneous: for all A from $\mathfrak{M}$, if $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ then $a \mapsto b$ extends to an automorphism of $\mathbb{U}$.

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[3]One proof of this is as follows. Suppose Γ is a subset of $\mathrm{L}_x(A)$ such that, for every finite subset Δ of Γ,

$$\bigcap_{\varphi\in\Delta} [\varphi] \neq \varnothing.$$

Since Δ is finite, this means $[\bigwedge_{\varphi\in\Delta} \varphi] \neq \varnothing$, that is, $\bigwedge_{\varphi\in\Delta} \varphi$ generates a proper filter of $\mathrm{L}_x(A)$ (namely the set of formulas implied by $\bigwedge_{\varphi\in\Delta} \varphi$). This being so for *every* finite subset Δ of Γ, the set Γ itself generates a proper filter (namely the set of formulas implied by $\bigwedge_{\varphi\in\Delta} \varphi$ for *some* finite subset Δ of Γ). This filter embeds in an ultrafilter p, for the same reason that proper ideals embed in maximal ideals. Thus

$$p \in \bigcap_{\varphi\in\Gamma} [\varphi].$$

Therefore $\mathrm{St}_x(A)$ is compact. For the compactness of first-order logic, see note 6 below.

2. Keisler measures

Now $\mathfrak{M}$ is just some structure, and A is a set of parameters from $\mathfrak{M}$. A function μ_x from $\mathrm{L}_x(A)$ to the closed interval $[0,\infty]$ of $\mathbb{R}\cup\{\infty\}$ is a **Keisler measure** if

$$\mu_x(\varphi\vee\psi)=\mu_x(\varphi)+\mu_x(\psi)$$

whenever φ and ψ are mutually contradictory, that is, $\varphi\wedge\psi=\bot$; in a word, μ_x is *additive.* Usually also

$$\mu_x(\top)=1,$$

in which case μ_x is a *probability measure* and

$$\mu_x(\neg\varphi)=1-\mu_x(\varphi).$$

We may consider μ_x also as having domain $\mathrm{Def}_A(M_x)$ (in which case the term *measure* is more suggestive). We may also consider Keisler measures on $\mathrm{Def}_A(X)$ for some A-open subset X of $\mathfrak{M}_x$.

Example 1. If $p\in\mathrm{St}_x(A)$, we can define μ_x on $\mathrm{L}_x(A)$ by

$$\mu_x(\varphi)=\begin{cases}1, & \text{if } \varphi\in p,\\ 0, & \text{otherwise,}\end{cases}$$

or on $\mathrm{Def}_A(M_x)$ by

$$\mu_x(X)=\begin{cases}1, & \text{if } p=\mathrm{tp}(a/A) \text{ for some } a \text{ in } X,\\ 0, & \text{otherwise.}\end{cases}$$

Example 2. If $\mathfrak{M}$ is finite and one-sorted, and $x=(x_i\colon i<n)$, we can define[4]

$$\mu_x^{\mathfrak{M}}(\varphi)=\frac{|\varphi^{\mathfrak{M}}|}{|M|^n}.$$

Thus $\mu_x^{\mathfrak{M}}$ is the **counting measure** on $\mathrm{L}_x(M)$. Given an infinite family $(\mathfrak{M}^i\colon i\in I)$ of finite structures (of the same signature), we can form an ultraproduct $\mathfrak{N}$ of the family, and then on $\mathrm{L}_x(N)$ we can define $\mu_x(\varphi)$ as the standard part of the image of $(\mu_x^i(\varphi)\colon i\in I)$ in ${}^*[0,1]$ or just ${}^*\mathbb{R}$. Indeed, in the Ravello volume, Hrushovski [4, Addendum, p. 209] says:

[4] This example, given by Hrushovski [5, §2.6, p. 197], is mentioned on the last of Pillay's slides [6].

> In an ultraproduct k of finite fields, one has the nonstandard counting measure; and one can let $\mu(V) = \operatorname{st}(|V|/|k|^{\dim(V)})$ (the standard part)... This recovers the generalization of Lang–Weil in Kieffe and [2][5]...

What is going on is the following. We select a non-principal ultrafilter of the Boolean algebra of subsets of I; elements of this ultrafilter will be considered *large.* Then $\mathfrak{N}$ is the Cartesian product of the structures $\mathfrak{M}^i$, but with two elements identified if their entries agree on a large set of indices. By the result called *Łoś's Theorem,* a sentence is true in $\mathfrak{N}$ if and only if it is true in $\mathfrak{M}_i$ for each i in a large set of indices.[6] In particular, ${}^*\mathbb{R}$ is the *ultrapower* that results when each $\mathfrak{M}^i$ is $\mathbb{R}$. In this case, ${}^*\mathbb{R}$ can be understood as the quotient of $\mathbb{R}^I$ by a non-principal maximal ideal P. By Łoś's Theorem, ${}^*\mathbb{R}$ is an ordered field, and it is non-Archimedean. If S is the ring of its finite elements, and $\mathfrak{m}$ is the ring of its infinitesimal elements, then $\mathfrak{m}$ is a maximal ideal of S, and the quotient map $x \mapsto x+\mathfrak{m}$ from S to $S/\mathfrak{m}$ is an isomorphism when restricted to the image of $\mathbb{R}$ in ${}^*\mathbb{R}$ under $x \mapsto (x\colon i \in I) + P$. Thus the standard part map from S to $\mathbb{R}$ is induced; this is a ring homomorphism, and in particular μ_x is additive.

To make μ_x *definable* in $\mathfrak{N}$, for each formula $\varphi(x,y)$, for each α in $\mathbb{Q}$, we introduce a new atomic formula, denoted by

$$\mathsf{Q}_\alpha x\ \varphi(x,y),$$

and we expand each $\mathfrak{M}^i$ so that

$$\mathsf{Q}_\alpha x\ \varphi(x,y)^{\mathfrak{M}^i} = \{b \in M^i_y \colon \mu^i_x(\varphi(x,b)) \leqslant \alpha\}.$$

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[5]Cited by Hrushovski as Chatzidakis, van den Dries, Macintyre, 'Definable sets over finite fields', Paris 7 Logique prepublication **23**; but I have not yet been able to obtain either version.

[6]Now we can prove compactness of first-order logic. Say Γ is a set of sentences, and every finite subset Δ of Γ has a model, $\mathfrak{M}_\Delta$. A certain ultraproduct of these $\mathfrak{M}_\Delta$ will be a model of Γ. Indeed, writing $\mathscr{P}_{\mathrm{f}}(\Gamma)$ for the set of these Δ, we let (Δ) be the set of all elements of $\mathscr{P}_{\mathrm{f}}(\Gamma)$ that include Δ. Then $(\Delta)\cap(\Delta') = (\Delta\cup\Delta')$. Thus the sets (Δ) generate a proper filter F of $\mathscr{P}(\mathscr{P}_{\mathrm{f}}(\Gamma))$. Now we take the ultraproduct $\mathfrak{N}$ of the $\mathfrak{M}_\Delta$ with respect to an ultrafilter that includes F. For each Δ in $\mathscr{P}_{\mathrm{f}}(\Gamma)$, the set (Δ) of indices is large, and Δ is true in $\mathfrak{M}_\Theta$ for each Θ in (Δ); thus Δ is true in $\mathfrak{N}$. This is the proof of Bell and Slomson [1, Thm 5.4.1], who trace it to a 1958 article by Morel, Scott, and Tarski.

Repeat ω times (so we have a formula $\mathsf{Q}_\alpha x\ \varphi(x,y)$ for every formula φ). For every formula φ and every α in $\mathbb{Q}$, the following are equivalent:

$$\mu_x(\varphi(x,b)) \leqslant \alpha,$$
$$\{i\colon \mu_x^i(\varphi(x,b)) \leqslant \alpha\} \text{ is large},$$
$$\mathfrak{N} \models \mathsf{Q}_\alpha x\ \varphi(x,b).$$

Thus
$$\mu_x(\varphi(x,b)) = \inf\{\alpha\colon \mathfrak{N} \models \mathsf{Q}_\alpha x\ \varphi(x,b)\}.$$

We can take this as the *definition* of μ_x, and then we can establish additivity as in the Dedekind construction of $\mathbb{R}$:

- If $\gamma, \delta \in \mathbb{R}$, then
$$\inf\{x \in \mathbb{Q}\colon \gamma \leqslant x\} + \inf\{y \in \mathbb{Q}\colon \delta \leqslant y\} = \inf\{z \in \mathbb{Q}\colon \gamma + \delta \leqslant z\}.$$
- If $\varphi(x,b) \wedge \psi(x,b) = \bot$, then the sentences
$$\mathsf{Q}_\alpha x\ \varphi(x,b) \wedge \mathsf{Q}_\beta x\ \psi(x,b) \to \mathsf{Q}_{\alpha+\beta} x\ (\varphi(x,b) \vee \psi(x,b),$$
$$\neg\mathsf{Q}_\alpha x\ \varphi(x,b) \wedge \neg\mathsf{Q}_\beta x\ \psi(x,b) \to \neg\mathsf{Q}_{\alpha+\beta} x\ (\varphi(x,b) \vee \psi(x,b))$$
are true in each $\mathfrak{M}^i$ and therefore in $\mathfrak{N}$.

A curiosity is that, while

$$\begin{aligned} \mu_x(\varphi(x,b)) < \alpha &\implies \mathfrak{N} \models \mathsf{Q}_\alpha x\ \varphi(x,b) \\ &\implies \mu_x(\varphi(x,b)) \leqslant \alpha, \end{aligned}$$

we need not have the converse of the second implication. That is, possibly $\mu_x(\varphi(x,b)) = \alpha$, although $\mathfrak{N} \models \neg\mathsf{Q}_\alpha x\ \varphi(x,b)$, because $\mu_x^i(\varphi(x,b)) > \alpha$ for a large set of i.

Suppose now μ_x is a Keisler measure on $\mathrm{L}_x(\mathbb{U})$. If for all y and all formulas $\varphi(x,y)$ in no parameters,[7] the value of $\mu_x(\varphi(x,b))$ depends only on $\operatorname{tp}(b/A)$, then μ_x is **A-invariant.** In this case, the function

$$b \mapsto \mu_x(\varphi(x,b))$$

[7] The qualification that φ must have no parameters is not made explicit by Hrushovski [5, §2.6, p. 197].

on $\mathbb{U}_y$ has the factor $b \mapsto \operatorname{tp}(b/A)$; the other factor can be called μ_φ,[8] so that we have the following commutative diagram (where we rely on the $|A|^+$-saturation of $\mathbb{U}$ to be able to define μ_φ on all of $\mathrm{St}_y(A)$):

$$\begin{tikzcd}
\mathbb{U}_y \arrow[r, "\varphi"] \arrow[d, "\mathrm{tp}"'] \arrow[dr] & \mathrm{L}_x(\mathbb{U}) \arrow[d, "\mu_x"] \\
\mathrm{St}_y(A) \arrow[r, "\mu_\varphi"'] & {[0,\infty]}
\end{tikzcd}$$

Theorem. *For a Keisler measure μ_x on $\mathrm{L}_x(\mathbb{U})$, the following are equivalent:*

1. *For each y disjoint from x, for each $\varphi(x,y)$ in $\mathrm{L}_{x,y}(\varnothing)$, the function $b \mapsto \mu_x(\varphi(x,b))$ from $\mathbb{U}_x$ to $[0,\infty]$ is continuous with respect to the A-topology on $\mathbb{U}_x$.*
2. *μ_x is A-invariant, and for each y disjoint from x, for each $\varphi(x,y)$ in $\mathrm{L}_{x,y}(\varnothing)$, the induced function μ_φ (from $\mathrm{St}_y(A)$ to $[0,\infty]$ such that $\mu_x(\varphi(x,b)) = \mu_\varphi(\operatorname{tp}(b/A))$) is continuous.*

Proof. In the presence of A-invariance, since the map $b \mapsto \operatorname{tp}(b/A)$ is continuous and open, continuity of the 'diagonal' maps $b \mapsto \mu_x(\varphi(x,b))$ with respect to the A-topology is equivalent to continuity of the μ_φ.

If A-invariance fails, that is, $\operatorname{tp}(c/A) = \operatorname{tp}(c'/A)$ for some c and c' in $\mathbb{U}_x$, but

$$\mu_x(\varphi(x,c)) \leqslant \alpha < \mu_x(\varphi(x,c')),$$

then $\{b\colon \mu_x(\varphi(x,b)) \leqslant \alpha\}$ is not A-closed, since every A-closed (or A-open) set contains both c and c', or neither. $\square$

Under the equivalent conditions of the theorem, μ_x is called ***A*-definable.**[9] If we work with an arbitrary structure $\mathfrak{M}$ instead of $\mathbb{U}$, we may

[8] Van den Dries [9, p. 11] uses μ_φ for the whole function $b \mapsto \mu_x(\varphi(x,b))$. I use it here for the factor mainly to have a label for the commutative diagram. Hrushovski just calls the factor g.

[9] Hrushovski just says μ_x is A-definable if it is A-invariant and *in addition* the maps μ_φ—his g—are continuous.

take the first condition as A-definability. This condition means just that for each φ and each α in $\mathbb{Q}$ the sets

$$\{b\colon \mu_x(\varphi(x,b)) \leqslant \alpha\}, \qquad\qquad \{b\colon \mu_x(\varphi(x,b)) \geqslant \alpha\}$$

are A-closed. So we have this continuity in Example 2.

In Example 1, where

$$\mu_x(\varphi) = \begin{cases} 1, & \text{if } \varphi \in p, \\ 0, & \text{otherwise,} \end{cases}$$

then we have A-definability of μ_x if and only if the sets

$$\{b\colon \varphi(x,b) \in p\}$$

are A-clopen, that is, A-definable. This condition is that p itself is *definable* over A.

More generally, suppose B is another parameter set. If $\mathbb{U}_x = \prod_{i<n} \mathbb{U}_{s(i)}$, we can let

$$B_x = \prod_{i<n} (\mathbb{U}_{s(i)} \cap B).$$

This set has the A-topology induced from $\mathbb{U}_x$, and then an **A-definable** subset of B_x can be understood as an A-clopen subset.

Now suppose $A \subseteq B$. An element p of $\mathrm{St}_x(B)$ is called **definable** over A [7, Defn 1.1] if for each y disjoint from x, for each $\varphi(x,y)$ in $\mathrm{L}_{x,y}(\varnothing)$, there is a formula $\mathrm{d}_x\,\varphi(x,y)$ in $\mathrm{L}_y(A)$ such that for all b in B_y

$$\varphi(x,b) \in p \iff \mathbb{U} \models \mathrm{d}_x\,\varphi(x,b).$$

We can define μ_x on $\mathrm{L}_x(B)$ as before; then this measure is A-definable as before if and only if p is A-definable.

Theorem. *Suppose $A \subseteq B$. All types in x over B are A-definable if and only if all $\mathbb{U}$-definable subsets of B_x are A-definable.*

Proof. Exercise, or see [3, §6.7, 'Definability of types']. □

The present situation can be depicted in one big commutative diagram:

$$\begin{tikzcd}
\mathrm{L}_{x,y}(\varnothing) \times B_y \arrow[rr] \arrow[dr] \arrow[dd, dotted] & & \mathrm{L}_x(B) \arrow[dd] \\
& \mathscr{P}(B_y) \times B_y \arrow[dr] & \\
\mathrm{L}_y(A) \times B_y \arrow[rr] \arrow[ur, dotted] & & \mathrm{L}(B)
\end{tikzcd}$$

Here $\mathrm{L}(B)$ is the set of sentences over B *modulo* T, so it is the 2-element Boolean algebra $\{\bot, \top\}$. The first component of the main diagonal map takes $(\varphi(x,y), b)$ to (C, b), where $C = \{c \in B_y \colon \varphi(x,c) \in p\}$; the next component takes this pair to $\top$ if and only if $b \in C$.

A type over A (that is, an element of some $\mathrm{St}_x(A)$) is called simply *definable* if it is A-definable. Then all types over $\varnothing$ are definable. The following are equivalent:

1. The theory T is *stable* (that is, κ-stable for some κ).
2. All types over *models* of T are definable [7, Cor. 1.21].
3. All types over all parameter sets are definable [3, Cor 6.7.11].

So here is a hint that definable Keisler measures are a generalization of types for unstable theories.

3. Ideals

if μ_x is a Keisler measure on $\mathrm{L}_x(A)$, then the set $\{\varphi \colon \mu_x(\varphi) = 0\}$ is an *ideal* of the Boolean ring $\mathrm{L}_x(A)$.

The forking ideal

Another example is the *forking ideal.*[10] A formula φ **forks** over A if $\varphi \neq \bot$ and there are finitely many formulas θ_i such that

[10] Gönenç talked about this on March 1.

- $\varphi \to \bigvee_i \theta_i = \top$, that is, $T \vdash \varphi \to \bigvee_i \theta_i$;
- each θ_i *divides* over A.

For present purposes, one might say that $\perp$ also forks, since as things are the *forking ideal* in x over A will be the set

$$\{\varphi \in \mathrm{L}_x(\mathbb{U})\colon \varphi \text{ forks over } A\} \cup \{\perp\}.$$

If $\varphi(x, y)$ is a formula[11] and $b \in \mathbb{U}_y$, the formula $\varphi(x, b)$ **divides** over A if b belongs to an *indiscernible* sequence $(b_i\colon i < \omega)$ over A such that $\{\varphi(x, b_i)\colon i < \omega\}$ is inconsistent. Recall that the indiscernibility means that, for all m in ω, if

$$n(0) < \cdots < n(m-1) < \omega, \tag{$*$}$$

then for all formulas ψ over A

$$T \vdash \psi(b_0, \ldots, b_{m-1}) \leftrightarrow \psi(b_{n(0)}, \ldots, b_{n(m-1)}).$$

So in $(\mathbb{Q}, <)$, every increasing sequence of elements is indiscernible over $\varnothing$. In a vector space, a basis is indiscernible as a *set* (under any ordering it is an indiscernible sequence).

If $\{\varphi(x, b_i)\colon i < \omega\}$ is inconsistent, then by compactness the subset $\{\varphi(x, b_i)\colon i < m\}$ is inconsistent for some m in ω; that is, the formula $\bigwedge_{i<m} \varphi(x, b_i)$ is ('identically') false (in T). By indiscernibility, if $(*)$ holds, then $\bigwedge_{i<m} \varphi(x, b_{n(i)})$ is false. In short, $\{\varphi(x, b_i)\colon i < \omega\}$ is **m-inconsistent.**

Easily, if φ divides, so does $\varphi \wedge \psi$. However, if $(b_i\colon i < \omega)$ and $(c_i\colon i < \omega)$ are indiscernible over A, it does not follow that $(b_i c_i\colon i < \omega)$ is indiscernible; so it is not immediate that if φ and ψ divide, so does $\varphi \vee \psi$.

However, the definition of *forking* ensures that the forking formulas, along with $\perp$, do compose an ideal. Also, in *stable* theories, forking and dividing are the same [7, ch. 6].

[11] Probably φ has no parameters.

Possible properties

Now say X is an A-definable set.[12] We may form the Boolean ring $\mathrm{Def}_{\mathbb{U}}(X)$ of definable subsets of X. If $X = \theta^{\mathbb{U}}$ for some θ in $\mathrm{L}_x(A)$, we can define

$$\mathrm{L}_X(\mathbb{U}) = \{\theta \wedge \varphi \colon \varphi \in \mathrm{L}_x(\mathbb{U})\}$$

and identify this with $\mathrm{Def}_{\mathbb{U}}(X)$. An ideal of the Boolean ring $\mathrm{L}_X(\mathbb{U})$ is **A-invariant** if for all formulas $\varphi(x, y)$ and all b in $\mathbb{U}_y$, the answer to the question of whether $\varphi(x, b)$ is in the ideal depends only on $\mathrm{tp}(b/A)$.

The forking ideal is invariant. If μ_x is A-invariant, then so is the ideal $\{P \colon \mu_x(P) = 0\}$ mentioned above.

Let I be an ideal of $\mathrm{Def}_{\mathbb{U}}(X)$. A subset Φ of $\mathrm{L}_X(\mathbb{U})$ (that is, a partial type in x defining a subset of X) is **I-wide** if Φ implies no formula in I, that is, the *filter* generated by Φ does not intersect I. If I is the zero ideal of a measure, then I-wideness of Φ means no element of Φ has measure 0. The filter

$$\{\theta \wedge \neg\varphi \colon \varphi \in I\}$$

is the maximal I-wide partial type, unless I is the improper ideal.

We generalize to the case where X is merely A-open. If $X = \bigcup_i X_i$, the X_i being definable over A, we let

$$\mathrm{L}_X(\mathbb{U}) = \bigcup_i \mathrm{L}_{X_i}(\mathbb{U}).$$

Then a subset I is an ideal if each $I \cap \mathrm{L}_{X_i}(\mathbb{U})$ is an ideal. This is Hrushovski's definition. Alternatively, it seems we could just define

$$\mathrm{L}_X(\mathbb{U}) = \{\varphi \in \mathrm{L}_x(\mathbb{U}) \colon \varphi^{\mathbb{U}} \subseteq X\}.$$

This may not be a Boolean *algebra.* It is still a Boolean *ring,* possibly without a unit; so it has ideals and filters. If I is a proper ideal, then the maximal I-wide partial type is

$$\{\psi \wedge \neg\varphi \colon \psi \in \mathrm{L}_X(\mathbb{U}) \text{ and } \varphi \in I\}.$$

[12] Van den Dries [9, p. 12] allows X to be A-open. Hrushovski, and therefore we, do this presently.

4. S_1 rank

In the Ravello volume, Hrushovski [4, Defn 4.1, p. 176] defines $\mathrm{S}_1(\theta)$ for formulas θ in $\mathrm{L}_x(\mathbb{U})$ (here $\mathbb{U}$ need only be ω-saturated):

1. $\mathrm{S}_1(\theta) > 0$ if $|\theta^{\mathbb{U}}| \geqslant \omega$.
2. $\mathrm{S}_1(\theta) \geqslant n+1$ if for some set A of parameters, $\theta \in \mathrm{L}_x(A)$ and there is an indiscernible sequence $(b_i : i \in I)$ over A and a formula $\varphi(x,y)$[13] such that

$$\mathrm{S}_1(\varphi(x,b_1) \wedge \varphi(x,b_2)) < n, \qquad \mathrm{S}_1(\theta \wedge \varphi(x,b_i)) \geqslant n$$

   for each i in I.

(Hrushovski has $>$ for $\geqslant$ and $\leqslant$ for $<$ in the second part of this definition. He notes that $\mathrm{S}_2(\theta)$ can be defined the same way, but with $\mathrm{S}_2(\theta) > 0$ if $\mathrm{S}_1(\theta) > n$ for all n in ω.)

Compare with Morley rank:

1. $\mathrm{RM}(\theta) \geqslant 0$ if $|\theta^{\mathbb{U}}| > 0$.
2. $\mathrm{RM}(\theta) \geqslant \alpha + 1$ if there is a sequence $(\psi_i : i < \omega)$ of formulas (with parameters) such that

$${\psi_i \wedge \psi_j}^{\mathbb{U}} = \varnothing, \qquad \mathrm{RM}(\theta \wedge \psi_i) \geqslant \alpha$$

   whenever $i < j < \omega$.

Then $\mathrm{S}_1(\theta)$ is the least n such that $\mathrm{S}_1(\theta) \not\geqslant n+1$.

(For a subset Γ of $\mathrm{L}_x(\mathbb{U})$, $\mathrm{S}_1(\Gamma)$ is the least of the $\mathrm{S}_1(\theta)$ such that θ is in the filter generated by Γ. Then $\mathrm{S}_1(a/B) = \mathrm{S}_1(\mathrm{tp}(a/B))$. Similarly for RM.)

Suppose θ in $\mathrm{L}_x(\mathbb{U})$ has Morley rank 1 and Morley *degree* 1 (that is, no two disjoint definable subsets have lower rank). Then θ or $\theta^{\mathbb{U}}$ is called **strongly minimal.** Every definable subset of $\theta^{\mathbb{U}}$ is finite or cofinite; moreover, by compactness, for every $\varphi(x,y)$, there is n in ω such that $\theta \wedge \varphi(x,a)^{\mathbb{U}}$ is smaller than n or infinite. Hrushovski asserts the same in case θ has S_1 rank 1:

[13] Presumably with no parameters?

Theorem ([4, Lem. 4.1, p. 176]). *Suppose θ in $\mathrm{L}_x(\mathbb{U})$ has S_1 rank 1. Then for every $\varphi(x,y)$, there is n in ω such that $\theta \wedge \varphi(x,a)^{\mathbb{U}}$ is smaller than n or infinite.*

Proof. Suppose not, so that there are a_m for infinitely many m such that $\theta \wedge \varphi(x,a_m)^{\mathbb{U}}$ has size m. Write $\theta \wedge \varphi(x,a_m)^{\mathbb{U}}$ as D_m. For every m, there is a least m' such that $m < m'$ and, for infinitely many n

$$D_m \cap D_{m'} = D_m \cap D_n. \tag{†}$$

Hence there is an infinite set of indices such that (†) holds whenever $m < m' \leqslant n$.[14] In this case

$$D_m \smallsetminus D_{m'} = D_m \smallsetminus D_n.$$

Thus the sets $D_m \smallsetminus D_{m'}$ are disjoint. Therefore their sizes are bounded, since $\mathrm{S}_1(\theta) = 1$.[15] This means that the sets $D_m \cap D_{m'}$ are unbounded in size. But by (†) they form a chain:

$$D_m \cap D_{m'} = D_m \cap D_{m'} \cap D_{m''} \subseteq D_{m'} \cap D_{m''}.$$

Perhaps by restricting the index set again[16], we may assume that the differences

$$(D_{m'} \cap D_{m''}) \smallsetminus (D_m \cap D_{m'})$$

are strictly increasing in size. Since they are disjoint, this contradicts $\mathrm{S}_1(\theta) = 1$. □

According to Hrushovski [4, p. 177],

> The fact that $\mathrm{S}_1(F) = 1$ in the case of pseudo-finite fields was shown in [2][17], using an extension of the Lang–Weil estimates.

[14] Hrushovski appeals to Ramsey's theorem for this.

[15] Apparently if the sizes were unbounded, by compactness we could assume that the parameters composed an indiscernible sequence.

[16] Hrushovski does not say this, but it seems to be needed.

[17] See note 5.

If X has ordinal[18] Morley rank α and has Morley *degree* 1 (that is, has no two disjoint definable subsets of its rank), then there is a Keisler measure on $\mathrm{L}_X(\mathbb{U})$ given by

$$\mu(\varphi) = \begin{cases} 1, & \text{if } \mathrm{RM}(\varphi) = \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

and this determines the ideal $\{\varphi\colon \mu(\varphi) = 0\}$. Even without the assumption that $\mathrm{dM}(X) = 1$, the set

$$\{\varphi\colon \mathrm{RM}(\varphi) < \alpha\}$$

is an ideal. Similarly

$$\{\varphi\colon \mathrm{S}_1(\varphi) < n + 1\}$$

is an ideal. If $\mathrm{S}_1(\theta) = n + 1$, then for all A such that $\theta \in \mathrm{L}_x(A)$, for all indiscernible $(b_i\colon i < \omega)$ over A and all $\varphi(x, y)$ [over A], if

$$\mathrm{S}_1(\varphi(x, b_0) \wedge \varphi(x, b_1)) < n,$$

then for some i in ω,

$$\mathrm{S}_1(\theta \wedge \varphi(x, b_i)) < n.$$

An arbitrary ideal I that is invariant over A is called an S_1 **ideal** (or $S1$ ideal) over A if it has the foregoing property, that is, for any $\varphi(x, y)$ over A and any indiscernible $(a_i\colon i < \omega)$ over A,

$$\text{if } \varphi(x, a_0) \wedge \varphi(x, a_1) \in I, \qquad \text{then } \varphi(x, a_0) \in I \tag{‡}$$

—equivalently, $\varphi(x, a_i) \in I$ for some and therefore all i in ω, by invariance of I and indiscernibility of the sequence.[19]

We can replace (‡) with the condition that for some or all n in ω

$$\text{if } \bigwedge_{i<2^n} \varphi(x, a_i) \in I, \qquad \text{then } \varphi(x, a_0) \in I.$$

Thus every S_1 ideal includes the forking ideal.[20]

[18] Piotr gave this example (with ‘finite’ for ‘ordinal’) on February 16.

[19] Hrushovski [5, Def. 2.8] just as the conclusion as $\varphi(x, a_i) \in I$ for some i in ω; van den Dries [9, p. 12] observes that it then holds for all i.

[20] Gönenç showed this on March 1 by the method of [9, Lem. 1.18, p. 12].

5. The stabilizer theorem

We are now ready to state what appears to be the central result (Theorem 3.5) of Hrushovski's paper. (It is van den Dries's [9, Thm 2.6].)

We let G be a group definable over a model $\mathfrak{M}_0$.

We let X be an arbitrary subset of G, and we define $\tilde{G} = \langle X \rangle$ (van den Dries calls this $\hat{X}$ [9, p. 17]). Then

$$\tilde{G} = \bigcup_{n \in \omega} (X \cup X^{-1})^{\leqslant n},$$

where $(X \cup X^{-1})^{\leqslant n}$ comprises the elements of $\tilde{G}$ that, as words in X, have length n or less.

We can form the isomorphic Boolean rings $\mathrm{Def}_{\mathbb{U}}(\tilde{G})$ and $\mathrm{L}_{\tilde{G}}(\mathbb{U})$. But (apparently) these are too big. Let

$$\mathrm{Def}_{\mathbb{U}}(\tilde{G})^* = \bigcup_{n \in \omega} (\mathrm{Def}_{\mathbb{U}}(\tilde{G}) \cap \mathscr{P}((X \cup X^{-1})^{\leqslant n})).$$

(The notation is mine, although van den Dries uses the star for restrictions to $XX^{-1}X$. Hrushovski refers to elements of $\mathrm{Def}_{\mathbb{U}}(\tilde{G})$ as definable, though to avoid confusion one might call them 'star-definable'.) We can let $\mathrm{L}_{\tilde{G}}(\mathbb{U})^*$ be the image of $\mathrm{Def}_{\mathbb{U}}(\tilde{G})^*$ in $\mathrm{L}_{\tilde{G}}(\mathbb{U})$.

Now let $\mathfrak{M}$ be another model (probably $\mathfrak{M}_0 \subseteq \mathfrak{M}$).

Suppose I is an ideal of $\mathrm{L}_{\tilde{G}}(\mathbb{U})^*$ that is

- $\mathfrak{M}$-invariant and
- S_1 over $\mathfrak{M}$.

Suppose also that I is invariant (or closed) under left and right translation by elements of $\tilde{G}$, that is, for all g in $\tilde{G}$,

$$\text{if } \varphi(x) \in I, \qquad \text{then } \varphi(g^{-1}x) \text{ and } \varphi(xg^{-1}) \text{ are in } I.$$

Let $q = \mathrm{tp}(a/\mathfrak{M})$ for some a in $\tilde{G}$, and suppose q is I-wide. That is, suppose $a \notin \varphi^{\mathbb{U}}$ for any φ in I, and let $q = \mathrm{tp}(a/\mathfrak{M})$.

Suppose further that for some other realization b of q, both $\mathrm{tp}(a/b\mathfrak{M})$ and $\mathrm{tp}(b/a\mathfrak{M})$ do not fork over $\mathfrak{M}$.

Then there will be a certain normal subgroup S of $\tilde{G}$ that is $\bigwedge$-definable over $\mathfrak{M}$.

S (or rather its defining partial type) will be I-wide.

Hrushovski says $S = (q^{-1}q)^2$, which we may write as

$$S = q^{-1}qq^{-1}q;$$

and he says $qq^{-1}q$ is a coset of S. He says moreover that, as a consequence of the theorem, $S \subseteq XX^{-1}XX^{-1}$, that is,

$$S \subseteq X^{-1}XX^{-1}X.$$

But where Hrushovski has q, van den Dries has $q(X)$, which would seem to mean the realizations of q that belong to X. (He does not write $q(\hat{X})$.)

A. Sorted structures

In the beginning, a **structure** is a set with distinguished *elements, operations,* and *relations.* The set is the **universe** of the structure. If this universe is A, then a **relation** is a subset of some Cartesian power A^n, where $n \in \omega$; and an **operation** is a function from some A^n to A. If R is a relation on A, and $\vec{b} \in R$, we write the **atomic sentence**

$$R\vec{b}.$$

In the binary case we usually write $b\ R\ c$ when $(b, c) \in R$. If f is an operation on A, then it is a certain kind of relation on A, namely a relation for which it is meaningful to write another **atomic sentence,**

$$f\vec{b} = c,$$

when $(\vec{b}, c) \in f$. In the ternary case we may write $b\ f\ c = d$ when $(b, c, d) \in f$. We may allow f to be a nullary operation, in which case f by itself denotes an element of A. We combine atomic formulas and quantify their variables in the usual way, for the sake of describing structures.

This definition of structure turns out to be needlessly limiting.

On a *field* K, division is not an operation; it is only a 'partial' operation, a function $(x, y) \mapsto x/y$ from $K \times (K \smallsetminus \{0\})$ to K. We normally want a binary operation $*$ to be 'total', so that $x * y$ is always meaningful. This is a notational convenience, which we can achieve in the present case by defining $x/0$ as 0. Alternatively, we can treat a field as *two* structures, abelian groups $(K, +)$ and $(K^\times, \cdot\,)$, with the appropriate interactions, including the function $(x, y) \mapsto x/y$ from $K \times K^\times$ to K, and the relation $\{(x, x) \colon x \in K \smallsetminus \{0\}\}$ from $K^\times$ to K.

A vector space is also a pair of structures, namely an abelian group V of vectors and a field K of scalars; and there is an action of the latter on the former, that is, a certain function from $K \times V$ to V. We could treat this pair as a structure whose universe is the disjoint union of K and V; then each of these components would be a singulary relation on the universe. It may however be considered ugly to introduce symbols for these relations. Convention already supplies a way to keep the two sets apart: we let boldface letters like $\mathbf{v}$ denote vectors, and plainface letters like a denote scalars.

Given a group G, we may want to consider it together with all of its quotient groups. If $M < N$, both being normal subgroups of G, then there is a function $xM \mapsto xN$ from G/M onto G/N. Here we probably do not want to treat the (disjoint) union of all of these quotients as the universe of a structure, because, if there are infinitely many quotients, then the Compactness Theorem would give us an elementary extension with elements not in any of the original quotients.

We might restrict the last example so that the quotients of G are all *definable* in G; then we might generalize so that, starting from a structure $\mathfrak{A}$, for every definable relation R on A, for every definable equivalence relation E on R, we consider a new structure whose universe is the quotient R/E.

Now we can generalize the original definition. On an indexed family $(A_s \colon s \in S)$ of sets, a **relation** is a subset of $\prod_{i \in I} A_i$ for some finite subset I of S, and an **operation** is a function from $\prod_{i \in I} A_i$ to A_s for some finite subset of I of S and some s in S. A **structure** then is an *indexed* family of sets with some relations and operations. The sets in the family are **sorts.** In a formula, the arguments of a relation symbol or a function symbol carry the information that they belong to the appropriate sort. Thus there are no variables simply; there are s**-variables** for the various indices s of sorts. There is also no requirement that the sorts be disjoint, since our symbolism refers to an element of a sort only through its index.

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