Descartes as model theorist

David Pierce

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Mathematics Department
Mimar Sinan Fine Arts University
dpierce@msgsu.edu.tr
http://mat.msgsu.edu.tr/~dpierce/

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Introduction

The present document is an exposition of three related topics:

1. The interpretation of a field in a Euclidean plane, which allowed Descartes to create what we call analytic geometry. Descartes did not supply the details, but we can do it by Pappus's Hexagon Theorem, as well as other means.
2. The interpretation of the scalar field in a vector space (of dimension at least 2) over the field by means of parallelism. This leads to the results of the paper, "Model-theory of vector-spaces over an unspecified field" [15].
3. The interpretability, by existential formulas, of fields with several derivations in certain Lie rings equipped with a group endomorphism. This leads, through the paper "Fields with several commuting derivations" [16], to the existence of model-complete theories of Lie rings with group endomorphism.

I alluded to the last item at [15, §1, p. 424]; but the present document is so far the most thorough account.

The document consisted originally of notes for a talk to be given on May 16, 2013. The abstract of the talk was as follows.

> In his *Geometry* of 1637, Rene Descartes gave a geometric justification of algebraic manipulations of symbols. He did this by interpreting a field in a vector-space with a notion of parallelism. At least this is how we might describe it today.

> I alluded to this in the abstract for my February 28 seminar, but did not actually talk about it.[1] Now I want to talk about it.
>
> By fixing a unit, Descartes defines the product of two line segments as another segment. He relies on a theory of proportion for this. Presumably this is the theory developed in Book V of Euclid's *Elements*—the theory that inspired Dedekind's definition of real numbers as "cuts" of rational numbers.
>
> But this theory has an "Archimedean" assumption: for any two given segments, some multiple of the smaller exceeds the larger.
>
> In fact this assumption is not needed, as Hilbert observed in Foundations of Geometry. Hilbert uses instead Pappus's Theorem. This work may be known now as "interpreting a field in a projective plane."
>
> I tracked down Pappus's original argument (from the 4th century), and [on May 13, 2013] I wrote an account of it on *Wikipedia* [in the "Pappus's hexagon theorem" article, in the "Origins" section].
>
> As for model theory, another result that comes out of these considerations is that there are model-complete theories of Lie-rings equipped with an endomorphism of the abelian-group-structure.

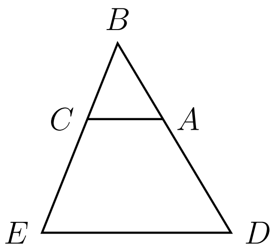
According to my records, in the actual talk, "I said something about all sections of the notes, mainly in the order 2 [vector spaces], 3 [Lie rings], 1 [geometry]." According to the notes of Ayşe Berkman, transcribed as an appendix below (but which may not be complete), I did not talk about geometry at all.

[1] The talk concerned mainly the work with Özcan Kasal that would appear in "Chains of theories and companionability" [8].

I revisited my notes in the following September, and now again in April, 2016. I have made some corrections and changes of style to the main text. More thoroughgoing revisions, reflecting the increase of my own knowledge (especially about Pappus and Dedekind), are in the footnotes. (All but one of these notes are from 2016.)

1. Geometry

By fixing a line segment in the Euclidean plane as a unit, Descartes defines multiplication of segments [2]. Thus he justifies algebra by interpreting it in geometry.



$$\text{Let } AC \parallel DE.$$
$$\text{If } AB = 1,$$
$$BD = a,$$
$$BC = b,$$
$$\text{then } BE := ab.$$

Figure 1.1.: Descartes's definition of multiplication

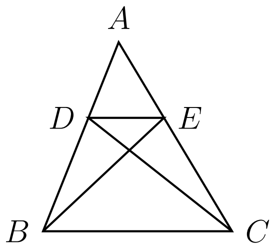
Hilbert will go the other way, using algebra to produce models of his geometric axioms.

Descartes needs Proposition VI.2 of Euclid's *Elements* [3], that a line parallel to the base of a triangle divides the sides proportionally, as in Figure 1.1. Descartes himself uses single minuscule letters for line segments. To denote equality, he uses the reverse of our $\propto$, instead of =.[1] Strictly we should probably consider these minuscule letters as *lengths* of line segments; and the **length** of a segment should be understood

[1] Nonetheless, though Descartes was writing in 1637, Robert Recorde had introduced our "equals" sign in 1557 [17]. (Footnotes were added in 2016, unless otherwise noted.)

as the class of all segments that are *congruent* to it. (In Euclid, equality *means* congruence; *sameness* is a different notion.)

The proof of VI.2 uses the auxiliary lines in Figure 1.2, along with VI.1, that triangles (and parallelograms) with the same height are to one another as their bases.



$$
\begin{aligned}
&\because DE \parallel BC,\\
&\therefore \triangle BDE = \triangle CDE,\\
\therefore BD : DA &:: \triangle BDE : \triangle ADE\\
&:: \triangle CDE : \triangle ADE\\
&:: CE : EA.
\end{aligned}
$$

Figure 1.2.: Euclid's Proposition VI.2

This follows easily from the definition of proportion in Book V of the *Elements.* This definition uses an "Archimedean" assumption: for any two magnitudes of the same kind (as line segments, or areas), some multiple of the smaller exceeds the larger. If A, B, C, and D are magnitudes, A and B being of the same kind, and likewise C and D, then

$$A : B :: C : D$$

means for all positive integers k and m,

$$
\begin{aligned}
kA > mB &\iff kC > mD,\\
kA = mB &\iff kC = mD,\\
kA < mB &\iff kC < mD.
\end{aligned}
$$

We then might understand

$$(A : B) = \{m/k \colon kA < mB\}.$$

Thus a ratio corresponds to a *Dedekind cut* of positive rational numbers.

Dedekind [1, I] does not say his definition (discovered November 24, 1858) of the real numbers is inspired by Euclid. But apparently he read Euclid in school [18, p. 47].[2]

Dedekind does not show explicitly that the real numbers defined by him satisfy the field axioms; but he says it can be done. His idea seems to be this: the operations of $+$ and $\times$ are *continuous* in each coordinate, and therefore every equation, like

$$(x + y) \cdot z = x \cdot z + y \cdot z,$$

that is satisfied by all rationals is satisfied by all reals as well.

The works that I know—Landau [10, Thm 144, p. 55] and Spivak [19, pp. 563–4]—do not take this approach, but work directly with the definitions of $+$ and $\times$ as cuts.

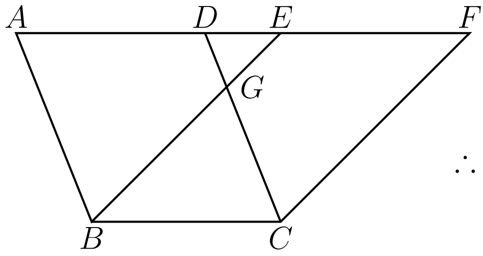
Descartes does not recognize a need to prove associativity and commutativity of his multiplication.

Note that addition is "obviously" commutative and associative, since equality of parts implies equality of the wholes. Consider for example Euclid's I.35 in Figure 1.3, that parallelograms on the same base and in the same parallels are equal, because they can be divided into parts that are respectively equal, though differently arranged.

We can prove commutativity of multiplication using a case of Pappus's Theorem.[3] In Figure 1.4, let

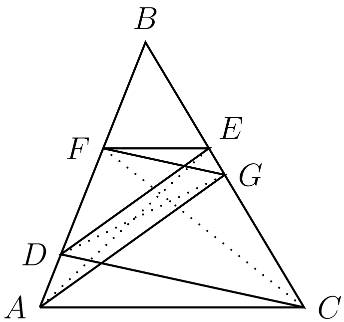
[2]Dedekind *does* cite Euclid in [1, II, Preface, pp. 36–40], where he observes in effect that every ratio of Euclidean magnitudes is a real number, but that critics have not understand that he (Dedekind) is doing something new, namely showing how to define real numbers *without* reference to magnitudes.

[3]What we now call Pappus's Theorem is that, if the vertices of a hexagon lie alternately on two straight lines, then the intersection points of



$$\begin{aligned} &\because AF \parallel BC, \\ &\therefore ABE = DCF, \\ &\therefore ABGD = GCFE, \\ \therefore ABCD &= ABGD + GBC \\ &= GCFE + GBC \\ &= EBCF. \end{aligned}$$

Figure 1.3.: Euclid's Proposition I.35



$$\begin{aligned} FGD &= FGC && \because FG \parallel DC \\ BGD &= FBC && \text{[add } BFG\text{]} \\ DEG &= DEA && \because DE \parallel AG \\ DBG &= BEA && \text{[add } DBE\text{]} \\ FBC &= BEA && \text{[2nd eqn]} \\ FEC &= FEA && \text{[subtract } FBE\text{]} \\ &\therefore FE \parallel AC \end{aligned}$$

Figure 1.4.: Pappus's Theorem and his proof

$$BF = 1, \qquad BE = 1, \qquad BD = a, \qquad BG = b.$$

Assume $DC \parallel FG$ and $DE \parallel AG$. Then

$$BA = ab, \qquad BC = ba.$$

These are equal, provided $AC \parallel FE$. To prove the parallelism, we may note that[4]

$$BF : BD :: BG : BC, \qquad BD : BA :: BE : BG,$$

each of the three pairs of opposite sides of the hexagon lie on a straight line. Here parallel straight lines are counted as meeting "at infinity." Thus, in the case now under consideration, each of two of the pairs of opposite sides of the hexagon are parallel; the conclusion is that the third pair are parallel. This is Lemma VIII of Pappus's 38 lemmas for Euclid's (now lost) book of *Porisms.* These lemmas are part of Book VII of Pappus's *Collection* [11, pp. 866–919]. Within Book VII, by the count of Hultsch, the lemmas are Propositions 127–64; thus Lemma VII is Proposition 134.

[4]Pappus's own proof uses not proportions, but areas; I have added it to Figure 1.4. In 2013, I did not know Pappus's proof, because I had accepted Heath's account [5, pp. 419–21], whereby Pappus's Theorem is found in Propositions 138, 139, 141, and 143 of Book VII, that is, Lemmas XII, XIII, XV, and XVII for Euclid's *Porisms.* Heath was following Chasles's analysis of the 38 lemmas. Thus, unless Heath misrepresents Chasles, he too overlooked the import of Proposition 134. Kline [9, p. 128] cites only Proposition 139 as being Pappus's Theorem. This proposition is the case of Pappus's Theorem where each pair of opposite sides of the hexagon are intersecting, and the two straight lines on which the vertices lie alternately are not parallel. In Proposition 138, those straight lines *are* parallel. Propositions 141 and 143 are the converse: if the straight line through the intersection points of two pairs of opposite sides of the hexagon meets a fifth side at a point, then the sixth side must pass through that point. As Alexander Jones observes, citing Jan Hogendijk, the result can be seen as the same as that of Propositions 138 and 139, if one relabels the points; but this is not how Pappus's proof goes [13, p. 562]. In 2015, I translated the first 19 of Pappus's lemmas from Greek to Turkish

and therefore, by Euclid's Proposition V.23, *ex aequali,*[5]

$$BF : BA :: BE : BC.$$

The proof of V.23 does *not* use commutativity of multiplication of integers, but uses the Archimedean property. In fact it uses a bit more than this: of two unequal magnitudes of the same kind, their *difference* is also of the same kind. The argument can be made as follows, where capital letters are now magnitudes.

V.8. Suppose $A > B$, and C is of the same kind as these. Then for some k we have $k(A-B) > C$, and so for some m we have

$$kA > mC > kB,$$

and thus $A : C > B : C$ (hence also $C : A < C : B$).

V.14. If $A : B :: C : D$ and $A > C$, then

$$C : D :: A : B > C : B,$$

so $B > D$.

---

for use in a course called *Geometriler.* Only late in the process did I learn about Jones's edition of Pappus's Book VII [12]. It seems that Pappus did not consider the case of the theorem named for him in which exactly one of the three pairs of opposite sides of the hexagon are parallel.

[5]Euclid's V.22 is, "If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio *ex aequali*" [4, p. 58]. That is, if $A_1 : A_2 :: B_1 : B_2, \dots, A_n : A_{n+1} :: B_n : B_{n+1}$, then $A_1 : A_{n+1} :: B_1 : B_{n+1}$. Proposition V.22 is a variant: "If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio *ex aequali.*"

v.21. Suppose

$$A : B :: E : F, \qquad B : C :: D : E.$$

If $A > C$, then $E : F :: A : B > C : B :: E : D$, so $D > F$.

v.23. Same supposition as v.21. Then for all k and m,

$$kA : kB :: mE : mF, \qquad kB : mC :: kD : mE,$$

and so $kA > mC \implies kD > mF$. Thus

$$A : C :: D : F.$$

For associativity, in the same Figure 1.4, suppose

$$BF = 1, \qquad \begin{matrix} BD = ab, & BE = b, \\ BA = ac, & BG = c. \end{matrix}$$

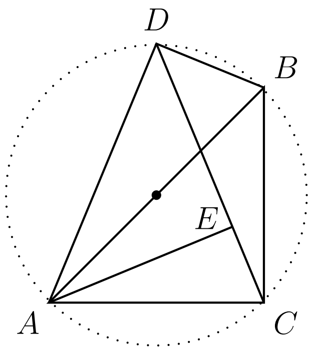
Then $DE \parallel AG$. Also

$$\begin{aligned} DC \parallel FG &\implies BC = c(ab), \\ AC \parallel FE &\implies BC = b(ac). \end{aligned}$$

The theorem we have used is that if the vertices of a hexagon lie alternately on two straight lines, and each of two pairs of opposite sides are parallel, then so are the third pair. More generally, the intersection points of the pairs of opposite sides lie on the same straight line—in our case, this is the "line at infinity". Pappus proved the finite case [11, VII.138–9].[6]

Pascal's Theorem is the generalization of Pappus's Theorem in which the vertices of the hexagon lie on a conic section. It

[6] As already noted, Pappus proved the "infinite" or parallel case as well.



$$\because \angle ACB = \angle ADB = \text{right}$$
$$\therefore \angle BAD = \angle CAE$$

Figure 1.5.: Hilbert's lemma for Pappus's Theorem

is enough to prove the case of a circle, since non-degenerate conic sections are projections of a circle.

One can prove Pappus's Theorem without using proportions (or the Archimedean property in any way). See Hilbert's *Foundations of Geometry* [6, §14, pp. 24–29], where the theorem is named for Pascal. Hilbert argues as follows. In Figure 1.5, the angles ACB and ADB are right, so the points $ABCD$ lie on a circle, and therefore the angles ABD and ACD are equal, so their complements BAD and CAE are equal. Considering now how AE is the result of two projections, in two different ways, from AB, we can write this as

$$c \cos\alpha \cos\beta = c \cos\beta \cos\alpha,$$

where $c = AB$, and $\alpha = \angle BAC$, and $\beta = \angle CAE$. Hilbert writes the conclusion as

$$\beta\alpha c = \alpha\beta c;$$

here αc just means the length of AC. Now apply this to Figure 1.6, which is just Figure 1.4 relettered. We assume $CB' \parallel BC'$ and $CA' \parallel AC'$. Perpendiculars dropped from O

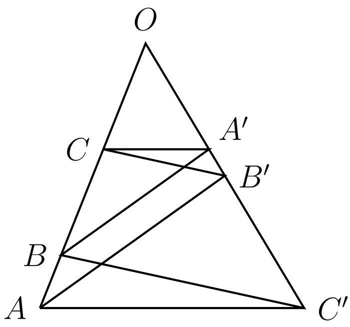


Figure 1.6.: Hilbert's first proof of Pappus's Theorem

to CB', CA', and BA' make angles λ', μ', and ν' with OA, and angles λ, μ, and ν with OC', respectively. Then with distances from O lettered in the obvious way, we have (in Hilbert's notation, as above)

$$\begin{aligned} \lambda b' &= \lambda' c, & \mu a' &= \mu' c, & \nu a' &= \nu' b, \\ \lambda' b &= \lambda c', & \mu' a &= \mu c', \end{aligned}$$

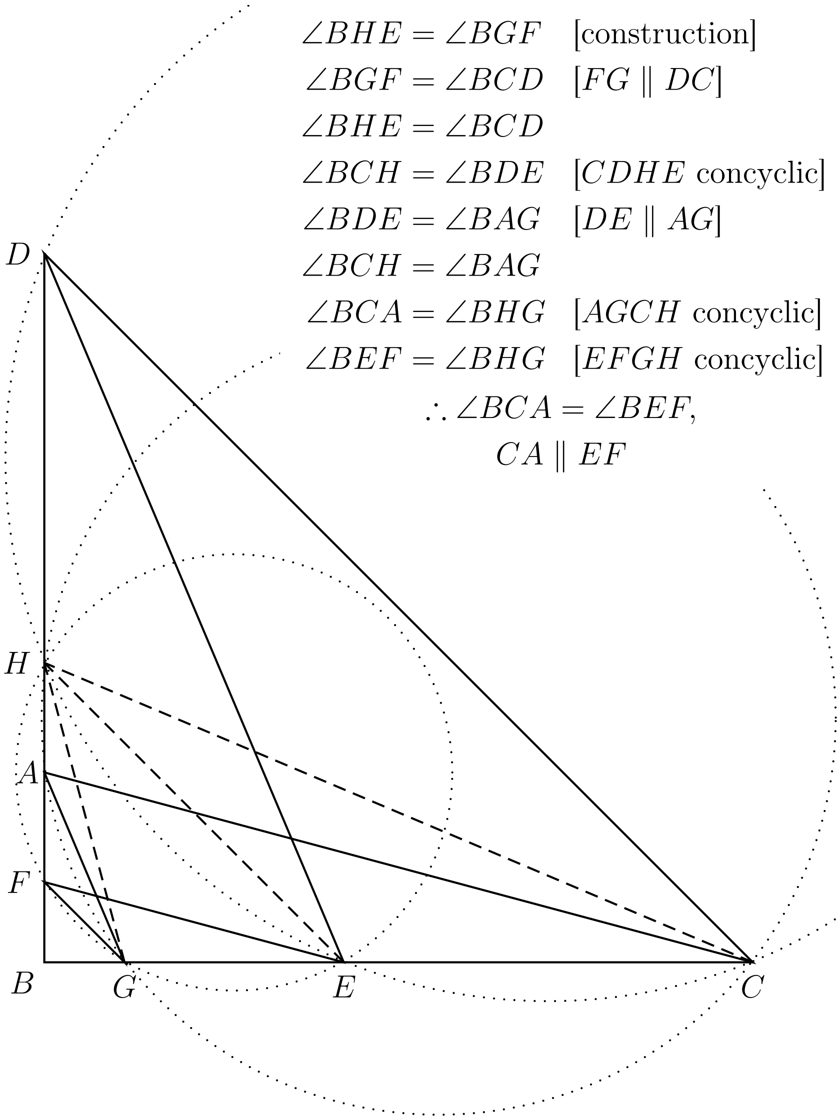
and therefore, since we can permute the angles,[7] we apply these equations in order to get

$$\nu'\mu'\lambda a = \nu'\mu\lambda c' = \nu'\mu\lambda' b = \nu\mu\lambda' a' = \nu\mu'\lambda' c = \nu\mu'\lambda b',$$
$$\nu' a = \nu b',$$

and therefore $BA' \parallel AB'$.

Hilbert gives also another argument, "in the following very simple manner, for which, however, I am indebted to another source."[8]

[7]We proved commutativity [of the angles]. Associativity is automatic, since the angles represent functions; but Hilbert does not seem to say



$$
\begin{aligned}
\angle BHE &= \angle BGF && \text{[construction]}\\
\angle BGF &= \angle BCD && [FG \parallel DC]\\
\angle BHE &= \angle BCD && \\
\angle BCH &= \angle BDE && [CDHE \text{ concyclic}]\\
\angle BDE &= \angle BAG && [DE \parallel AG]\\
\angle BCH &= \angle BAG && \\
\angle BCA &= \angle BHG && [AGCH \text{ concyclic}]\\
\angle BEF &= \angle BHG && [EFGH \text{ concyclic}]
\end{aligned}
$$

$$\therefore \angle BCA = \angle BEF,$$
$$CA \parallel EF$$

Figure 1.7.: Hilbert's second proof of Pappus's Theorem

With Pappus's Theorem, Hilbert develops an "algebra of segments," more or less along the lines of Descartes. In short, he interprets a field.

There is an alternative approach to interpreting a field, using only Book I of the *Elements*.[9] Fix a unit segment. By Propositions I.42 and I.44, in effect, every rectangle is equal to a rectangle with unit side. The other side of this rectangle can then be defined as the product of the two sides of the first rectangle. This multiplication is immediately commutative, as well as distributive over addition.

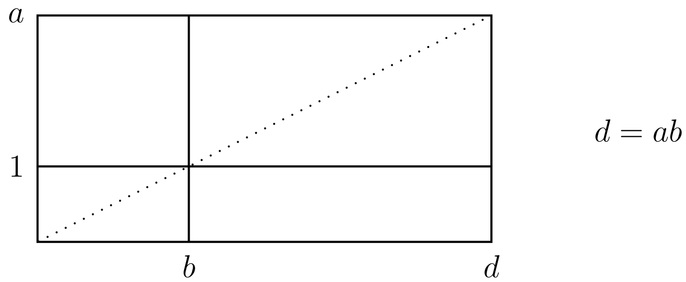


Figure 1.8.: Multiplication defined by equal rectangles

More precisely, multiplication is effected as in Figure 1.8, where points are labelled with their distances from the lower left vertex. Then (by I.43 and its converse) $d = ab$ if and only if the diagonal passes through the intersection of the vertical and

this explicitly. (Note added, September 2013.)

[8] I have now supplied this proof as Figure 1.7 (lettered like Figure 1.4, not like Hilbert's own figure). The point to observe is that it uses *angles,* like Hilbert's first proof, and not *areas,* like Pappus's own proof.

[9] Again, Pappus's own proof of the parallel case of Pappus's Theorem has already provided this alternative approach.

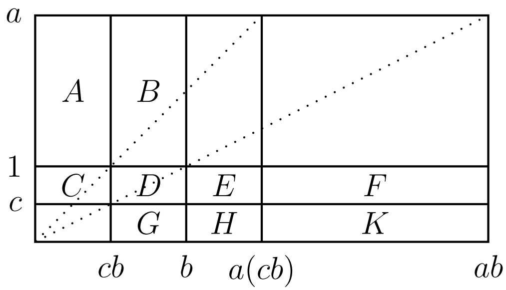


Figure 1.9.: Associativity of multiplication

horizontal lines. Associativity of multiplication thus defined can be established by means of Figure 1.9. Again points are labelled with their distances from the lower left vertex. The longer diagonal gives us both cb and ab. Then the shorter diagonal then gives us $a(cb)$. This is equal to $c(ab)$, provided

$$C + D + E = K.$$

The longer diagonal gives us

$$A + B = E + F + H + K,$$
$$B = E + F,$$

and therefore

$$A = H + K.$$

The shorter diagonal gives us

$$A = D + E + G + H,$$

and therefore

$$D + E + G = K.$$

We finish by noting (from the longer diagonal)

$$C = G.$$

Therefore $c(ab) = a(cb)$. We have assumed $c < 1 < a$ and $b < a(cb)$. Strictly we should consider four more cases:

(1) $c < 1 < a$, but $a(cb) = b$;
(2) $c < 1 < a$, but $a(cb) < b$;
(3) $c < a < 1$; and
(4) $1 < c < a$.

2. Vector spaces

Descartes's idea for a geometric definition of multiplication lets us interpret the scalar field in a vector space (of dimension at least two) by means of *parallelism.* This is worked out in my paper [15]. Given two parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we define $[\boldsymbol{a} : \boldsymbol{b}]$ as the class of pairs $(\boldsymbol{c}, \boldsymbol{d})$ of parallel vectors such that

$$\boldsymbol{a} - \boldsymbol{c} \parallel \boldsymbol{b} - \boldsymbol{d}$$

—assuming $\boldsymbol{a} \nparallel \boldsymbol{c}$; otherwise we must be able to find a third pair $(\boldsymbol{e}, \boldsymbol{f})$ of parallel vectors with the same relation to the first two pairs, as in Figure 2.1. Then the field is the set of

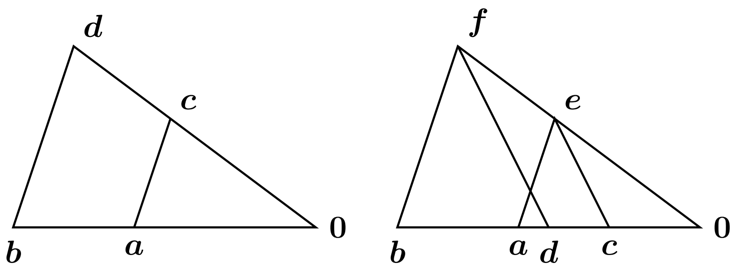


Figure 2.1.: Scalar field defined in a vector space

these classes $[\boldsymbol{a} : \boldsymbol{b}]$, where $\boldsymbol{b} \neq \boldsymbol{0}$. Equality and inequality of these, and addition and multiplication of these, are defined by *existential* formulas. Hence we obtain an equivalence of the categories of:

1) vector spaces with scalar field as a separate sort,

2) vector spaces with scalar field only as interpreted above, where in each case the morphisms are merely *embeddings,* not just elementary embeddings.[1]

[1]In other words, if one vector space embeds in another, merely as an abelian group with the relation of parallelism, then the embedding preserves the whole structure of the vector space.

3. Lie rings

Suppose K is a field. Let $\operatorname{Der}(K)$ be the set of derivations of K. Then this is both

1. a vector space over K, and
2. a *Lie ring,* the multiplication being the Lie bracket, $(X, Y) \mapsto [X, Y]$, where

$$[X, Y] = X \circ Y - Y \circ X.$$

For example,

$$K = \mathbb{Q}(x^0, \dots, x^{m-1}), \qquad \operatorname{Der}(K) = \langle \partial_0, \dots, \partial_{m-1} \rangle_K,$$

where $\partial_i = \partial/\partial x^i$. Suppose V is both a subspace and sub-ring of $\operatorname{Der}(K)$. Then (K, V) is a **Lie–Rinehart pair.**

Since V is a vector space over K, we may suppose $K \subseteq \operatorname{End}(V, +)$. In particular, we have

$$(f, D) \mapsto fD \colon K \times V \to V.$$

Since $V \subseteq \operatorname{Der}(K)$, we have

$$(D, f) \mapsto Df \colon V \times K \to K.$$

Two compatibility conditions are satisfied. First, if $f, g \in K$ and $D \in V$, then

$$(fD)g = f(Dg). \tag{$*$}$$

Thus the expression

$$fDg$$

is unambiguous. Next, if $f, g \in K$ and $D, E \in V$, then

$$\begin{aligned}[D, fE]g &= D(fEg) - fE(Dg) \\ &= (Df)(Eg) + fD(Eg) - fE(Dg) \\ &= ((Df)E)g + f[D, E]g,\end{aligned}$$

so

$$[D, fE] = (Df)E + f[D, E]. \qquad (\dagger)$$

Suppose $D \in V$ and $t \in K$ and $Dt \neq 0$. For every f in K, we have

$$\Big(\frac{f}{Dt}D\Big)t = f.$$

Thus

$$K = \{Dt \colon D \in V\}.$$

That is, under the assumption that there is a nonzero derivative, then every element of the field is a derivative.

Let b denote the Lie bracket operation. I propose to call the structure $(V, +, -, 0, \mathsf{b}, t)$ a **Lie ring of vectors.** The class of these is elementary.[1] For, first of all, there are axioms as follows:

1. $(V, +, -, 0)$ is an abelian group.
2. b makes this a lie ring: b distributes over $+$, and

$$X \mathbin{\mathsf{b}} X = 0,$$
$$X \mathbin{\mathsf{b}} (Y \mathbin{\mathsf{b}} Z) + Y \mathbin{\mathsf{b}} (Z \mathbin{\mathsf{b}} X) + Z \mathbin{\mathsf{b}} (X \mathbin{\mathsf{b}} Y) = 0.$$

[1] That is, the class of such structures for which there is a field K making (V, K) a Lie–Rinehart pair as above is elementary.

3. t is an endomorphism of the group:

$$t(X+Y) = tX + tY.$$

Next, rearranging the second compatibility condition (†), we obtain

$$(Df)E = D \mathsf{b} (fE) - f(D \mathsf{b} E). \qquad (‡)$$

If f is replaced by t, then the right hand side is a term in our signature. We then take the left hand side as an abbreviation of this. By the axioms so far, each operation $X \mapsto (Dt)X$ or Dt is an endomorphism of $(V, +)$. Let

$$K = \{Xt \colon X \in V\}.$$

Then this is a group under

$$Xt + Yt = (X+Y)t.$$

The map $X \mapsto Xt$ is a group homomorphism from K to $\mathrm{End}(V, +)$. We want it to be a *ring monomorphism.* So the axioms say further:

4. The action is faithful:

$$(Xt)Y = 0 \to Y = 0 \lor (Xt)Z = 0.$$

5. K is closed under multiplication:

$$\exists W \ (Xt)((Yt)Z) = (Wt)Z.$$

(Here the outer universal quantifiers are suppressed.) Then multiplication is associative and distributes over addition, by what we already have; so K is an associative ring. Expressions like

$$(Xt)(Yt)Z$$

are now unambiguous.

6. K is commutative:

$$(Xt)(Yt)Z = (Yt)(Xt)Z.$$

7. K has inverses:[2]

$$\exists Z\ \big((Zt)(Xt)Y = Y \vee (Xt)Y = 0\big).$$

   In particular, since K is closed under multiplication, it contains 1, which is different from 0, since the action is faithful.

We also need K to be closed under the action of V. Again by the second compatibility condition, rearranged as (‡), we have

$$(DFt)E = D\,\mathsf{b}\,(FtE) - (Ft)(D\,\mathsf{b}\,E),$$

the right hand being a term of the signature; we use the left as an abbreviation. So we now require:

8. That K be closed under $x \mapsto Dx$, for all D in V:

$$\exists W\ (XYt)Z = (Wt)Z.$$

9. That the first compatibility condition $(*)$ hold:

$$\Big(\big((Xt)Y\big)(Zt)\Big)W = \big((Xt)(YZt)\big)W.$$

This is it. We have not shown that V acts on K as a Lie ring of derivations, but this is automatic from the definition of the action, since K is now established as a sub-ring of $\operatorname{End}(V,+)$.

If $0 < m < \omega$, let LV^m be the theory of m-dimensional Lie rings of vectors. Let $(V,+,\mathsf{b},t)$ be a model, with scalar field

[2]Originally I forgot the condition that $(Xt)Y$ should not be 0.

K. Then V has a basis of *commuting* derivations $\partial_0, \dots, \partial_{m-1}$ of K, so

$$(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}.$$

The structure $(K, \partial_0, \dots, \partial_{m-1})$ has a one-dimensional interpretation in $(V, +, \mathsf{b}, t)$ with coordinate map $X \mapsto Xt$ from (all of) V to K (see Hodges [7, §5.3, p. 212]).[3] To show this, we need to find, for the appropriate formulas φ of the signature $\{+, \cdot, \partial_0, \dots, \partial_{m-1}\}$, formulas φ^* of the signature $\{+, \mathsf{b}, t\}$ such that

$$(V, +, t) \models \varphi^*(X, \dots) \iff (K, +, \cdot, \partial_0, \dots, \partial_{m-1}) \models \varphi(Xt, \dots).$$

These are as follows.

φ	φ^*
$x = y$	$(Xt)\partial_0 = (Yt)\partial_0$
$x + y = z$	$(Xt)\partial_0 + (Yt)\partial_0 = (Zt)\partial_0$
$x \cdot y = z$	$(Xt)(Yt)\partial_0 = (Zt)\partial_0$
$\partial_i x = y$	$(\partial_i Xt)\partial_0 = (Yt)\partial_0$
$x \neq y$	$(Xt)\partial_0 \neq (Yt)\partial_0$

The existence of all but the last formula φ^* ensures the interpretation. That *all* of the φ^* are quantifier-free (existential would be enough) ensures that, if $(V, +, \mathsf{b}, t)$ is existentially closed, then so is $(K, +, \cdot, \partial_i \colon i < m)$.[4]

[3] As will be seen, the interpretation will use the parameter ∂_0.

[4] That is, if some model $(V, +, \mathsf{b}, t)$ of LV^m is an elementary substructure of every model of which it is a substructure, then the same is true of every model of m-DF interpreted in $(V, +, \mathsf{b}, t)$ as above.

To derive this conclusion directly, we need only note that, if

$$(K, +, \cdot, \partial_0, \dots, \partial_{m-1}) \subseteq (L, +, \cdot, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1}),$$

then, letting

$$\tilde{V} = \langle \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1} \rangle_L,$$

and letting $\tilde{\mathsf{b}}$ be the Lie bracket, we have an embedding $\partial_i \mapsto \tilde{\partial}_i$ of $(V, +, \mathsf{b}, t)$ in $(\tilde{V}, +, \tilde{\mathsf{b}}, t)$.

We can go the other way. There is an m-dimensional interpretation of $(V, +, \mathsf{b}, t)$ in $(K, +, \cdot, \partial_0, \dots, \partial_{m-1})$, with coordinate map

$$(x^0, \dots, x^{m-1}) \mapsto \sum_{i<m} x^i \partial_i.$$

As before, if $(K, +, \cdot, \partial_0, \dots, \partial_{m-1})$ is existentially closed, so must $(V, +, \mathsf{b}, t)$ be.

Since m-DF has the model companion m-DCF [16], the theory LV^m also has a model companion.[5] Indeed, note that if V and K correspond as above, Then the dual space V^* has a basis $(\mathrm{d}\, t^i \colon i < m)$ for some t^i in K, and this basis is dual to a basis $(\partial_i \colon i < m)$ of V given by

$$\partial_i t^j = \delta_i^j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

This is a commuting basis, since

$$[\partial_i, \partial_j]\, \mathrm{d}\, t^k = [\partial_i, \partial_j] t^k = 0$$

[5] The details of this paragraph concerning the dual space V^* were lacking originally, but are found at [14, p. 925]—where there is a blanket assumption that fields have characteristic 0, but this does not affect the present point. Apparently I gave the details in the talk as well: see the appendix.

in each case. Then the axioms of the model-companion of LV^m say that if $\{\partial_0, \dots, \partial_{m-1}\}$ is a subset of V such that

$$\partial_i D_j t = \delta_i^j$$

for all i and j in m, for some $(D_j : j < m)$ in V^m, then the structure $(K, +, \cdot, \partial_i : i < m)$ is a model of m-DCF.[6]

[6] One could avoid mentioning the D_j. But then one must require both that, in each case, $\partial_i \mathsf{b} \partial_j = 0$, and also that the ∂_i span V. The resulting axioms for the model companion of LV^m would not then be $\forall\exists$, though one could still assert that such axioms must exist.

A. Transcript

I transcribe the following from Ayşe Berkman's handwritten notes of my talk. She does not guarantee that her notes are complete. I add the labelling and captioning of figures.

Descartes:

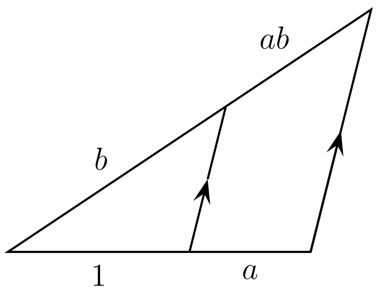


Figure A.1.: Descartes's definition of multiplication

Say (V, K) is a vector space, where $\dim_K V \geqslant 2$. Let

$$A = \{(x, y) \in V^2 \colon x \parallel y \wedge y \neq 0\}.$$

We have

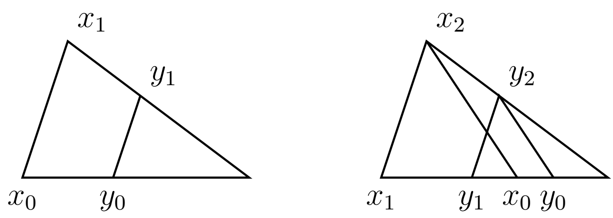
$$\begin{aligned} f \colon A &\twoheadrightarrow K \\ (x, y) &\mapsto [x : y], \qquad \text{where} \qquad x = [x : y]y. \end{aligned}$$

This f is the coordinate map of an interpretation of $(K, +, \cdot)$ in $(V, +, \parallel)$. One checks:

- A is definable;
- for certain φ of $\{+, \cdot\}$ there are φ^* of $\{+, \|\}$ such that for all (x, y) in A,

$$V \models \varphi^*(x, y) \iff K \models \varphi(f(x, y)) :$$

φ	φ^*
$x_0 = x_1$	$x_0 - x_1 \parallel y_0 - y_1$ as in Fig. A.2a $\vee \big(y_0 \parallel y_1 \wedge \exists (x_2, y_2)$ as in Fig. A.2b$\big)$
$x + y = z$	
$x \cdot y = z$	
$x \neq y$	



(a) $y_0 \nparallel y_1$ (b) $y_0 \parallel y_1$

Figure A.2.: Equality of pairs (x_i, y_i) of parallel vectors

Remark: These are all $\exists$ formulas.[1]
Result: If

$$T = \text{Th(two sorted vector spaces)},$$
$$T' = \text{Th(abelian groups with } \|),$$

[1]In the table, in the formula $(x_0 = x_1)^*$, the first disjunct needs the condition $y_0 \nparallel y_1$. Also, the formulas φ for which apparently I did not write down φ^* should be $x_0 + x_1 = x_2$, $x_0 \cdot x_1 = x_2$, and $x_0 \neq x_1$. Alternatively, they can stay as they are, but then $x_0 = x_1$ should be $x = y$, so that the elements of A used in $(x = y)^*$ would be (x_0, x_1), (y_0, y_1), and (z_0, z_1), rather than (x_0, y_0), (x_1, y_1), and (x_2, y_2).

then $\mathbf{Mod}_{\subseteq}(T)$ and $\mathbf{Mod}_{\subseteq}(T')$ are equivalent. (Morphisms are embeddings, not necessarily elementary.)

Again (V, K) is a vector space, but now say also $V \subseteq \mathrm{Der}(K)$. For example,

$$K = \mathbb{Q}(x_0, \dots, x_{m-1}), \qquad V = \left\langle \frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_{m-1}} \right\rangle_K.$$

$\mathrm{Der}(K)$ is always a vector space over K; it is also a Lie ring under $[\ ,\ ]$, where

$$[D, E] = D \circ E - E \circ D.$$

So, say V is both a subspace and sub-ring of $\mathrm{Der}(K)$. Then (K, V) is a Lie–Rinehart algebra. If $f \in K$, then

$$\begin{aligned}[D, fE] &= D \circ (fE) - fE \circ D \\ &= Df \cdot E + f \cdot D \circ E - fE \circ D \\ &= Df \cdot E + f \cdot [D, E].\end{aligned}$$

Also if $g \in K$ then[2]

$$(fD)g = f \cdot (Dg).$$

Suppose $t \in K$, and for some D in V, $Dt \neq 0$. Then

$$\begin{aligned} V &\twoheadrightarrow K \\ D &\mapsto Dt \end{aligned}$$

because

$$\left(\frac{g}{Dt} \cdot D\right) t = g \qquad \text{if} \qquad Dt \neq 0.$$

[2]The following identity was used in the previous identity, where for example $fE \circ D$ is understood first as $(fE) \circ D$, that is, $x \mapsto (fE)(Dx)$, and then as $f(E \circ D)$, that is, $x \mapsto f \cdot (E(Dx))$.

Now suppose $\dim_K V = m$. We have

$$\begin{aligned} K &\to V^* \\ x &\mapsto \mathrm{d}\, x \end{aligned}$$

given by

$$D(\mathrm{d}\, x) = Dx, \qquad \text{where} \qquad D \in V.$$

The range of d spans V^*. So for some t^0, …, t^{m-1} in K,

$$V = \langle \mathrm{d}\, t^0, \dots, \mathrm{d}\, t^{m-1} \rangle_K.$$

Let $(\partial_0, \dots, \partial_{m-1})$ be the dual basis of V, so

$$\partial_i(\mathrm{d}\, t^j) = \delta_i^j.$$

Then $[\partial_i, \partial_j] = 0$ in each case. So

$$(K, \partial_0, \dots, \partial_{m-1}) \models m\text{-DF}$$

(differential fields with m commuting derivations). The map $D \mapsto Dt$ allows us to interpret $(K, \partial_0, \dots, \partial_{m-1})$ in (V, b, t) (where $t \in \mathrm{End}(V)$ and b is the bracket) by $\exists$ formulas. Similarly

$$\begin{aligned} K^m &\to V \\ (x^0, \dots, x^{m-1}) &\mapsto \sum_{i<m} x^i \partial_i \end{aligned}$$

is the coordinate map of an interpretation of (V, b, t) in $(K, \partial_0, \dots, \partial_{m-1})$, again by $\exists$ formulas. So we get model-complete theories of Lie rings with a group endomorphism.

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