Model-theory of differential fields

Tabriz, August, 2012

David Pierce

Mathematics Department
Mimar Sinan Fine Arts University
Istanbul
http://mat.msgsu.edu.tr/~dpierce/
dpierce@msgsu.edu.tr

Model-theory:

- is ‘foundational’, like category theory or set theory;
- is a kind of mathematics done self-consciously;
- pays attention to the language of mathematics;
- is a study of **structures** as **models** of **theories**;
- is a study of the relation of **truth** between structures and **sentences.**

The truth-relation is usually symbolized by a double **‘turnstile’,**

$$\models \tag{1}$$

(which is `\models` in LaTeX).

Examples of the truth-relation.

$$\mathbb{C} \vDash \exists x\ x^2 = -1, \tag{2}$$

but

$$\mathbb{R} \nvDash \exists x\ x^2 = -1, \tag{3a}$$

that is,

$$\mathbb{R} \vDash \neg\exists x\ x^2 = -1, \tag{3b}$$

or equivalently

$$\mathbb{R} \vDash \forall x\ x^2 \neq 1. \tag{3c}$$

Also,

$$\mathbb{Q} \vDash \forall x\ \forall y\ \bigl(x < y \to \exists z\ (x < z \wedge z < y)\bigr), \tag{4}$$

but

$$\mathbb{Z} \vDash \exists x\ \exists y\ \bigl(x < y \wedge \forall z\ (x \nless z \vee z \nless y)\bigr). \tag{5}$$

In general, if $\mathfrak{A}$ is a structure, and σ is a sentence, and it is meaningful to write

$$\mathfrak{A} \models \sigma \tag{6}$$

(that is, if it is true or false that σ is true in $\mathfrak{A}$), then the **signature** of σ must be included in the **signature** of $\mathfrak{A}$.

Examples. The signature of

- groups is $\{\,\cdot\,, {}^{-1}, 1\}$;
- orders, $\{\leqslant\}$;
- ordered fields, $\{+, -, \,\cdot\,, 0, 1, \leqslant\}$.

The signature of the sentence σ is the set of **non-logical symbols** occurring in σ.
The signature of the structure $\mathfrak{A}$ is the set of non-logical symbols for which $\mathfrak{A}$ has an **interpretation.**

Non-logical symbols can be:

- **operation-symbols** (or *function-symbols,* as $+$, $-$, and $\cdot$),
- **constants** (or *constant-symbols,* as 0 and 1),
- **predicates** (or *relation-symbols,* as $\leqslant$).

Logical symbols are:

- the predicate $=$ (always interpreted as equality),
- **variables,** as x and y, standing for *individuals* (not sets as such);
- **Boolean connectives,** as $\neg$, $\land$, $\lor$, $\to$, and $\leftrightarrow$;
- **quantification symbols,** as $\exists$ and $\forall$;
- **brackets** (parentheses), as needed.

Such symbols are combined into **formulas,** according to a recursive definition. The formulas in which every variable is **bound**—the formulas without **free** variables—are **sentences.**

For example, the formula

$$\exists y \; x \cdot y = 1 \tag{7}$$

defining the group of units of a commutative ring has the free variable x; so it is not a sentence.
Our formulas and sentences are in **first-order logic.**

Non-examples.

1. The induction axiom for $\mathbb{N}$,

$$\forall X \; (1 \in X \wedge \forall y \; (y \in X \to y + 1 \in X) \to \forall y \; y \in X) \tag{8}$$

(here X ranges over subsets of $\mathbb{N}$, not individual elements of $\mathbb{N}$).

2. The axiom distinguishing *torsion* groups among abelian groups,

$$\forall x \; (x = 0 \vee 2x = 0 \vee 3x = 0 \vee \cdots) \tag{9}$$

(sentences must be finite).

Let $\mathscr{S}$ be a signature. The set of sentences with non-logical symbols from this signature can be denoted by

$$\operatorname{Sn}(\mathscr{S}). \tag{10}$$

There is also a *class*, denoted by

$$\operatorname{Mod}(\mathscr{S}), \tag{11}$$

of **'models'** of $\mathscr{S}$, that is, **structures** with signature $\mathscr{S}$. Such a structure can be written as $\mathfrak{A}$, and then $\mathfrak{A}$ consists of:

- a set A, called the **universe** of the structure (or there may be several 'universes', as in a vector-space; these are **sorts**);
- an **interpretation** of $\mathscr{S}$: a function $s \mapsto s^{\mathfrak{A}}$ on $\mathscr{S}$ assigning to each operation-symbol in $\mathscr{S}$ an operation on A, and so on.

This is just a formalization of the idea that, for example, the same symbol $+$ denotes a different operation in different abelian groups.

Then $\vDash$ is a relation between $\mathrm{Mod}(\mathscr{S})$ and $\mathrm{Sn}(\mathscr{S})$, namely

$$\{(\mathfrak{A},\sigma)\in\mathrm{Mod}(\mathscr{S})\times\mathrm{Sn}(\mathscr{S})\colon \sigma \text{ is true in } \mathfrak{A}\}. \tag{12}$$

Therefore we obtain a **Galois correspondence** as follows.

- If $\mathcal{K}\subseteq\mathrm{Mod}(\mathscr{S})$, we define the **theory** of $\mathcal{K}$:

  $$\mathrm{Th}(\mathcal{K})=\bigcap_{\mathfrak{A}\in\mathcal{K}}\{\sigma\in\mathrm{Sn}(\mathscr{S})\colon \mathfrak{A}\vDash\sigma\}, \tag{13}$$

  the set of sentences true in every structure in $\mathcal{K}$.

- If $\Gamma\subseteq\mathrm{Sn}(\mathscr{S})$, we define the class of **models** of Γ:

  $$\mathrm{Mod}(\Gamma)=\bigcap_{\sigma\in\Gamma}\{\mathfrak{A}\in\mathrm{Mod}(\mathscr{S})\colon \mathfrak{A}\vDash\sigma\}, \tag{14}$$

  the class of structures in which every sentence in Γ is true. Such a class is called an **elementary class.**

So there is a one-to-one correspondence between *theories* and *elementary classes.*

We restrict our attention to first-order logic so that we have the

Compactness Theorem. For all subsets Γ of $\mathrm{Sn}(\mathscr{S})$, if every finite subset of Γ has a model, then so does Γ itself.

By this, the theory of finite fields has an infinite model. In fact we can construct such a model as a quotient

$$\prod_{p \text{ prime}} \mathbb{F}_p/M,$$

where M is a maximal ideal of the product. Moreover, the Compactness Theorem can be proved by a generalization of this method.

The Compactness Theorem fails in second-order logic. For example, the **Peano axioms**

$$\forall X\ (1 \in X \wedge \forall y\ (y \in X \to y+1 \in X) \to \forall y\ y \in X), \tag{15a}$$

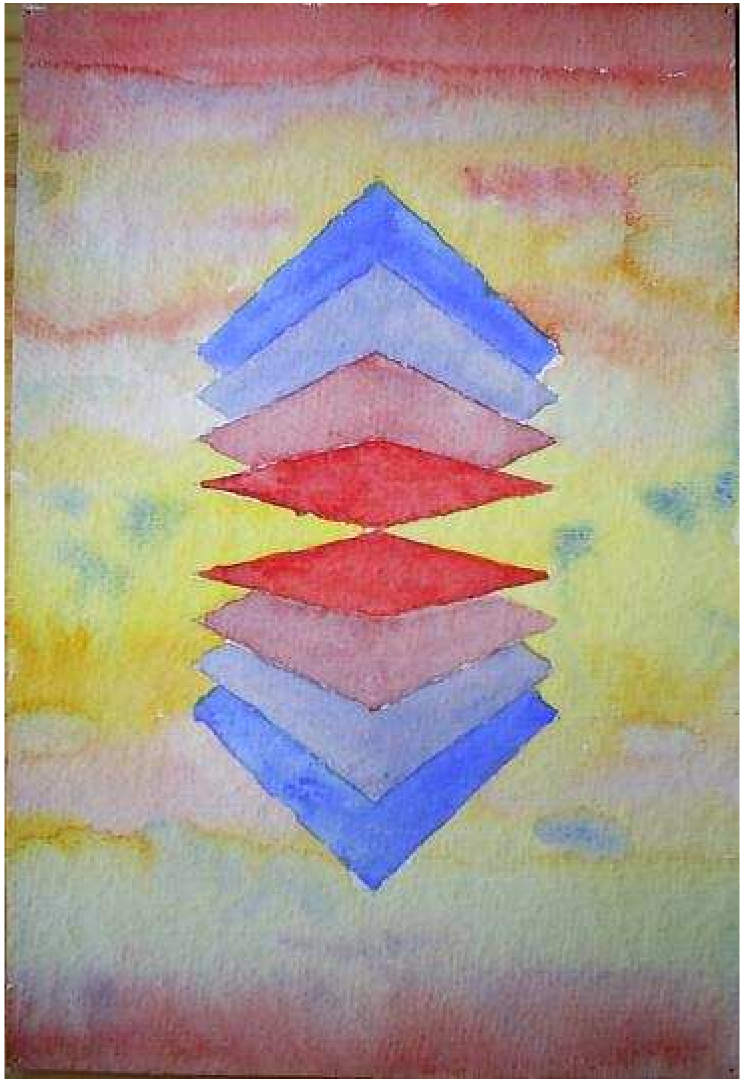
$$\forall x\ x+1 \neq 1, \tag{15b}$$

$$\forall x\ \forall y\ (x+1 = y+1 \to x = y), \tag{15c}$$

along with

$$c \neq 1, \quad c \neq 1+1, \quad c \neq 1+1+1, \quad \cdots, \tag{16}$$

say there is an infinite natural number (denoted by c). These sentences have no model, but every finite subset of them does. Also by Compactness, the class of torsion groups is not elementary.



In the following table, $\mathcal{K}$ and $\mathcal{K}^*$ are subclasses of $\mathrm{Mod}(\mathscr{S})$, and $\mathcal{K}^*$ is the class of **existentially closed** members of $\mathcal{K}$:

$\mathscr{S}$	$\mathcal{K}$	$\mathcal{K}^*$
$\varnothing$	sets	infinite sets
$\{<\}$	linear orders	dense linear orders w/o endpoints
$\{+,-,0\} \cup \mathbb{F}_p$	vector-spaces/$\mathbb{F}_p$	infinite vector-spaces/$\mathbb{F}_p$
$\{+,-,\cdot,0,1\}$	fields	algebraically closed fields

In general, $\mathfrak{A} \in \mathcal{K}^*$ means that if

$$\mathfrak{A} \subseteq \mathfrak{B}, \qquad \mathfrak{B} \in \mathcal{K}, \tag{17}$$

then every *existential* sentence *with parameters from* A that is true in $\mathfrak{B}$ is already true in $\mathfrak{A}$: for example, every system of equations and inequations over A with a solution in $\mathfrak{B}$ already has a solution in $\mathfrak{A}$.

For another example, let

$$\mathscr{S} = \{+, -, \cdot, 0, 1, F\},$$

and let $\mathcal{K}$ be the class of fields with an algebraically closed subfield called F. Then $\mathcal{K}^*$ is the class of (G, F) or G/F in $\mathcal{K}$ such that

- G is algebraically closed, and
- $\text{tr-deg}(G/F) = 1$.

Indeed, say $F_0 = {F_0}^{\mathrm{alg}}$, and α, β, and γ are algebraically independent over F, but $\delta = \alpha \cdot \gamma + \beta$. Let $F_1 = F_0(\gamma, \delta)^{\mathrm{alg}}$. Then

$$F_0(\alpha, \beta) \cap F_1 = F_0. \tag{18}$$

Therefore $(F_0(\alpha, \beta), F_0) \subseteq (F_1(\alpha, \beta), F_1)$. The sentence

$$\exists x\ \exists y\ (x \in F \land y \in F \land \alpha \cdot x + \beta = y)$$

is true in $(F_1(\alpha, \beta), F_1)$, so that

$$\text{tr-deg}(F_1(\alpha, \beta)/F_1) = 1. \tag{19}$$

For yet another example, let

$$\mathscr{S} = \{\boldsymbol{+}, \boldsymbol{-}, \mathbf{0}\} \cup \{+, -, \cdot, 0, 1\} \cup \{*\},$$

and let $\mathcal{K}$ be the class of vector-spaces, considered as two-sorted structures. (There is a sort of vectors and a sort of scalars.) Then $\mathcal{K}^*$ is the class of (V, F) in $\mathcal{K}$ such that

- F is algebraically closed, and
- $\dim_F(V) = 1$.

Again as F grows, $\dim_F(V) = 1$, if positive, can only go down.

If $T = \mathrm{Th}(\mathrm{Mod}(\Gamma))$, then T is the theory **axiomatized** by Γ.

Example. The classes of groups, rings, and fields are axiomatized by sets of $\forall\exists$ sentences, such as the field axiom

$$\forall x\ \exists y\ (x = 0 \lor xy = 1).$$

The classes in the example are also **closed under unions of chains.** In fact we have the following (example of) a **preservation theorem:**

Theorem (Chang, Łoś–Suszko, 1950s)**.** Let T be a theory. TFAE:

1. $\mathrm{Mod}(T)$ is closed under unions of chains.
2. T is axiomatized by a set of $\forall\exists$ sentences.

We may define

$$T_{\forall\exists} = \mathrm{Th}(\mathrm{Mod}(\{\sigma \in T \colon \sigma \text{ is } \forall\exists\})). \tag{20}$$

Then the second condition of the theorem is $T = T_{\forall\exists}$. In this case, we may say T is **inductive.**

Suppose T is inductive, that is $T = T_{\forall\exists}$, so that T is preserved under unions of chains. If T has models at all, then it has existentially closed models: these are unions of the appropriate chains.
If, further, the class of existentially closed models of T is elementary, then its theory is called the **model-companion** of T. This is what we shall be interested in.

Non-example. The theory of fields with an algebraically closed subfield has no model-companion: having transcendence-degree 1 is not an 'elementary' or first-order property.

If a model $\mathfrak{A}$ of T is existentially closed, this means for all quantifier-free formulas $\varphi(\vec{x}, \vec{y})$, for all $\vec{a}$ from A, if

$$\mathfrak{A} \subseteq \mathfrak{B}, \qquad \mathfrak{B} \in \operatorname{Mod}(T), \qquad \mathfrak{B} \models \exists \vec{y}\ \varphi(\vec{a}, \vec{y}) \tag{21}$$

then $\mathfrak{A} \models \exists \vec{y}\ \varphi(\vec{a}, \vec{y})$.

Stronger than being a *model-companion* is being a **model-completion.**

Theorem (Robinson, $\leqslant$1963)**.** Let T be a theory. TFAE:

- T has a model-completion.
- $T = T_{\forall\exists}$, and there is a function $\exists \vec{y}\ \varphi(\vec{x}, \vec{y}) \mapsto \widehat{\varphi}(\vec{x})$, where φ and $\widehat{\varphi}$ are quantifier-free, such that, if $\mathfrak{A} \models T$ and $\vec{a}$ is from A, then

$$\mathfrak{A} \models \widehat{\varphi}(\vec{a}) \iff \text{for some } \mathfrak{B},\ \mathfrak{A} \subseteq \mathfrak{B} \models T \cup \{\exists \vec{y}\ \varphi(\vec{a}, \vec{y})\}.$$

- the model-completion of T is axiomatized by

$$T \cup \Big\{ \forall \vec{x}\ \big(\widehat{\varphi}(\vec{x}) \to \exists \vec{y}\ \varphi(\vec{x}, \vec{y})\big) : \varphi \text{ quantifier-free} \Big\}.$$

Examples.

1. The theory ACF of algebraically closed fields is the model-completion of field-theory. If $\varphi(\vec{x}, y)$ is

$$x_n \cdot y^n + \cdots + x_1 \cdot y + x_0 = 0, \tag{22}$$

then $\widehat{\varphi}(\vec{x})$ can be

$$(x_n = 0 \wedge \cdots \wedge x_1 = 0) \to x_0 = 0. \tag{23}$$

2. The theory RCF of real-closed fields is the model-completion of the theory of ordered fields. If $\varphi(\vec{x}, y)$ is

$$x_2 \cdot y^2 + x_1 \cdot y + x_0 = 0, \tag{24}$$

then $\widehat{\varphi}(\vec{x})$ can be

$$\begin{aligned}&(x_2 = 0 \wedge x_1 = 0 \to x_0 = 0) \wedge \\ &\qquad \wedge (x_2 \neq 0 \to {x_1}^2 - 4x_2 \cdot x_0 \geqslant 0).\end{aligned} \tag{25}$$

Another example is:

Theorem (P.)**.** The theory of vector-spaces of dimension 1 over an algebraically closed field is:

1) the model-companion of the theory of vector-spaces,
2) *not* the model-completion of the theory of vector-spaces,
3) the model-completion of the theory of vector-spaces of dimension at most 1.

Here if $\varphi(\boldsymbol{x}_0, \boldsymbol{x}_1, y_0, y_1)$ is

$$\boldsymbol{x}_0 \cdot y_0 + \boldsymbol{x}_1 \cdot y_1 = \mathbf{0} \wedge (y_0 \neq 0 \vee y_1 \neq 0),$$

then $\widehat{\varphi}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ can be just $\boldsymbol{x}_0 = \boldsymbol{x}_0 \wedge \boldsymbol{x}_1 = \boldsymbol{x}_1$. But if $\psi(\boldsymbol{x}_0, \boldsymbol{x}_1, y_0, y_1)$ is

$$\varphi(\boldsymbol{x}_0, \boldsymbol{x}_1, y_0, y_1) \wedge f(y_0, y_1) = 0$$

for some nonzero polynomial f, then $\widehat{\psi}(\boldsymbol{x}_0, \boldsymbol{x}_1)$ cannot be found.

A **differential field** is a pair (K, D), where

- K is a field (for simplicity, of characteristic 0 here),
- D is a **derivation** of K, that is,

$$D(x+y) = Dx + Dy, \qquad D(xy) = Dx \cdot y + x \cdot Dy. \tag{26}$$

Example. $(\mathbb{C}(X), f \mapsto f')$.

The theory of differential fields will be denoted by

$$\mathrm{DF}\,. \tag{27}$$

Theorem (Seidenberg)**.** $\exists \vec{y}\ \varphi(\vec{x}, \vec{y}) \mapsto \widehat{\varphi}(\vec{x})$ as in Robinson's Theorem exists when T is DF.

Corollary (Robinson)**.** DF has a model-completion.

The model-completion of DF is called

$$\mathrm{DCF} \tag{28}$$

for **(theory of) differentially closed fields.**

For more comprehensible axioms for DCF, we can use another preservation theorem. If T is a theory, we define

$$T_\forall = \mathrm{Th}(\mathrm{Mod}(\{\sigma \in T \colon \sigma \text{ is universal}\})). \tag{29}$$

Theorem. The class of substructures of models of T is elementary, and its theory is $T_\forall$.

For example, if T is field-theory, then $T_\forall$ is the theory of integral domains.

Corollary. TFAE:

1. $\mathrm{Mod}(T)$ is closed under substructures.
2. $T = T_\forall$.

Theorem (Blum, ⩽1977)**.** Suppose $T = T_\forall$. For T to have a model-completion, the function $\exists\vec{y}\,\varphi(x,\vec{y}) \mapsto \widehat{\varphi}(x)$ as in Robinson's Theorem need only be well-defined on existential formulas with only *one* free variable x.

Since $\mathrm{DF}_\forall$ is 'close enough' to DF (because a derivation on an integral domain extends uniquely to the quotient field), Blum gets nice axioms for DCF:

Theorem (Blum, ⩽1977)**.** $(K, D) \in \mathrm{Mod}(\mathrm{DCF})$ if and only if:

- $(K, D) \in \mathrm{Mod}(\mathrm{DF})$,
- for all ordinary polynomials f and g over K, where
$$f \in K[X_0, \dots, X_n], \qquad g \in K[X_0, \dots, X_{n-1}], \tag{30}$$
if $g \neq 0$ and $\partial f/\partial X_n \neq 0$, then the sentence
$$\exists x \ \bigl(f(x, Dx, \dots, D^n x) = 0 \wedge g(x, Dx, \dots, D^{n-1} x) \neq 0\bigr) \tag{31}$$
is true in (K, D). (In particular, $K = K^{\mathrm{alg}}$.)

So Blum's approach to DCF is to consider only systems of equations and inequations in one variable; and this suffices (as it does in the axiomatization of the theory of algebraically closed fields).

There is an alternative, 'geometric' approach to simplifying the axioms of DCF. We use the following observations (found for example in Lang's *Algebra*).
Suppose $(K, D) \in \mathrm{Mod}(\mathrm{DF})$.

1. If $a \in K^{\mathrm{alg}}$, then D extends uniquely to $\tilde{D}$ on K^{alg} so that $(K, \tilde{D}) \in \mathrm{Mod}(\mathrm{DF})$. Indeed, $f(a) = 0$ for some irreducible f in $K[X]$; and then we obtain $\tilde{D}a$ by formal differentiation:
$$0 = D\big(f(a)\big) = f'(a) \cdot \tilde{D}a + f^D(a), \tag{32}$$
where f^D is the result of differentiating (by D) the coefficients of f.

2. If $a \notin K^{\mathrm{alg}}$, then D extends uniquely to $\tilde{D}$ on $K(a)$ so that $(K(a), \tilde{D}) \in \mathrm{Mod}(\mathrm{DF})$, once $\tilde{D}a$ is chosen in $K(a)$ (and this can be done arbitrarily).

We want to understand when systems of differential polynomial equations and inequations have solutions in some extension.
We can eliminate inequalities by adding variables:

$$x \neq 0 \iff \exists y\ xy = 1. \tag{33}$$

Then we can eliminate all higher-order derivatives. Indeed, over a model (K, D) of DF, TFAE:

$$\exists \vec{x}\ \bigwedge_f f(\vec{x}, D\vec{x}, \ldots, D^n\vec{x}) = 0, \tag{34}$$

$$\exists(\vec{x}_0, \ldots, \vec{x}_n)\ \Big(\bigwedge_f f(\vec{x}_0, \ldots, \vec{x}_n) = 0 \wedge \bigwedge_{i<n} D\vec{x}_i = \vec{x}_{i+1}\Big). \tag{35}$$

The latter is an instance of

$$\exists \vec{x}\ \Big(\bigwedge_f f(\vec{x}) = 0 \wedge \bigwedge_{i<k} Dx_i = g_i(\vec{x})\Big), \tag{36}$$

where $\vec{x} = (x_0, \ldots, x_{n-1})$ and $k \leqslant n$.

Theorem (P.–Pillay 1998, P. 2004). $(K, D) \in \mathrm{Mod}(\mathrm{DCF})$ if and only if:

- $(K, D) \in \mathrm{Mod}(\mathrm{DF})$,
- $K = K^{\mathrm{alg}}$,
- for all f and g_i in $K[X_0, \dots, X_{n-1}]$,

$$\exists \vec{x} \left(\bigwedge_f f(\vec{x}) = 0 \wedge \bigwedge_{i<k} Dx_i = g_i(\vec{x}) \right), \tag{37}$$

provided the f impose no algebraic condition on $(x_0, \dots, x_{k-1})$.

In the last condition, it is enough that the f define an irreducible variety with generic point $\vec{a}$ such that $(a_0, \dots, a_{k-1})$ is a transcendence-basis of $K(\vec{a})/K$.

How do these ideas work in case of several derivations?

Let DF^m be the theory of fields (of characteristic zero) with m commuting derivations: these are **'partial' differential fields** (with m derivations).

Example. $(\mathbb{C}(X_0, \dots, X_{m-1}), \partial/\partial X_0, \dots, \partial/\partial X_{m-1})$.

So DF^m is the theory of $(m+1)$-tuples $(K, \partial_0, \dots, \partial_{m-1})$, where

- $\partial_i \in \mathrm{Der}(K)$, that is, $(D, \partial_i) \in \mathrm{Mod}(\mathrm{DF})$;
- $[\partial_i, \partial_j] = 0$.

Theorem (McGrail, 2000)**.** DF^m has a model-completion, called DCF^m.

The proof uses complicated differential algebra of Kolchin and others. The aim here is to make this more 'geometric'.
We can use Blum's theorem, considering only systems in one variable.
Let
$$\omega = \{0, 1, 2, \dots\}. \tag{38}$$

If $\sigma \in \boldsymbol{\omega}^m$, this means

$$\sigma = (\sigma(0), \ldots, \sigma(m-1)). \tag{39}$$

We then write

$$\partial^\sigma x = {\partial_0}^{\sigma(0)} \cdots {\partial_{m-1}}^{\sigma(m-1)} x. \tag{40}$$

We define a **differential polynomial ring:**

$$K\{X\} = K[\partial^\sigma X \colon \sigma \in \boldsymbol{\omega}^m]. \tag{41}$$

Suppose

$$(K, \partial_0, \ldots, \partial_{m-1}) \subseteq (L, \tilde{\partial}_0, \ldots, \tilde{\partial}_{m-1}), \tag{42}$$

both being models of DF, and $a \in L$. Then we can define

$$I(a) = \{f \in K\{X\} \colon f(a) = 0\};$$

this is a prime differential ideal. Kolchin identifies a kind of finite subset Λ of this ideal, called a **characteristic set** of the ideal, that determines the ideal in the following sense.

Again if $\sigma \in \boldsymbol{\omega}^m$, define the **height** of σ by

$$|\sigma| = \sigma(0) + \cdots + \sigma(m-1). \tag{43}$$

Since the characteristic set Λ of $I(a)$ is finite, for some n in $\boldsymbol{\omega}$,

$$\Lambda \subseteq K[\partial^\sigma X \colon |\sigma| \leqslant n].$$

For each f in this ring there is an ordinary polynomial $\widehat{f}$ in $K[X_\sigma \colon |\sigma| \leqslant n]$ such that

$$f = \widehat{f}(\partial^\sigma X \colon |\sigma| \leqslant n).$$

A zero $(a_\sigma \colon |\sigma| \leqslant n)$ of $\widehat{f}$ from some field-extension of K can be called an **algebraic zero** of f, as opposed to a **(true) zero,** which will be an element (like a) of some differential-field-extension of $(K, \partial_0, \ldots, \partial_{m-1})$.
Then a *generic* algebraic zero of (the elements of) Λ does consist of the derivatives of a true zero of Λ. This is the sense in which the characteristic set Λ determines $I(a)$.

Now McGrail's axioms for DCF^m are as follows.
Again Λ is a characteristic set of $I(a)$. Suppose $g \in K\{X\} \smallsetminus I(a)$. The formula

$$\bigwedge_{f\in\Lambda} f(y) = 0 \wedge g(y) \neq 0 \tag{44}$$

is a formula $\varphi(\vec{a}, y)$ for some list $\vec{a}$ of parameters from K. McGrail shows there is $\widehat{\varphi}(\vec{x})$ such that

$$(K, \partial_0, \ldots, \partial_{m-1}) \vDash \varphi(\vec{a}), \tag{45}$$

and also, whenever $(L, \partial_0, \ldots, \partial_{m-1}) \in \mathrm{Mod}(\mathrm{DF})$ and $(L, \partial_0, \ldots, \partial_{m-1}) \vDash \varphi(\vec{b})$, then the differential system

$$\varphi(\vec{b}, y)$$

is soluble in some extension of $(L, \partial_0, \ldots, \partial_{m-1})$. Then one of the axioms of DCF is

$$\forall \vec{x}\ \big(\widehat{\varphi}(\vec{x}) \to \exists x\ \varphi(\vec{x}, y)\big). \tag{46}$$

We can see what is going on as follows. In case $m = 2$, we may ask whether the set of differential polynomials

$$\partial^{(2,2)}X - X, \qquad \partial^{(1,1)}X - \partial^{(0,2)}X \tag{47}$$

has a zero. These polynomials belong to $K[\partial^\sigma X : |\sigma| \leqslant 4]$. We depict a potential zero as

$$\begin{matrix} \partial^{(0,0)}x & \partial^{(0,1)}x & \partial^{(0,2)}x & \partial^{(0,3)}x & \partial^{(0,4)}x \\ \partial^{(1,0)}x & \partial^{(1,1)}x & \partial^{(1,2)}x & \partial^{(1,3)}x & \\ \partial^{(2,0)}x & \partial^{(2,1)}x & \partial^{(2,2)}x & & \\ \partial^{(3,0)}x & \partial^{(3,1)}x & & & \\ \partial^{(4,0)}x & & & & \end{matrix} \tag{48}$$

Now consider the polynomials

$$X_{(2,2)} - X_{(0,0)}, \qquad X_{(1,1)} - X_{(0,2)}. \tag{49}$$

A generic zero of the polynomials

$$X_{(2,2)} - X_{(0,0)}, \qquad\qquad X_{(1,1)} - X_{(0,2)} \tag{50}$$

in $K[X_\sigma\colon |\sigma| \leqslant 4]$ can be depicted as:

$$\begin{array}{lllll} a & * & b & * & * \\ * & b & * & * & \\ * & * & a & & \\ * & * & & & \\ * & & & & \end{array} \tag{51}$$

Closing under differentiation imposes further conditions:

$$\begin{array}{lllll} a & * & b & c & \underline{a} \\ * & \underline{b} & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{52}$$

But the underlined entries should have a common derivative—say d—outside the triangle:

$$\begin{array}{ccccc} a & * & b & c & \underline{a} \\ * & \underline{b} & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \qquad \text{leads to} \qquad \begin{array}{ccccc} a & * & b & c & \underline{a} \\ * & \underline{b} & c & a & d \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{53}$$

This imposes further conditions *inside* the triangle:

$$\begin{array}{ccccc} a & * & b & c & \underline{a} \\ * & \underline{b} & c & a & d \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \qquad \text{leads to} \qquad \begin{array}{ccccc} a & d & b & c & a \\ d & b & c & a & d \\ b & c & a & & \\ c & a & & & \\ a & & & & \end{array} \tag{54}$$

That's fine, we can extend this diagram indefinitely; so the original differential polynomials have a zero.

But suppose we started instead with

$$\partial^{(2,2)}X - X, \qquad\qquad \partial^{(1,1)}X - \partial^{(0,2)}X - 1. \tag{55}$$

This gives us (writing b' for $b+1$):

$$\begin{array}{ccccc} a & * & b & * & * \\ * & b' & * & * & \\ * & * & a & & \\ * & * & & & \\ * & & & & \end{array} \qquad \begin{array}{ccccc} a & * & b & c & a \\ * & b' & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \qquad \begin{array}{ccccc} a & d & b & c & a \\ d & b' & c & a & d \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{56}$$

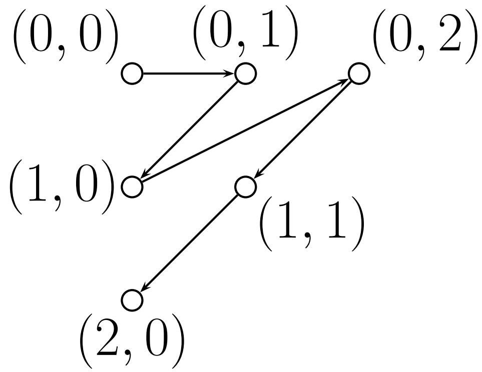
so $\partial_1 d$ must be both b and b', which is absurd.

This test works generally as follows.

We define a strict partial ordering $\lessdot$ on ω^2 by

$$\sigma \lessdot \tau \iff (|\sigma|, \sigma(0)) <_\ell (|\tau|, \tau(0)) \tag{57}$$

where $<_\ell$ is the left lexicographic ordering on ω^2. So $\lessdot$ is thus:



If $\sigma \lessdot \tau$, let us say σ is a **predecessor** of τ.
Now let $\leqslant$ be the product ordering of ω^2, so

$$\sigma \leqslant \tau \iff \sigma(0) \leqslant \tau(0) \wedge \sigma(1) \leqslant \tau(1). \tag{58}$$

Then

$$\sigma \leqslant \tau \iff \text{for some } \rho \text{ in } \omega^2,\ \sigma + \rho = \tau. \tag{59}$$

In this case, let us say τ is **above** σ.

Suppose

$$(K, \partial_0, \partial_1) \subseteq (L, \tilde{\partial}_0, \tilde{\partial}_1), \tag{60}$$

both being models of DF, and $a \in L$. Let

$$A = \{\sigma \in \omega^2 \colon \tilde{\partial}^\sigma a \in K(\tilde{\partial}^\rho a \colon \rho \lessdot \sigma)^{\mathrm{alg}}\}, \tag{61}$$

the set of σ such that $\tilde{\partial}^\sigma a$ is algebraic over its predecessors (so to speak).

This set A is closed under $\geqslant$, that is,

$$\sigma \in A \ \& \ \sigma \leqslant \tau \implies \tau \in A.$$

Then $I(a)$ is 'determined' by those $\partial^\sigma a$ such that σ is $\leqslant$-minimal in A.

This is the idea behind *characteristic* subsets of $I(a)$.

A tuple $(a_\sigma\colon |\sigma|\leqslant n)$ of elements of some field-extension of K will be called **soluble** if the a_σ belong to some L as above so that

$$\tilde{\partial}^\rho a_\sigma = a_{\sigma+\rho} \tag{62}$$

whenever $|\sigma+\rho|\leqslant n$. For solubility, it is *necessary* that, for all f in $K[\partial^\sigma X\colon |\sigma|<n]$,

$$\widehat{f}(a_\sigma\colon |\sigma|<n)=0 \implies \widehat{\partial_i f}(a_\sigma\colon |\sigma|\leqslant n)=0. \tag{63}$$

Call this the **differential condition.**
The last example shows the differential condition is not *sufficient* for solubility: The tuple depicted as

$$\begin{array}{lllll} a & * & b & c & a \\ * & b' & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{64}$$

meets the condition, but is not soluble.

Again given a tuple $(a_\sigma \colon |\sigma| \leqslant n)$ of elements of some field-extension of K, we call σ a **leader** if it is $\leqslant$-minimal among those τ such that

$$a_\tau \in K(a_\rho \colon \rho \lessdot \tau)^{\mathrm{alg}}.$$

In the last example, there are two leaders, with corresponding terms underlined:

$$\begin{array}{lllll} a & * & b & c & \underline{a} \\ * & \underline{b}' & c & a & \\ * & c & a & & \\ * & a & & & \\ * & & & & \end{array} \tag{65}$$

For solubility, it is necessary that equating the common derivatives of two leaders must not introduce new conditions. This (with the differential condition) is also sufficient:

Theorem (P.). For $(a_\sigma \colon |\sigma| \leqslant n)$ to be soluble, it is sufficient that

1) it meet the differential condition, and
2) for every leader a_σ, we have $|\sigma| \leqslant n/2$.

Moreover, if $(a_\sigma \colon |\sigma| \leqslant n)$ is soluble, then n can be chosen large enough so that the foregoing conditions are met; and the new n depends only on the original n.

We can make adjustments to allow each a_σ to be a *tuple* $(a_\sigma^j \colon j < k)$ and to allow K to have arbitrary characteristic. This leads to a model-companion of the theory of fields (of unspecified characteristic) with m commuting derivations.

END