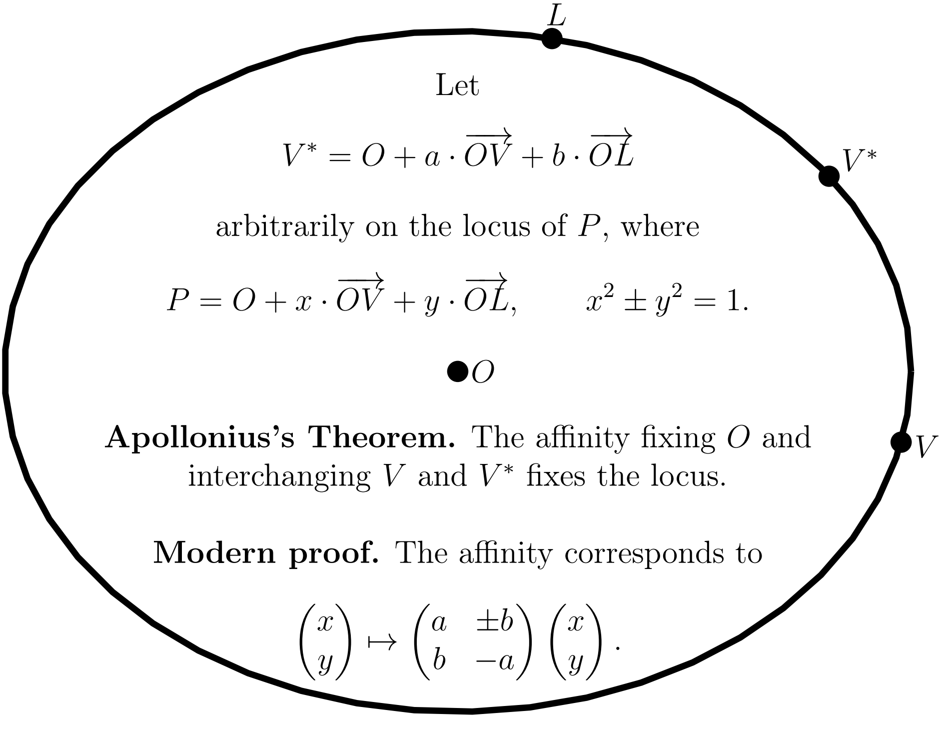
Affine Planes with Polygons

David Pierce
Mimar Sinan Güzel Sanatlar Üniversitesi
mat.msgsu.edu.tr/~dpierce, polytropy.com

Batumi (ბათუმი), September, 2019
საქართველოს მათემატიკოსთა კავშირის
X ყოველწლიური საერთაშორისო კონფერენცია
X Annual International Conference
of the Georgian Mathematical Union



Let

$$V^* = O + a \cdot \overrightarrow{OV} + b \cdot \overrightarrow{OL}$$

arbitrarily on the locus of P, where

$$P = O + x \cdot \overrightarrow{OV} + y \cdot \overrightarrow{OL}, \qquad x^2 \pm y^2 = 1.$$

Apollonius's Theorem. The affinity fixing O and interchanging V and V^* fixes the locus.

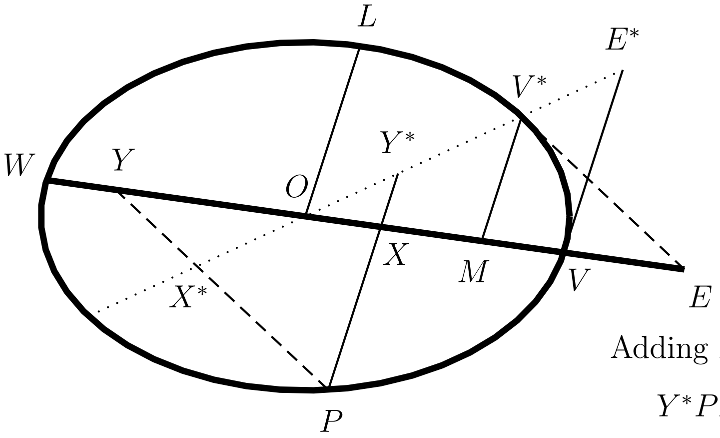
Modern proof. The affinity corresponds to

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & \pm b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Apollonius's Proof. The locus is given by $\boxed{XPY = VXY^*E^*,}$

because this is true when P is V^*, and

$$XPY \propto XP^2 \propto XV \cdot XW \propto VXY^*E^*.$$



Adding XYX^*Y^* yields

$$Y^*PX^* = VYX^*E^*,$$

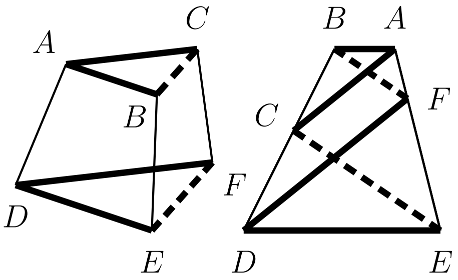
then $\boxed{Y^*PX^* = EYX^*V^*.}$

The foregoing happens in an **affine plane,** satisfying

1) two points determine a line;

2) through a point not on a line, a single parallel passes;

3) there is a proper triangle.



The plane is K^2 for some field K, if also, assuming

$$AB \parallel DE \ \ \& \ \ AC \parallel DF,$$

4) **Desargues's Theorem:**

$$BC \parallel EF,$$

if AD, BE, and CF either

a) are mutually parallel or
b) have a common point;

5) **Pappus's Theorem:**

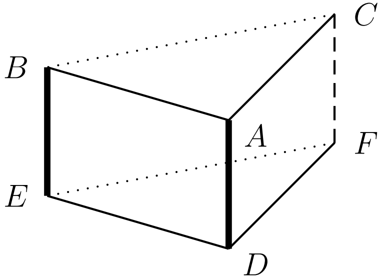
$$BF \parallel CE,$$

if
- D lies on BC and
- A lies on EF.

Of Desargues, case (a), **"Prism,"** lets us define, for non-collinear directed segments,

$$\overrightarrow{AD} = \overrightarrow{BE} \iff$$

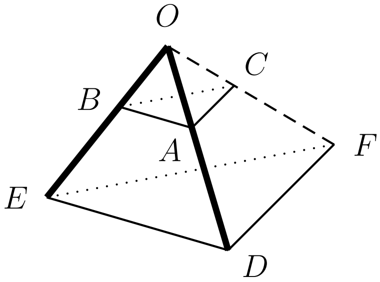
$$ABED \text{ is a parallelogram;}$$



case (b), **"Pyramid,"** for non-parallel pairs of parallel vectors,

$$\overrightarrow{OA} : \overrightarrow{OD} :: \overrightarrow{OB} : \overrightarrow{OE}$$

$$\iff AB \parallel DE.$$



On the plane, ratios of vectors act as a field or skew-field.

With Pappus, it is a field.

For Pappus and Desargues to be *Theorems,* I propose axioms:

Addition. The polygons compose an abelian group where

$$-ABC\cdots = \cdots CBA,$$
$$AAB = 0, \qquad A* \;=\; *A,$$
$$A*B + B\dagger A = A*B\dagger,$$

$*$ and $\dagger$ being strings of vertices.

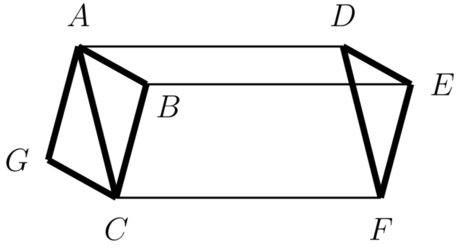
Linearity.

$$ABC \neq 0 \wedge BCD = 0 \\ \wedge C \neq D \Rightarrow ACD \neq 0.$$

Parallels . . .

Translation. $ABED$ and $BCFE$ being parallelograms,

$$ABC = DEF.$$

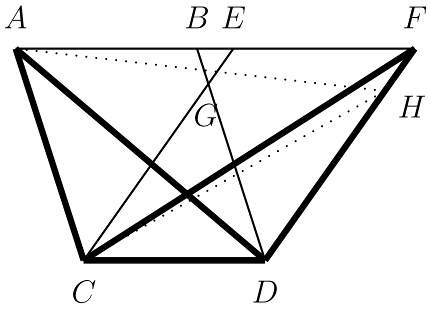


Bisection. $ABCG$ being a parallelogram,

$$CGA = ABC.$$

Halving. All nonzero polygons have the same order, not 2.

Hence **Euclid I.37, 39:**



• Assuming $AF \parallel CD$, by **Parallels** we may let

$$AC \parallel BD, \qquad CE \parallel DF;$$

by **Translation,**

$$ACE = BDF;$$

by **Addition,**

$$ACDB = ACGB + CDG =$$
$$ACE - BGE + CDG =$$
$$BDF - BGE + CDG = ECDF;$$

by **Bisection,**

$$ACDB = 2ACD,$$
$$ECDF = 2FCD;$$

by **Halving,**

$$ACD = FCD.$$

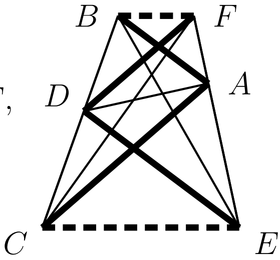
• If $AF \not\parallel CD$, let $AH \parallel CD$.

$$ACD = HCD,$$
$$FCD = HCD + FCH,$$

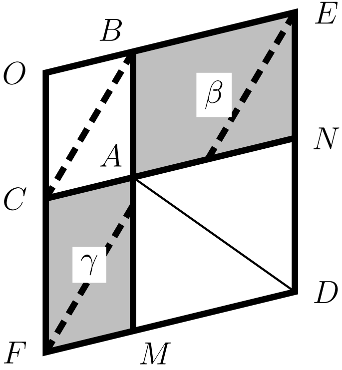
so $ACD \neq FCD$ by **Linearity.**

We can now prove:

- **Prism,** by Translation, I.39, and Parallels.
- **Pappus.** From $AB \parallel DE$ and $AC \parallel DF$, we obtain $BF \parallel CE$, since, by I.37,



$$BFC = BFAD = BFE.$$



- **Pyramid, special case.** Assume
  - BE and CF meet at O,
  - $AB \parallel DE$ and $AC \parallel DF$,
  - $AB \parallel OC$ and $AC \parallel OB$.

Then

$$\begin{aligned} AD \text{ contains } O &\iff \beta = \gamma \\ &\iff BC \parallel DF. \end{aligned}$$

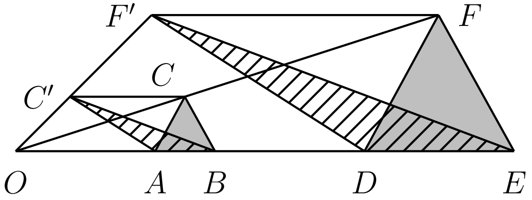
Pyramid, less special case. If

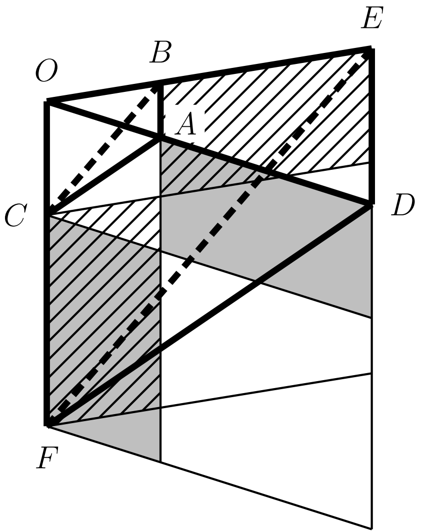
- AD, BE, and CF meet at O,
- $AB \parallel DE$ and $AC \parallel DF$,
- $AB \parallel OC$, then $BC \parallel DF$.

General case. Assuming $AC' \parallel DF'$,

$$ABC \sim DEF \implies ABC' \sim DEF',$$

provided $CC' \parallel OA$; we can remove this; putting C' on OC is enough.





Given $\boxed{ABC \sim DEF,}$ we let

$$BG \parallel AC, DF,$$
$$HG, DF' \parallel AC'.$$

By Pappus,

1) from $ACHGBC'$, $HC \parallel BC'$;

2) from $BCDFEG$, $DC \parallel EG$;

3) from $HCDF'EG$, $HC \parallel EF'$.

Thus

$$BC' \parallel EF',$$

$\boxed{ABC' \sim DEF';}$ also $\boxed{ADC \sim BEG.}$