# The Sense of Proportion in Euclid

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## Preface

Here are notes from a talk given in the mathematics department of Gebze Institute of Technology on Friday, May 8, 2015, at 3:00 p.m. Physically, the talk was in the Molecular Biology and Genetics conference room, to provide room for students to attend. I prepared a draft of the present notes by way of preparing to give the talk. Now I have edited the notes so that they are closer to what I actually said. Additional material is in footnotes. In order to include that material in a future talk, I may shorten or remove §1, "Geometry."

The abstract that I submitted for the talk was:


> A proportion is an identification of ratios. In the Elements, Euclid (c. 300 B.C.E.) gives two definitions of a proportion: a clear definition for arbitrary magnitudes, and an unclear definition for numbers. A positive real number, as defined by Richard Dedekind (1831–1916), can be understood as a ratio of magnitudes in Euclid's sense. However, unlike Euclid, Dedekind establishes the *existence* of all of the so-called real numbers: this has been overlooked, at least by some of Dedekind's contemporaries. It has also been thought that Euclid's

> ratios of numbers are just fractions in the modern sense; but this makes Euclid wrong in ways that he is not likely to have been wrong. Euclid is *more* careful than we often are today with the foundations of number theory. He proves rigorously that in an ordered ring whose positive elements are well-ordered, multiplication is commutative. Seeing this can be helped by treating the reading of Euclid as an instance of doing history: history in the sense worked out by the philosopher R. G. Collingwood (1889–1943) in several of his books.

The talk is based mostly on a long (draft) essay of mine [6].

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# Introduction

We are going to look at the two definitions of **proportion** in Euclid's *Elements*:

1. the geometric,
2. the arithmetic.

The first is clear, but hard to understand; the second is obscure, but thought to be easy to understand. So people make mistakes about both of them. A way to avoid mistakes is to treat Euclid *historically.* We should not assume that his mathematics is the same as ours.

# 1 Geometry

## 1.1 Dedekind

We start with the question of how Dedekind's account, in "Continuity and Irrational Numbers" (1872), of the real numbers depends on Euclid's geometric definition of proportion.

Given $(\mathbb{Q}^+, <)$ (the linear order of positive rational numbers), following Dedekind, we define a **cut** as a pair $(A, B)$ of nonempty disjoint subsets of $\mathbb{Q}^+$ such that

$$A \cup B = \mathbb{Q}^+, \qquad A < B.$$

For example, every $c$ in $\mathbb{Q}^+$ determines the two cuts

$$(\{x \in \mathbb{Q}^+ : x < c\}, \{x \in \mathbb{Q}^+ : c \leqslant x\}),$$
$$(\{x \in \mathbb{Q}^+ : x \leqslant c\}, \{x \in \mathbb{Q}^+ : c < x\})$$

(which for Dedekind are "not essentially different"). But there are also cuts such as

$$(\{x \in \mathbb{Q}^+ : x^2 < 2\}, \{x \in \mathbb{Q}^+ : 2 < x^2\}),$$

not determined by a rational number: we consider it as defining a new, *irrational* number (called $\sqrt{2}$ in this case), which, conversely, determines the cut.

Then we can define $\mathbb{R}^+$ as $\mathbb{Q}^+ \cup \{\text{irrationals}\}$.[1]

[1] Thus $(\mathbb{R}, <)$ is a **complete** linear order (every nonempty subset with an upper

## 1.2 Bertrand

In the preface to "The Nature and Meaning of Numbers" (1887), Dedekind reports claims that his (and others') idea is already found in Bertrand, *Traité d'Arithmétique* (1849).

However, given nonsquare $N$ in $\mathbb{Q}^+$, Bertrand defines $\sqrt{N}$ *geometrically*: If a unit length and a ray from an origin $O$ have been chosen, then

$$\sqrt{N} = |OA|,$$

where $A$ is the unique point on the ray such that

$$OB < OA < OC \implies |OB|^2 < N < |OC|^2.$$

Probably $|OB|$ and $|OC|$ can be assumed to be rational, but Bertrand

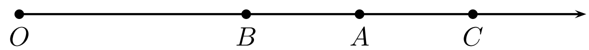


**Figure 1:** Bertrand's definition of $\sqrt{N}$

is not clear. Unlike Dedekind's, Bertrand's account does not show how to multiply possibly irrational numbers.[2] It also does not show why the point $A$ should exist.

bound has a least upper bound). The algebraic structure of $\mathbb{Q}^+$ has not been needed. Given $(\mathbb{Q}^+, <, +)$, if $C$ and $D$ are cuts, we define

$$C + D = \{x + y \colon x \in C \land y \in D\}.$$

Then $C + D \in \mathbb{R}$, and $(X, Y) \mapsto X + Y$ is continuous, and $(\mathbb{R}, +, <)$ is an abelian ordered semigroup. Similarly we obtain $(\mathbb{R}^+, \times, <)$ as an abelian ordered group. Introducing 0 and negative numbers, we obtain $(\mathbb{R}, +, \times, <)$ as a complete ordered field.

[2] For example, it does not show $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ (this is Dedekind's example).

## 1.3 Euclid

As Dedekind observes, it has been understood since Euclid's *Elements,* Book V, that an irrational number is defined by the cut of rational numbers that it determines. In Euclid, line segments[3] $A$, $B$, $C$, and $D$ are **proportional** if, for all natural numbers $k$ and $m$,

$$\begin{aligned} kA > mB &\iff kC > mD, \\ kA = mB &\iff kC = mD, \\ kA < mB &\iff kC < mD. \end{aligned}$$

In this case, one may say

- $A$ is to $B$ as $C$ is to $D$,
- $A$ has the **same ratio** to $B$ that $C$ has to $D$.

In Euclid:

- Two distinct line-segments can be equal (for example, an **isosceles** triangle has two equal sides).
- Two ratios are never equal, but they may be the *same.*

Thus, if $A$ is to $B$ as $C$ is to $D$, I prefer to write

$$A : B :: C : D,$$

rather than $A : B = C : D$. Then the ratio $A : B$ is defined by the cut

$$\left(\left\{\frac{x}{y} \in \mathbb{Q}^+ : xB \leqslant yA\right\}, \left\{\frac{x}{y} \in \mathbb{Q}^+ : xB > yA\right\}\right)$$

(and by "essentially the same" cut, if $A$ and $B$ are commensurable), as in Bertrand. But Dedekind defines cuts in terms of rational numbers alone, not magnitudes.

Euclid's *Elements* had been the foundational mathematical textbook for over two thousand years. Mathematics had changed, but this may

[3]They could be arbitrary **magnitudes,** such as planar regions, or solids. In any case, $A$ and $B$ **have a ratio,** which means some multiple of either magnitude exceeds the other, so that they generate an *archimedean* ordered semigroup; also $C$ and $D$ must have a ratio.

have been hard to see. It had been assumed that the real number line was a *geometric* object. Finally, Dedekind was able to see that it was not. Everything in Euclid requires only algebraic numbers.

## 2 Arithmetic

In the *Elements,* Books VII, VIII, and IX concern arithmetic. They are headed by some definitions:

**Definition 1.** **Unity** (or a **unit**) is that by virtue of which every existing thing is called one.

**Definition 2.** A **number** is a multitude composed of units.[4]

**Definition 3.** A number is a **part** of a number, the less of the greater, when it measures the greater.

**Definition 4.** But **parts,** when it does not measure.

**Definition 5.** And the greater [number] is a **multiple** of the less when it is measured by the less.

**Definition 15.** A number is said to **multiply** a number when, however many units are in it, so many times is the multiplicand composed, and some number comes to be.

We may write

$$a \cdot b = \underbrace{a + \cdots + a}_{b}.$$

Here $a$ and $b$ are what we call *positive integers,* elements of $\mathbb{N}$. Thus

- $a \cdot b$ is $a$ **multiplied by** $b$, or $b$ **times** $a$.
- $a$ **measures** $a \cdot b$.
- $a$ is a **part** of $a \cdot b$.
- $a \cdot b$ is a **multiple** of $a$.
- $b$ **divides** $a \cdot b$. (Euclid does not use this.)

[4]Unity is normally not a number, though sometimes must be allowed to be.

If $a < b$, and $a$ is **parts** of $b$, does this mean,[5] for some $c$, $k$, and $m$,

$$a = c \cdot k, \qquad b = c \cdot m, \qquad 1 < k < m?$$

Euclid's **Definition 20** (of Book VII) is,

> Numbers are **proportional** when the first is of the second, and the third is of the fourth, equally multiple, or the same part, or the same parts.

Can we interpret this to mean $a : b :: c : d$ if and only if, for some $e$, $f$, $k$, and $m$,

$$\begin{aligned} a &= e \cdot k, & b &= f \cdot k, \\ b &= e \cdot m, & d &= f \cdot m? \end{aligned}$$

As Pengelley and Richman (2006) observe, by this interpretation, the relation " :: " between $a : b$ and $c : d$ is not obviously transitive.

I would go further: the interpretation does not give a way to extract a definition of the ratios $a : b$ and $c : d$ in the first place. But again, one way Euclid reads our expression $a : b :: c : d$ is,

$a$ has the **same ratio** to $b$ that $c$ has to $d$.

Therefore the above interpretation must be wrong. We could say

$$(a : b) = \{(x, y) \colon \text{for some } e,\ a = e \cdot x \ \ \&\ \ b = e \cdot y\}.$$

[5]The term "parts" *may* allude to Egyptian fractions. There were tables from which one could learn, for example,

$$\frac{12}{17} = \frac{1}{2} + \frac{1}{12} + \frac{1}{17} + \frac{1}{34} + \frac{1}{51} + \frac{1}{68}.$$

The example is from David Fowler, *The Mathematics of Plato's Academy* (1999), where it is said, "We have no evidence for any conception of common fractions $p/q$ and their manipulations such as, for example, $p/q \times r/s = pr/qs$ and $p/q + r/s = (ps + qr)/qs$, in Greek mathematical, scientific, financial, or pedagogical texts before the time of Heron and Diophantus. . . "

Then, by the above interpretation,

$$a : b :: c : d \iff (a : b) \cap (c : d) \neq \varnothing.$$

This does not say $a : b$ and $c : d$ are the same.

Consider **Proposition 4** of Book VII:

> Every number is of every number, the less of the greater, either part or parts.

Euclid's proof is not, "Immediate from the definitions." It considers cases. Assume $a < b$.

1. If $a$ and $b$ are coprime, then

$$a = 1 \cdot a, \qquad\qquad b = 1 \cdot b.$$

2. Suppose $a$ and $b$ are not coprime.
   a) If $a$ measures $b$, then $a$ is a part of $b$.
   b) If not, let $c$ be the greatest common measure of $a$ and $b$. Then for some $k$ and $m$,

$$a = c \cdot k, \qquad\qquad b = c \cdot m.$$

This is not really a proof.[6] Euclid's "proof" of Proposition 4 follows the pattern of **Propositions 1** and **2,** in which the so-called **"Euclidean Algorithm"** is shown to produce the **greatest common measure** of two numbers.[7] (In Proposition 3, the greatest common measure of three numbers is found.)

[6] Euclid never promised it would be. We learned the "statement-proof" style of presenting mathematics from Euclid. We cannot complain if he does not always use the style in the way we expect.

[7] Let $A_1$ and $A_2$ be numbers or magnitudes, where $A_1 > A_2$. By the so-called **Euclidean Algorithm,** we obtain sequences

- $A_1, A_2, A_3, \dots$, of numbers or magnitudes,
- $n_1, n_2, \dots$, of multipliers,

such that

$$A_k = \underbrace{A_{k+1} + \dots + A_{k+1}}_{n_k} + A_{k+2}, \qquad A_{k+1} > A_{k+2}.$$

In case $A_1$ and $A_2$ are numbers, the sequence ends with some $A_m$.

Then the "proof" of Proposition 4 shows that $a : b :: c : d$ means the Euclidean Algorithm has the same steps, whether applied to $a$ and $b$ or $c$ and $d$.[8] For example,

$$\begin{aligned} 57 &= 21 \cdot 2 + 15 & \qquad 38 &= 14 \cdot 2 + 10 \\ 21 &= 15 \cdot 1 + 6 & 14 &= 10 \cdot 1 + 4 \\ 15 &= 6 \cdot 2 + 3 & 10 &= 4 \cdot 2 + 2 \\ 6 &= 3 \cdot 2 & 4 &= 2 \cdot 2 \end{aligned}$$

and therefore

$$57 : 21 :: 38 : 14.$$

The Euclidean Algorithm involves "alternate subtraction": in Greek, **anthyphaeresis.**[9] By the anthyphaeretic definition, it is clear that a straight line dividing a parallelogram into two parallelograms divides

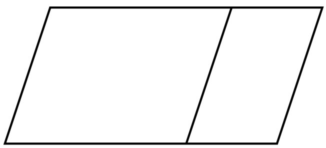

**Figure 2:** Parallelograms in the same parallels are as their bases

the base in the same ratio.[10] In the *Topics,* Aristotle uses this result

[8] Thus $a : b :: c : d$ means for some $k$ and $m$,

$$\begin{aligned} a &= \mathrm{gcm}(a, b) \cdot k, & \qquad c &= \mathrm{gcm}(c, d) \cdot k, \\ b &= \mathrm{gcm}(a, b) \cdot m, & d &= \mathrm{gcm}(c, d) \cdot m, \end{aligned}$$

where gcm means greatest common measure.

[9] Strictly, Euclid uses only the verb ἀνθυφαιρέ-ω "alternately subtract"; the related noun is ἀνθυφαίρεσις.

[10] According to Proposition vi.1 of the *Elements,* "Triangles and parallelograms that are under the same height are to one another as their bases"—that is, they have the same ratio as their bases.

as an example of something that is immediately clear, once one has the correct definition; and the definition of "same ratio" is "having the same **antanaeresis** (ἀνταναίρεσις)." In a comment on the passage, Alexander of Aphrodisias observes that Aristotle uses the word "antanaeresis" for anthyphaeresis. In 1933, Oskar Becker observed that Aristotle and Alexander could be alluding to the Euclidean Algorithm.[11]

By the earliest definition, it seems, $A : B :: C : D$ means, that, whether applied to $A$ and $B$ or to $C$ and $D$, the Euclidean Algorithm gives us the same pattern of subtractions.

However, when applied to arbitrary magnitudes, the Euclidean Algorithm may never end. [12]

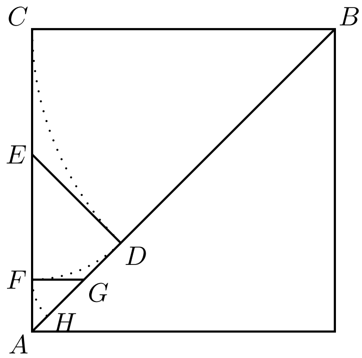


**Figure 3:** Incommensurability of side and diagonal

[11] My source is Ivor Thomas, at the end of the first of the two Loeb Classical Library volumes, *Selections Illustrating the History of Greek Mathematics.*

[12] In this case, the magnitudes are **incommensurable,** as Euclid shows in Proposition X.2. For example, in the square in Figure 3,

$$AB = AC \cdot 1 + AD,$$
$$AC = AD \cdot 2 + AF,$$
$$AD = AF \cdot 2 + AH,$$

In any case, it is difficult to prove general geometric results with. Thus Euclid prefers the Book-V definition of proportion for arbitrary magnitudes.

But the anthyphaeretic definition of proportion is still behind the scenes in the arithmetical books. By this definition,

$$1 : a :: b : b \cdot a$$

(since 1 is $a$ parts of $a$, and $b$ is $a$ parts of $b \cdot a$). Also, if $a : b :: c : d$, then

$$a : b :: a + c : b + d.$$

In particular, since $1 : a :: 1 : a$, we obtain

$$1 : a :: \underbrace{1 + \cdots + 1}_{b} : \underbrace{a + \cdots + a}_{b},$$

that is,

$$1 : a :: b : a \cdot b.$$

Therefore

$$b \cdot a = a \cdot b$$

and so on. In modern terms, we start with

$$\sqrt{2} = 1 + (\sqrt{2} - 1), \qquad 0 \leqslant \sqrt{2} - 1 < 1,$$

that is, 1 measures $\sqrt{2}$, one time, with $\sqrt{2} - 1$ remaining. Then

$$\frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 = 2 + (\sqrt{2} - 1),$$

so $\sqrt{2} - 1$ measures 1 twice, with $(\sqrt{2} - 1)^2$ remaining, and so on. Thus, formally,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}$$

—a result not normally proved, but only assumed, in number-theory courses today.[13]

[13]The commutativity result can be understood as being that, in any well-ordered set that is closed under *ordinal* addition and multiplication, if addition is commutative, then so is multiplication. We may assume that that well-ordered set is an ordinal itself. Every nonzero ordinal has a Cantor normal form

$$\omega^{\alpha_0} \cdot b_0 + \cdots + \omega^{\alpha_n} \cdot b_n,$$

where

$$\omega = \{0, 1, 2, \dots\}, \qquad \alpha_0 > \dots > \alpha_n, \qquad \{b_0, \dots, b_n\} \subseteq \omega \smallsetminus \{0\}.$$

The usual rules of arithmetic apply, except that

- addition is not commutative:

$$1 + \omega = \omega < \omega + 1.$$

- multiplication is not commutative, and distributes over addition only from the left, not the right:

$$(1 + 1) \cdot \omega = 2 \cdot \omega = 2 + 2 + \cdots = \omega < \omega + \omega = \omega \cdot 2.$$

Then the ordinals that are closed under addition and multiplication are precisely the ordinals of the form

$$\omega^{\omega^{\alpha}}.$$

The only one of these where addition is commutative is $\omega$, that is, $\omega^{\omega^0}$; and here multiplication is commutative as well. Since every ordinal equation

$$\alpha + \xi = \beta$$

has a unique solution, provided $\alpha \leqslant \beta$, we can extend the operations of addition and multiplication to

$$\omega^{\omega^{\alpha}} \cup \{-\xi \colon 0 < \xi < \omega^{\omega^{\alpha}}\},$$

just as we extend them from $\mathbb{N}$ to $\mathbb{Z}$ in school. In general, multiplication will not distribute over addition in either sense. Again though, if addition is commutative, then so will multiplication be.

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