Definable soluble and nilpotent envelopes "around" subgroups in simple theory

1. Motivation

Theorem. G is an infinite group with small theory. Then G has an infinite abelian subgroup.

Conjecture (Smidt). Every infinite group has an infinite abelian subgroup.

False. (1968 Adian Novikov).

Definition. A **Tarski Monster** is an infinite (countable) group such that every proper subgroup is either $\{1\}$ or cyclic of order a prime p.

Fact (Ol'shanskii 1979). For every prime $p > 10^{75}$, there are $2^{\aleph_0}$ non-isomorphic Tarski monsters.

Corollary. Any Tarski Monster has $2^{\aleph_0}$ countable models sharing its theory, up to isomorphisms.

1. Motivation

Theorem. G is an infinite group with small theory. Then G has an infinite abelian subgroup.

Corollary (Wagner). G is a group with small and stable theory. Then G has a definable infinite abelian subgroup.

Proof. A infinite abelian. Take $Z(C(A))$.

Question. When can one find definable abelian groups around abelian subgroups?

Question. When can one find definable nilpotent/soluble groups around nilpotent/soluble subgroups?

2. What is known

Remark. If G has **dcc on centralisers**, and $A \leq G$ is abelian H, then $Z(C(A))$ is a definable abelian envelope of H.

Fact (Poizat). If G is **stable** and $H \leq G$ is n-nilpotent/n-soluble, H has a definable n-nilpotent/n-soluble envelope.

Fact (Shelah). If G has **NIP** and $A \leq G$ is abelian, A has a definable abelian envelope.

Fact (Aldama). If G has **NIP** and $H \leq G$ is n-nilpotent/normal n-soluble, H has a definable envelope with same property.

Fact (Altınel, Baginsky). If G has **dcc on centralisers** and $H \leq G$ is n-nilpotent, H has a definable n-nilpotent envelope.

3. Question

What happens if G has merely a simple theory ? Can one find a definable abelian/nilpotent/soluble envelope of an abelian/nilpotent /soluble $H \leq G$?

The answer is no. But :

Proposition. If G is simple and $A \leq G$ is abelian, then A has a definable envelope which is abelian-by-finite.

Theorem A. If G is simple and $N \leq G$ is n-nilpotent, there is a definable $2n$-nilpotent group finitely many translates of which cover N.

Theorem B. If G is simple and $S \leq G$ is n-soluble, there is a definable $2n$-soluble group finitely many translates of which cover S.

4. Stable and simple definitions and properties

Definition. X is a definable subset of G, $\phi(x, y)$ a formula. The ϕ-**Cantor-Bendixson rank** of X :

- $CB(X, \phi) \geq 0$ if $X \neq \emptyset$,
- $CB(X, \phi) \geq n + 1$ if there are infinitely many 2-disjoint ϕ-sets $X_1, X_2, \ldots$ with $CB(X_i \cap X, \phi) \geq n$.

Definition. G is **stable** if $CB(G, \phi)$ is finite for every formula ϕ.

Definition. X is a definable subset of G, $\phi(x, y)$ a formula, k a natural number. The $D(\, . \, , \phi, k)$-**Cantor rank** of X :

- $D(X, \phi, k) \geq 0$ if $X \neq \emptyset$,
- $D(X, \phi, k) \geq n + 1$ if there are infinitely k-disjoint sets defined by $\phi(x, a_1), \phi(x, a_2), \ldots$ with $D(X_i \cap X, \phi, k) \geq n$.

Definition. G has a **simple** theory if $D(G, \phi, k)$ is finite for every formula ϕ and natural number k.

4. Stable and simple definitions and properties

Remark. $D(X, \phi, k) \leq CB(X, \phi)$: stability implies simplicity.

Fact (Baldwin Saxl's chain condition). G is a group with stable theory, $\phi(x, y)$ a formula. There is some n such that every descending chain of subgroups defined by ϕ-formulae has no more than n elements.

Fact (Wagner's chain condition). G is a group with simple theory, $\phi(x, y)$ a formula. There is some n such that every descending chain of subgroups defined by ϕ-formulae has no more than n elements, up to finite index.

4. Stable and simple definitions and properties

In a stable theory	Analogue in a simple theory
Uniform dcc	Uniform dcc up to finite index
abelian groups	FC-groups (eg finite, abelian)
$C_G(H)$	$FC_G(H) = \{g \in G : g^H$ is finite$\}$ (Haimo, 1953)
$Z(H)$	$FC(G) = FC_G(G))$
$Z_{n+1}(G)$	$FC_{n+1}(G)$ $(FC_{n+1}(G)/FC_n(G) = FC(G/FC_n(G))$
n-nilpotent	n-FC-nilpotent ($FC_n(G) = G$, Haimo) (eg finite, nilpotent)
n-soluble	n-FC-soluble (Duguid, McLain, 1956) $G_0 = G \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$ with $G_i \trianglelefteq G$ and G_i/G_{i+1} an FC-group (eg finite, soluble, virtually-soluble)

Proposition. G is a saturated group with simple theory, and H is a definable subgroup. Then $FC_G(H)$ is definable.

5. Main results

Theorem. Let G be a group with simple theory and N a subgroup of G. If N is FC-nilpotent of class n, then it is contained in a definable FC-nilpotent group of class n.

Theorem. Let G be a group with simple theory, and let S be a subgroup of G. If S is FC-soluble of class n, then it is contained in a definable FC-soluble group of class n the members of whose FC series are definable subgroups.

Fact (Wagner). *In a group with simple theory, an FC-nilpotent definable subgroup is virtually-m-nilpotent, with $m \leq 2n$.*

Proposition. In a group with simple theory, an FC-soluble definable subgroup is virtually-m-soluble, with $m \leq 2n$.

5. Main results

Corollary. If G is simple and N is n-nilpotent, there is a definable $2n$-nilpotent group finitely many translates of which cover N.

Corollary. If G is simple and S is n-soluble, there is a definable $2n$-soluble group finitely many translates of which cover S.

Corollary. In a group with simple theory, let N be a normal nilpotent subgroup of class n. There is a definable normal nilpotent group of class at most $3n$ containing N.

Corollary. In a group with simple theory, let S be a normal soluble subgroup of class n. There is a definable normal soluble group of class at most $3n$ containing S.

6. Next questions : nilpotent and soluble radical

In a group G, the **Fitting subgroup** $Fit(G)$ is the subgroup generated by all normal nilpotent subgroups of G.The **soluble radical** $R(G)$ is generated by all normal solvable subgroups of G.

Remark (Ould Houcine).

1. $Fit(G)$ is definable if and only if it is nilpotent.
2. $R(G)$ is definable if and only if it is solvable.

Question. In a group with simple theory, are $R(G)$ and $Fit(G)$ definable?

Fact (Wagner). *If G is stable, $Fit(G)$ is definable.*

Remark. Known for algebraic groups, groups of finite RM (Nesin).

Fact (Baudish). *If G is superstable, $R(G)$ is definable.*

Remark. Known for groups of finite RM (Belegradek), and groups of finite U-rank (Baldwin-Pillay).

6. Next questions : nilpotent and soluble radical

Fact (Elwes, Jaligot, Macpherson, Ryten). *G is a supersimple group of finite SU-rank such that T^{eq} eliminates $\exists^\infty$. Then $R(G)$ is definable and soluble.*

Question (Elwes, Jaligot, Macpherson, Ryten). G is a supersimple group of finite SU-rank such that T^{eq} eliminates $\exists^\infty$. Is $Fit(G)$ definable and nilpotent?

Proposition. Yes, and one does not need to assume that T^{eq} eliminates $\exists^\infty$.

Proposition. G is a supersimple group of finite SU-rank. Then $R(G)$ is definable and soluble.