Apollonian Proof

Slides with commentary

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We shall look at a proof by Apollonius that seems to have been overlooked, even by scholars of Apollonius, be they historians like Fried and Unguru or mathematicians like Rosenfeld.

I shall put my slides and notes for this talk on my departmental webpage; my blog at `polytropy.com` already has an article called "Elliptical Affinity" from April, with animations illustrating the proof.

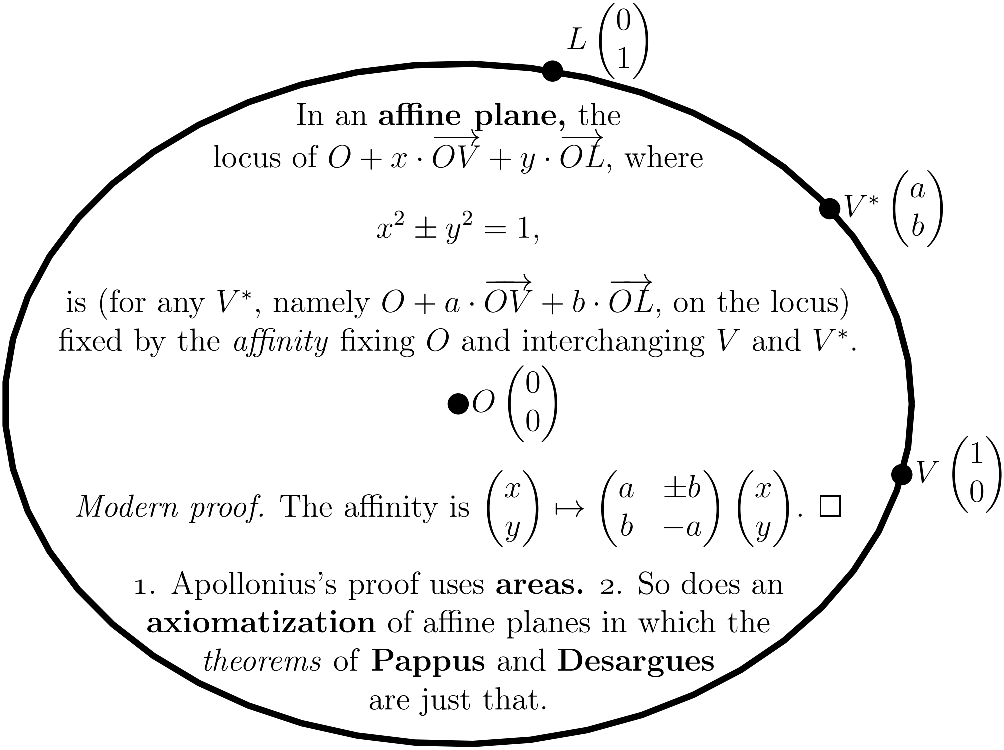
The proof uses *areas* and works in an **affine plane,** namely a principle homogeneous space of a 2-dimensional vector space over some field (not of characteristic 2). This means the space acts *simply* (or *sharply*) *transitively* on the plane.

I am going to offer an axiomatization of affine planes based on areas.

[Slide 2] In an affine plane, we choose non-collinear points O, V, and L, determining a coordinate system in which, by definition,

- $\overrightarrow{OV}$ is a unit vector in the x-direction;
- $\overrightarrow{OL}$ in the y-direction.

A **conic section** with center O is given by $x^2 + y^2 = 1$ or $x^2 - y^2 = 1$.



In an **affine plane,** the locus of $O + x \cdot \overrightarrow{OV} + y \cdot \overrightarrow{OL}$, where

$$x^2 \pm y^2 = 1,$$

is (for any V^*, namely $O + a \cdot \overrightarrow{OV} + b \cdot \overrightarrow{OL}$, on the locus) fixed by the *affinity* fixing O and interchanging V and V^*.

Modern proof. The affinity is $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & \pm b \\ b & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. □

1. Apollonius's proof uses **areas.** 2. So does an **axiomatization** of affine planes in which the *theorems* of **Pappus** and **Desargues** are just that.

In modern terms, Apollonius's theorem is that, for an arbitrary point V^* on the curve, the affine transformation that fixes O and interchanges V and V^* fixes the whole curve setwise.

The modern proof involves plugging and chugging with the given rule. This took centuries of development after Descartes's *Geometry* (1637).

In modernizing Euclid, Hilbert reduces the theory of areas to a theory of lengths, which compose a field. Michael Beeson continued this work on Tuesday, *defined* equality of rectangles. To prove this transitive, he needed Hilbert's theory of proportion, as simplified by Bernays.

We shall develop a theory of proportion *from* a theory of areas. Apollonius's proof will use all of this.

In *Geometric Algebra* (1957), Artin shows how to obtain a field from an affine space, axiomatized by:

1. Two points determine a line.
2. **Playfair's Axiom:** through a point not on a line, a single parallel passes.
3. There are three non-collinear points.
4. **Desargues's Theorem:** except in cases of collinearity, the following are transitive:
   a) being opposite sides of a parallelogram
   b) being sides of a triangle cut by a line parallel to the base.

   We can now define **vectors,** and **ratios** of parallel vectors; the ratios compose a field or skew-field, acting on the plane.
5. **Pappus's Hexagon Theorem,** to be seen later.

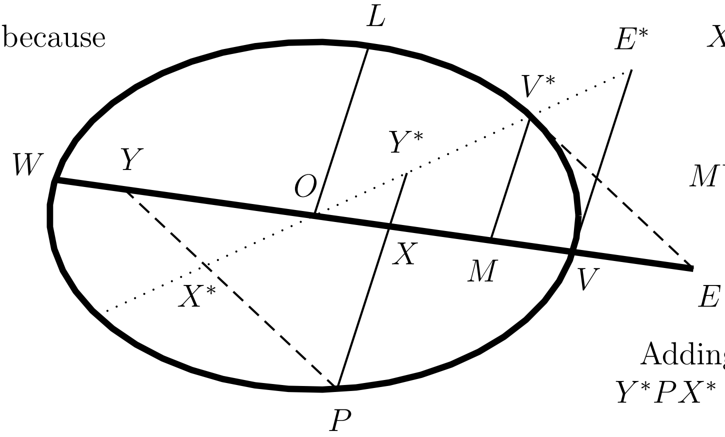
Though Hilbert calls it Pascal's, the Hexagon Theorem is Pappus's, as its *Wikipedia* article explains, in the section called "Origins"—which I added a few years ago. I forgot the "parallel" case that we are using now: Lemma VIII in the relevant part of the *Collection.* In his *History of Greek Mathematics,* Heath omits to discuss Lemma VIII too.

[Slide 3] We first establish the equation of triangle and trapezoid at the top. In the figure:

Proof of Apollonius. The locus of P is given by

$$XPY = VXY^*E^*,$$

because



$$\begin{aligned} XPY &\propto XP^2 \\ &\propto XV \cdot XW \\ &\propto VXY^*E^*, \\ MV^*E &= VMV^*E^*. \end{aligned}$$

Adding XYX^*Y^* yields $Y^*PX^* = VYX^*E^*$, then

$$Y^*PX^* = EYX^*V^*.$$

- P is a random point on the conic section.
- PX, V^*M, and VE^* are parallel to OL.
- OE is a third proportional to OM and OV.
- Hence the given equation $MV^*E = VMV^*E^*$ holds.
- $PY \parallel EV^*$.

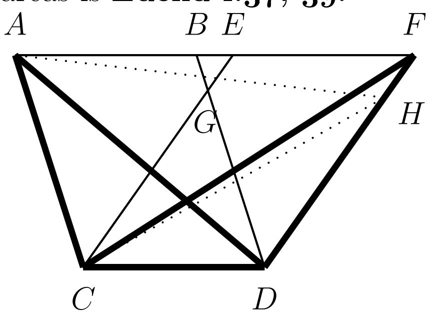
We conclude:

- XPY varies as the square on its side XP.
- By the property that Apollonius derives from the cone itself, the square on the *ordinate* XP varies jointly as the two *abscissas* XW and XV.
- The trapezoid VXY^*E^* also varies jointly as the abscissas, since in particular

$$XW = XO + VO \propto XY^* + VE^*.$$

[Slide 4] Michael Beeson reviewed the proof of Euclid's Proposition I.35, which yields as a consequence 37 and its converse, 39: triangles on the same base are equal if and only if the line joining their apices is parallel to the common base.

Fundamental to the geometry of areas is **Euclid I.37, 39.**



1. Assuming $AF \parallel CD$, let
$$AC \parallel BD, \quad CE \parallel DF.$$
2. By translation,
$$ACE = BDF.$$
3. By polygon algebra,
$$ACDB = ECDF.$$
4. By bisection,
$$ACDB = 2ACD,$$
$$ECDF = 2FCD.$$
5. By halving,
$$ACD = FCD. \qquad (*)$$
6. If $AF \nparallel CD$, let $AH \parallel CD$.
$$ACD = HCD,$$
$$FCD = HCD + FCH,$$
so $(*)$ fails.

I analyze the proof of 37 and 39 into six parts, corresponding to six axioms for an affine plane, consisting, as a structure, of

- a sort for points,
- a sort for polygons,
- for each n greater than 2, a map sending an n-tuple of points to the polygon with those points as vertices,
- a relation of equality of polygons,
- operations of an abelian group on the sort of polygons.

In order 3, 6, 1, 5, 4, 2, the steps are justified by:

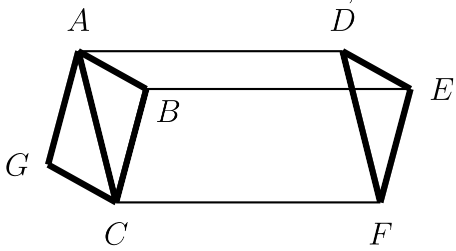
Axiom 1. The polygons compose an abelian group Π where, $*$ and $\dagger$ being strings of vertices,

$$A* = A*A = *A,$$
$$A*B + B\dagger A = A*B\dagger,$$
$$-ABC\cdots = \cdots CBA.$$

Axiom 2. $ABC = 0$ means that A, B, and C are collinear.

Axiom 3. Playfair's Axiom.

Axiom 4. All nonzero elements of Π have the same order, not 2.



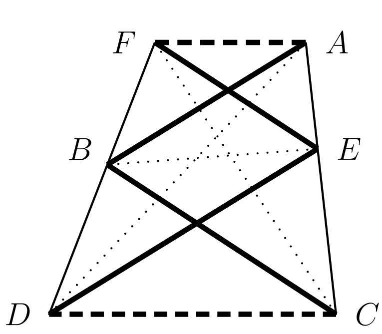
Axiom 5, 6. If $ABCG$, $ABED$, and $BCFE$ are parallelograms, then

$$CGA = ABC = DEF.$$

[Slide 5] In Axiom 1, the polygons shown as equal can be taken as identical; but equality as in Axioms 2, 5, and 6 will be a *congruence* with respect to the abelian-group operations on polygons.

In Axiom 2, we use equations $ABC = 0$ to express that the set of points, each collinear with two points, is determined in this way by any two of its points.

In Axiom 4, the common order, if finite, is automatically prime.



From Euclid I.*37, 39:*

- **Pappus's Hexagon Theorem,** by his proof: In hexagon $ABCDEF$, if

$$AB \parallel DE, \qquad BC \parallel EF,$$

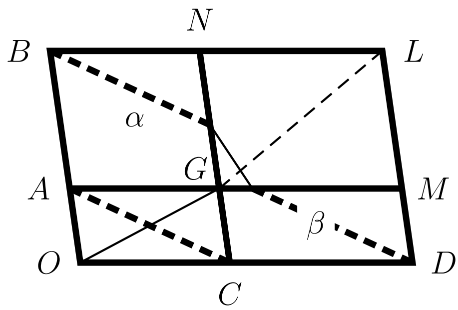
then $FAD = FAEB = FAC$, so

$$CD \parallel FA.$$

- **Euclid** I.**43 plus.**

$$OGL = 0 \iff \alpha = \beta$$
$$\iff BD \parallel AC.$$

- *Desargues's Theorem.*



[Slide 6] By design, our six axioms yield Euclid I.37 and 39, and these in turn give us the Hexagon Theorem, by Pappus's own proof (except he uses an intersection point of the bounding lines DF and CA).

It remains to prove Desargues's Theorem.

Michael Beeson used the diagram of I.43 to define $\alpha = \beta$ in the rectangular case when G lies on OL.

We strengthen I.43 with its converse and more, in order to establish first a special case of Desargues.

Desargues's Theorem. If

$$AB \parallel DE \ \& \ AC \parallel DF,$$

then $BC \parallel EF$, so $ABC \sim DEF$.
Proof.

- True when $AB \parallel OC$, by I.43+.
- Enough now that, since

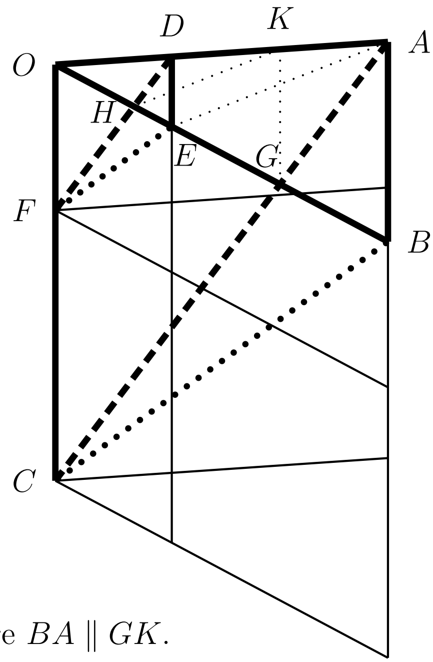
$$BAG \sim EDH,$$

  for all X (not shown) on OA,

$$BXG \sim EYH$$

  for some Y on OA.

Note $BAE \sim GKH$ by Pappus, where $BA \parallel GK$.



[Slide 7] Now we obtain Desargues's Theorem, that in triangles ABC and DEF, where the bold solid and bold dashed sides are parallel, the bold dotted sides are also parallel, so that the triangles are **similar.**

The converse will follow, that similar triangles are *perspective from a point.*

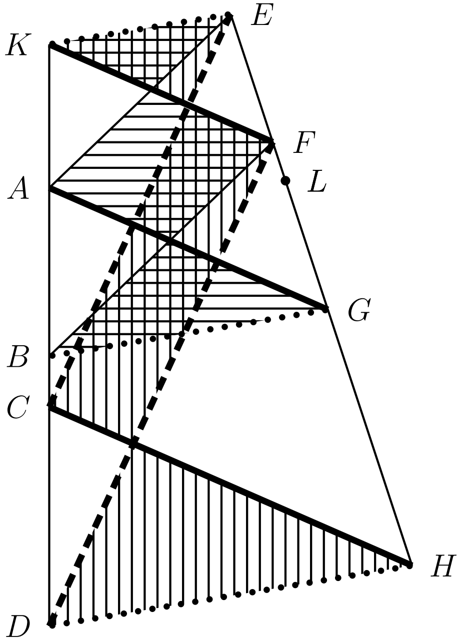
When we assume $AB \parallel OC$, the result follows by I.43 plus.

To continue, by Pappus, $BAG \sim EDH$ yields $BAE \sim GKH$, where $GK \parallel BA$.

Thus when BG and EH are bases of similar triangles with apices on OA, so are BE and GH.

We show that we can maintain similarity while moving the apices along OA.

Then the special case of Desargues yields the general.



Lemma. Given

$$AEC \sim BFD,$$

we noted

$$AEB \sim CLD$$

for some L on EF. Now let

$$KF \parallel AG \parallel CH.$$

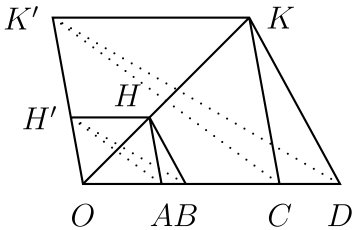
By Pappus twice,

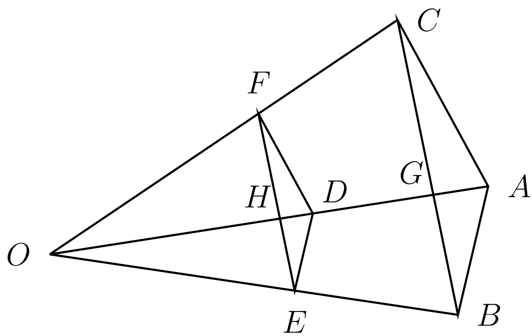
$$BG \parallel KE \parallel DH,$$

whence

$$AGB \sim CHD.$$

[Slide 8] In two steps now, if $AHB \sim CKD$, then $AH'B \sim CK'D$.





Hence if $EF \parallel BC$ and $DF \parallel CA$, so $HFD \sim GCA$, then $HED \sim GBA$, so $DE \parallel AB$.