Galois correspondences

David Pierce

2004.12.16

This note gives a unified treatment to several mathematical observations.

For notation, let 2 be considered as the universe $\{0,1\}$ of an abelian group. For each e in 2, let A_e be a set, and let $*$ be a relation from A_0 to A_1. This means that $*$ is a subset of $A_0 \times A_1$; if $*$ contains (x_0, x_1), then we write

$$x_0 * x_1.$$

If $x_e \in A_e$, let

$$[x_e] = \{x_{e+1} \in A_{e+1} : x_0 * x_1\} \in \mathcal{P}(A_{e+1}).$$

If $X \in \mathcal{P}(A_e)$, let

$$X' = \bigcap\{[x] : x \in X\} \in \mathcal{P}(A_{e+1}).$$

Let A^*_{e+1} be the image of $\mathcal{P}(A_e)$ under the map $X \mapsto X'$.

Theorem 1. *Let $e \in 2$.*

(∗) *If $X, Y \in \mathcal{P}(A_e)$ and $X \subseteq Y$, then $Y' \subseteq X'$.*

(†) *If $X \in \mathcal{P}(A_e)$, then $X \subseteq X''$.*

(‡) *The map $X \mapsto X'$ is a bijection from A^*_e to A^*_{e+1} with inverse $X \mapsto X'$.*

Proof. Exercise. (For the last point, see [3, ch. V, Lemma 2.6].) □

Regardless of how the maps $X \mapsto X'$ are originally defined, if they meet the conditions established in the theorem, they constitute a **Galois correspondence** between A^*_0 and A^*_1. (This definition is in [4, p. 35].) There are several examples, as you should verify:

Field-theory

The usual Galois correspondence in field-theory is the case when A_0 is a field L that is a finite Galois extension of a field K, and A_1 is $\mathrm{Aut}(L/K)$, and

$$x * \sigma \iff x^\sigma = x.$$

Then A^*_0 comprises the subfields F of L that include K, and A^*_1 comprises the subgroups H of $\mathrm{Aut}(L/K)$, and $F' = \mathrm{Aut}(L/F)$, and $H' = \mathrm{Fix}(H)$.

The Zariski topology

Suppose A_0 is a ring R (commutative with 1), and A_1 is $\operatorname{Spec} R$, that is, the set of prime ideals of R. Let $*$ be $\in$. Then

$$[x] \cup [y] = [xy]$$

if $x, y \in R$. Hence the sets $[x]$ are the basic closed sets for a topology, the **Zariski topology** on $\operatorname{Spec} R$. (See for example [1, pp. 54–55] or [2, § II.2].) The topology is compact, although possibly not Hausdorff. In this topology, if $X \subseteq \operatorname{Spec} R$, then X'' is the closure of X. If $X \subseteq R$, then X'' is the radical (in the sense of [3, ch. VIII, Definition 2.5]) of the (possibly improper) ideal (X). In general, A_0^* comprises the radical ideals of R, and A_1^* comprises the closed subsets of $\operatorname{Spec} R$.

The Stone space

Now suppose in particular that A_0 is a Boolean ring or algebra B, and A_1 is its **Stone space** $S(B)$, the set of ultrafilters of B. The ultrafilters are dual to the prime ideals, all of which are maximal. Let $*$ be $\in$ again. Then

$$[x] \cap [y] = [xy] = [x \wedge y]$$

when $x, y \in B$, and also

$$[x]^{\mathrm{c}} = [x + 1] = [\neg x],$$

so that

$$[x] \cup [y] = [x + y + xy] = [x \vee y].$$

Hence the sets $[x]$ are basic open and closed sets for a topology on $S(B)$. This topology is compact as before, but also Hausdorff. The elements of A_1^* are still just the closed subsets of $S(B)$; the elements of A_0^* are just the filters of B. If $X \subseteq B$, then X'' is the filter generated by X; if $X \subseteq S(B)$, then X'' is its closure.

Model-theory

Suppose $\mathcal{L}$ is a signature for first-order logic. Let A_0 be the *class* $\mathrm{Mod}(\mathcal{L})$ of $\mathcal{L}$-structures, let A_1 be $\mathrm{Sn}_{\mathcal{L}}$, and let $*$ be $\models$. Then A_1^* is the set of theories of $\mathcal{L}$, and A_0^* is the set of **elementary classes** of $\mathcal{L}$-structures. (See [4, § 3.4].)

References

[1] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[2] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[3] Thomas W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980. Reprint of the 1974 original.

[4] Philipp Rothmaler. *Introduction to model theory*, volume 15 of *Algebra, Logic and Applications*. Gordon and Breach Science Publishers, Amsterdam, 2000. Prepared by Frank Reitmaier, Translated and revised from the 1995 German original by the author.