A Method for Companionability, Applied to Group Actions and Valuations

with Ayşe Berkman and with
Özlem Beyarslan, Daniel Max Hoffmann, and Gönenç Onay

David Pierce

Mimar Sinan Güzel Sanatlar Üniversitesi, İstanbul

Delphi, July, 2017

Ἢν στρατεύηται ἐπὶ Πέρσας, μεγάλην ἀρχὴν μιν καταλύσειν

—Oracle to Croesus, as reported by Herodotus (i.53)

Three theories of education:

1. Learn the word of God.
2. Learn skills.
3. Learn freedom.

Mathematics teaches freedom by being

(a) *personal* (none can command you to accept a theorem),

(b) *universal* (disagreement must be settled peacefully).

> [Formal proof] was one of the great discoveries of the early 20th century, largely due to Frege, Russell, and Whitehead . . . This discovery has had a profound impact on mathematics, because it means that *any dispute about the validity of a mathematical proof can always be resolved.*
>
> —Timothy Gowers, *Mathematics: A Very Short Introduction*

Nesin Mathematics Village, Şirince
Selçuk, İzmir (Ephesus, Ionia), July 9, 2016

A PROBLEM OF MODEL THEORY: identify complete theories and their properties, such as being axiomatizable or not.

Presburger 1930. $\mathrm{Th}(\mathbb{N}, +)$ is axiomatizable.

Gödel 1931. $\mathrm{Th}(\mathbb{N}, +, \times)$ is not.

USEFUL DEFINITIONS:

$$\begin{aligned} \mathrm{diag}(\mathfrak{A}) &= \mathrm{Th}(\text{structures in which } \mathfrak{A} \text{ embeds}), \\ T_\forall &= \mathrm{Th}(\text{structures embedding in models of } T). \end{aligned}$$

A. Robinson 1956. A theory T is **model-complete** if, whenever $\mathfrak{A} \models T$, then $T \cup \mathrm{diag}(\mathfrak{A})$ is complete.

"Eli Bers" 1969. A model-complete theory T^* is the **model-companion** of any theory T for which ${T^*}_\forall = T_\forall$.

Fields may have

1) a valuation ring $\mathcal{O}$ (with max. ideal $\mathcal{M}$),
2) a derivation δ,
3) an automorphism σ.

the theory of fields	has model companion	source
—	ACF	Tarski 1948
with $\mathcal{O}$	ACVF	Robinson 1956
of char. 0 with δ	DCF_0	Robinson 1959
of char. p with δ	DCF_p	Wood 1973
with σ	ACFA	Macintyre 1997; Chatzidakis–Hrushovski 1999
of char. 0 with δ, σ	yes	P. 2004
of char. p with δ, σ	no	P. 2004
with $\mathcal{O}$, σ	yes	Beyarslan–Hoffmann–Onay–P. 201?

THEOREM (generalizing **Robinson 1957**). If it exists, the model-companion T^* of a theory T is axiomatized by $T_\forall$ and sentences

$$\forall \boldsymbol{x}\ \forall \boldsymbol{y}\ \exists \boldsymbol{z}\ \bigl(\vartheta(\boldsymbol{x}, \boldsymbol{y}) \Rightarrow \varphi(\boldsymbol{x}, \boldsymbol{z})\bigr),$$

where

- φ is a system of atomic and negated atomic formulas,
- ϑ is from a set Θ_φ of formulas,
- for all models $\mathfrak{M}$ of $T_\forall$ with parameters $\boldsymbol{a}$,

$$\vartheta(\boldsymbol{a}, \boldsymbol{y}) \text{ is soluble in } \mathfrak{M} \text{ for some } \vartheta \text{ in } \Theta_\varphi \iff$$
$$\varphi(\boldsymbol{a}, \boldsymbol{z}) \text{ is soluble in a model of } T_\forall \cup \operatorname{diag}(\mathfrak{M}).$$

Not every system φ need be considered, but "enough" of them.

To axiomatize DCF_0, Robinson considered all systems.

Blum 1968: one-variable systems are enough.

For the model-companion of any theory T, it is enough to consider *unnested* systems.

EXAMPLE. Over a field K with σ, $\mathcal{O}$, and $\mathcal{M}$, one need only understand systems

$$\bigwedge_{f\in I_0} f = 0 \wedge \bigwedge_{i<m} X_i{}^{\sigma} = X_{\tau(i)} \wedge \bigwedge_{\ell\in\lambda} X_\ell \in \mathcal{O} \wedge \bigwedge_{k\in\kappa} X_k \in \mathcal{M},$$

where, for some n in ω,

- I_0 is a finite subset of $K[X_j : j < n]$,
- $m \leqslant n$ and $\tau\colon m \rightarrowtail n$,
- $\kappa \subseteq \lambda \subseteq n$.

EXAMPLE. A **group action** is (P, A), where

1) $P = \{\text{functions}\}$,
2) $A = \{\text{points}\}$,
3) there is $(\xi, y) \mapsto \xi\, y$ from $P \times A$ to A whereby
   (a) functions have inverses: $\forall\xi\; \exists\eta\; \forall z\; (\xi\,\eta\, z = z \wedge \eta\,\xi\, z = z)$;
   (b) two functions have a composite: $\forall\xi\; \forall\eta\; \exists\zeta\; \forall v\; \xi\,\eta\, v = \zeta\, v$;
   (c) there is an identity: $\exists\xi\; \forall y\; \xi\, y = y$.

Let $\mathsf{GA} = \mathrm{Th}(\text{group actions})$.

Then $\mathsf{GA}_\forall$ is that functions are injective:

$$\forall\xi\; \forall y\; \forall z\; (y \neq z \Rightarrow \xi\, y \neq \xi\, z).$$

KEY OBSERVATION: Compositions not preserved in extensions.

To find GA^*, one need only consider systems

$$\bigwedge \alpha\, x = y \wedge \bigwedge \xi\, t = u,$$

where t and u are point variables or constants.

GA^* is complete and says,

1) any $n!$ distinct functions act like $\mathrm{Sym}(n)$ on some n points;

2) on any n distinct points, some $n!$ functions act like $\mathrm{Sym}(n)$;

3) there are at least two points.

GA includes Th(**parametrized permutations**), axiomatized by

$$\forall\xi\ \forall y\ \forall z\ (y \neq z \Rightarrow \xi\, y \neq \xi\, z), \qquad \forall\xi\ \forall y\ \exists z\ \xi\, z = y.$$

Shelah 1993: $\mathsf{T}_{\mathrm{feq}}$, namely

Th(**parametrized equivalence relations**).

$\mathsf{T_{feq}}^* = \mathrm{Th}(\text{Fraïssé limit of the class of finite models of } \mathsf{T_{feq}})$.

It was shown that $\mathsf{T_{feq}}^*$ has TP_2 and, ultimately, NSOP_1.

Thus $\mathsf{T_{feq}}^*$ occupies an undivided region of the *Map of the Universe* (**Conant 2013–**, `forkinganddividing.com`).

As for GA^*, so for for $\mathsf{T_{feq}}^*$, one can obtain axioms:

1) a partition of n points is effected by some relation,

2) the intersection of classes of n distinct relations is nonempty,

3) there are n relations and n classes of each.

Like $\mathsf{T_{feq}}^*$, GA^* has TP_2 and NSOP_1.

Chernikov–Ramsey 2016: In a finite relational signature,

if the theory of the Fraïssé limit of a Fraïssé class with Strong Amalgamation is simple,

then the theory of *parametrized* models has NSOP_1,

because it has a certain independence relation $\mathop{\smile\!\!\!\!\!\!\,|}$ with independent amalgamation of types.

THEOREM. GA^* also has NSOP_1, because of $\mathop{\smile\!\!\!\!\!\!\,|}$ given by

$$(\mathrm{P}, A) \mathop{\smile\!\!\!\!\!\!\,|}_{(\mathrm{T},C)} (\Sigma, B) \iff \mathrm{P} \cap \Sigma \subseteq \mathrm{T} \ \& \ \langle A \cup C\rangle_{\mathrm{P}\cup\mathrm{T}} \cap \langle B \cup C\rangle_{\Sigma\cup\mathrm{T}} \subseteq \langle C\rangle_{\mathrm{T}},$$

where $\langle X\rangle_{\Xi} = \{\xi^n x : \xi \in \Xi \ \& \ n \in \mathbb{Z} \ \& \ x \in X\}$.

Τέττιξ on tree, Marmara Island (Proconnesus), July 26, 2012

τῇ δὲ πρεσβυτάτῃ Καλλιόπῃ καὶ τῇ μετ᾽ αὐτὴν Οὐρανίᾳ τοὺς ἐν φιλοσοφίᾳ διάγοντάς τε καὶ τιμῶντας τὴν ἐκείνων μουσικὴν ἀγγέλλουσιν, αἳ δὴ μάλιστα τῶν Μουσῶν περί τε οὐρανὸν καὶ λόγους οὖσαι θείους τε καὶ ἀνθρωπίνους ἱᾶσι καλλίστην φωνήν —Plato, *Phaedrus* 259D